A (new) method to compute multileg one-loop cross sections

Germán Rodrigo

IFIC Valencia

in collaboration with S. Catani, T. Gleisberg, F. Krauss and J. Winter

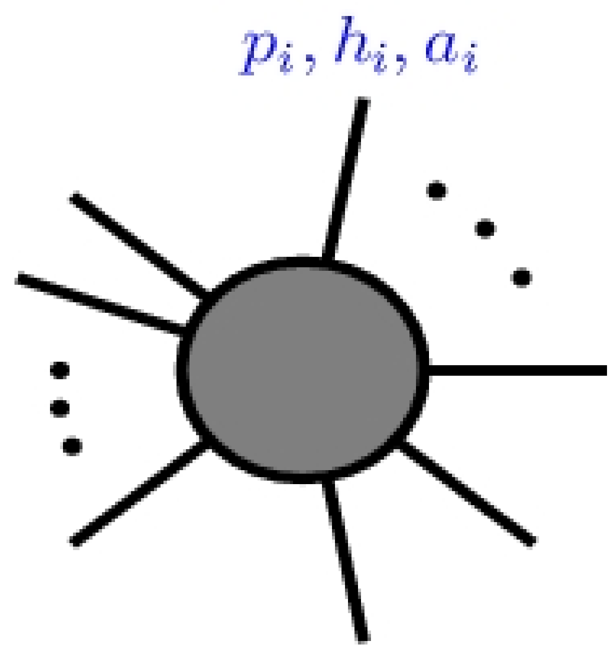
ILC Workshop, 6-11 November 2006, Valencia

ILC is a precision machine

- at least NLO prediccions (NNLO even better)
- higher energies → higher multiplicities

technical difficulties increases with the
number of loops and the **number of legs**

I. twistor inspired methods
II. numerical or seminumerical approaches
(much recent progress)



leading order

$$\sigma^{LO} = \int_m d\sigma^B = \int d\Phi^{(m)}(\{p_i\}) \quad M^{(m)}(\{p_i\}) \quad F^{(m)}(\{p_i\})$$

phase-space:
multidimensional integral

tree-level Feynman graphs
can be obtained by
analytical/numerical methods

kinematics: momentum
conservation + observable
dependent function

computable numerically by using e.g. MC methods
practical limitation: $m_{max} \sim 8$ at present

next-to-leading order

$$\sigma^{NLO} = \int_{m+1} d\sigma^{R} + \int_{m} d\sigma^{V}$$

new feature wrt LO:
combine m with m+1

real radiation

virtual contribution

next-to-leading order

$$\sigma^{NLO} = \int_{m+1} d\sigma^R + \int_m d\sigma^V$$

real radiation

virtual contribution

new feature wrt LO:
combine m with m+1

A) real radiation

$$\int_{m+1} d\sigma^R = \int d\Phi^{(m+1)}(\{p_i\}) \quad \underbrace{M^{(m+1)}(\{p_i\}) \quad F^{(m+1)}(\{p_i\})}$$

several well known/tested working methods (subtraction, dipole, slicing, mixed, ...)

split phase-space integrand in two parts:

$(...)_{fin} + (...)_{div}$

IR finite: computable numerically as LO

IR singular: analytically computable up to $O(\varepsilon)$

B) virtual contribution

$$\int_m d\sigma^V = \int d\Phi^{(m)}(\{p_i\}) \quad \underbrace{\int d^d q \quad M^{(m)}(\{p_i\})} \quad F^{(m)}(\{p_i\})$$

loop integral: in multiparton processes ($m \geq 5$) regarded as main practical bottleneck

- hard to get in analytic form
- numerical methods based
  or reduction formalism (pentagons, hexagons -> boxes)
  have to eliminate/control numerical instabilities
  (many terms, Gram determinants)

(Feynman parametrization costs one extra Feynman parameter per extra parton)

our general goal: two steps

I transform loop integral in customary phase space integral for real radiation

$$\int_{loop} d^d q \quad M^{(m)}(\{p_i\}, q) = \int d\Phi(q) \quad M^{(m+q)}(\{p_i\}, q)$$

$$d^d q \quad \delta_+(q^2)$$

II then treat $\int_{m+q}(\ldots)$

similarly to the real emission contribution $\int_{m+1}(\ldots)$

from loop integrals to phase space integrals

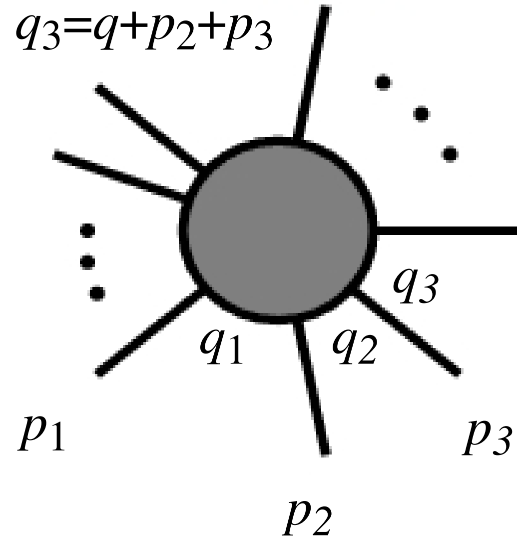
Multileg (N-point) one-loop integral
(to simplify presentation only masless and scalar)

$$L^{(N)}(p_1, \ldots, p_N) = -i\,\mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \prod_{i=1}^{N} \frac{1}{q_i^2 + i0}$$

bubles (N=2)
triangles (N=3)

well known, simple for arbitrary d

$q_1 = q$
$q_2 = q + p_2$
$q_3 = q + p_2 + p_3$

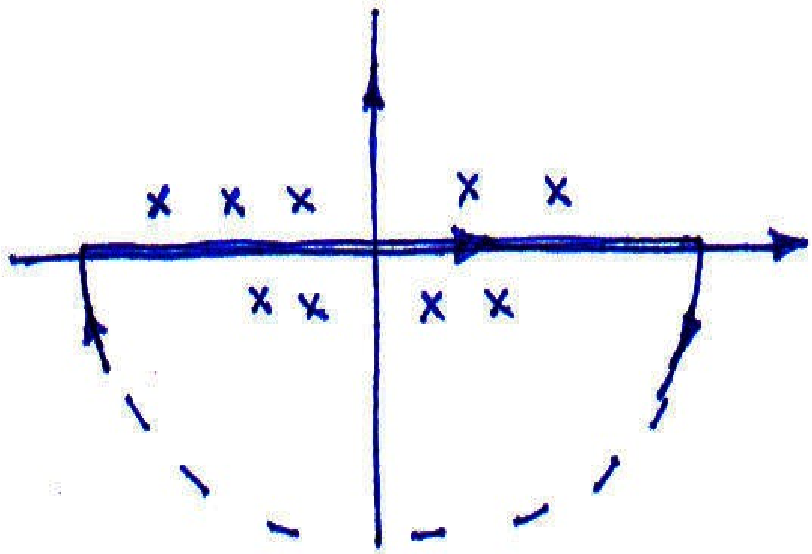


boxes(N=4) well known, relatively simple up to O(ε)

technical difficulties start from pentagons (N≥5)

basic complex analysis: residue theorem

consider $q^\mu=(q^0,q^{d-1})$ and go to q^0-complex plane: integration contour



$$L^{(N)} \sim \int_{-\infty}^{\infty} dq^0 \int d^{d-1} q \, (\dots)$$

close the contour at ∞ on the lower half plane → select residues with **positive** definite energies

$$L^{(N)} \sim (-2\pi i) \int d^{d-1} q \sum_{Res_i} (\dots)$$

In practice:

(i) $\boldsymbol{Res_i}$ equivalent to replacement $\dfrac{1}{q_i^2 + i0} \to -2\pi i \;\; \delta_+(q_i^2)$

(ii) then exchange $\int \sum_{Res_i} = \sum_{Res_i} \int$ (allowed if N>4)

(iii) then shift $q_i \longrightarrow q$ in each term

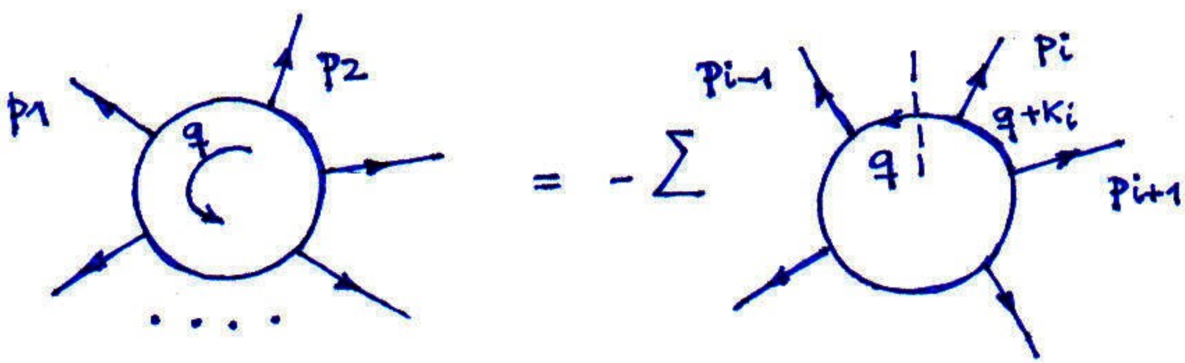
phase-space duality

general result

$$L^{(N)}(p_1,\ldots,p_N) = -\sum_{i=1}^{N} I^{(N-1)}(k_i,\ldots,k_{i+N-2})$$

loop integral

phase-space integral: named **DUAL INTEGRAL (single cut)**



$$I^{(n)}(k_1,\ldots,k_n) = \mu^{4-d}\int \frac{d^d q}{(2\pi)^{d-1}}\,\delta_+(q^2)\left[\prod_{j=1}^{n}\frac{1}{2q\cdot k_j + k_j^2}\right]$$

if all external legs are off-shell ($p_i^2 \neq 0$)

perform directly (numerically) phase space integration in d=4

counting number of terms

one-loop integral (N legs) $\Leftrightarrow$ sum of N dual integrals

but all the dual integrals have the same phase-space

one loop integral | one dual integral

$$\int d^d q \ \ (.....) \qquad = \qquad \int d^d q \ \ \delta_+(q^2) \ \ (.......)$$

one term | N-terms

in general,
multileg case
N large

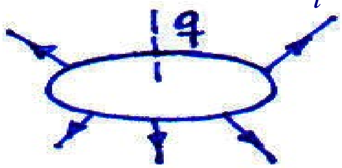
(#) dual $\gtrsim$ (#) real $\gtrsim$ (#) virtual

but always comparable

IR classification of dual integrals

IR behaviour depends on the two external momenta joined by the cut line

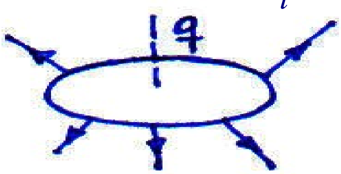
$p_{i-1}^2 \neq 0$ $\quad$ $p_i^2 \neq 0$



$$I_{finite}^{(N-1)} = \mu^{4-d} \int \frac{d^d q}{(2\pi)^{d-1}} \delta_+(q^2) \left[\prod_{j=1}^{N-1} \frac{1}{2q \cdot k_j + k_j^2} \right]$$

IR finite (numerically integrable in d=4)

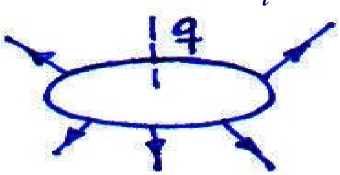
$p_{i-1}^2 \neq 0$ $\quad$ $p_i^2 = p^2 = 0$



$$I_{collinear}^{(N-2)} = \mu^{4-d} \int \frac{d^d q}{(2\pi)^{d-1}} \delta_+(q^2) \frac{1}{p \cdot q} \left[\prod_{j=1}^{N-2} \frac{1}{2q \cdot k_j + k_j^2} \right]$$

IR divergent in collinear region $q // p$, single $1/\varepsilon$ poles

$p_{i-1}^2 = \bar{p}^2 = 0$ $\quad$ $p_i^2 = p^2 = 0$



$$I_{soft}^{(N-3)} = \mu^{4-d} \int \frac{d^d q}{(2\pi)^{d-1}} \delta_+(q^2) \frac{p \cdot \bar{p}}{p \cdot q \; \bar{p} \cdot q} \left[\prod_{j=1}^{N-3} \frac{1}{2q \cdot k_j + k_j^2} \right]$$

IR divergent in collinear regions $q // p$ and $q // \bar{p}$
double $1/\varepsilon^2$ and single $1/\varepsilon$ poles

e.g. Box=$L^{(4)}$: 6 different types of boxes from IR point of view (4m, 3m, 2m-easy, 2m-hard, 1m, 0m)
3 different types of dual integrals from the IR point of view ($I_f^{(3)}, I_c^{(2)}, I_s^{(1)}$)

IR poles of dual integrals

computable in closed (and simple) analytic from for arbitrary number of external legs

collinear integrals

$$a_i = \frac{k_i^2}{2\, p \cdot k_i} \qquad l_i = \ln(a_i)$$

$$I_c^{(n)\,div.} = \frac{2 c_\Gamma}{\epsilon \prod_{i-1}^n k_i^2} \sum_{j=1}^n \; l_j \;\; a_j \left(\prod_{k \neq j} \frac{a_k}{a_k - a_j} \right)$$

soft integrals

$$I_s^{(n=1)\,div.} = \frac{2 c_\Gamma}{k_1^2} \left\{ \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \left(\frac{2\, p \cdot \bar{p}}{\mu^2} \right) - \frac{1}{\epsilon} (l_1 + \bar{l}_1) \right\}$$

$$I_s^{(n \geq 2)\,div.} = \frac{2 c_\Gamma}{\prod_{i-1}^n k_i^2} \left\{ \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \left(\frac{2\, p \cdot \bar{p}}{\mu^2} \right) - \frac{1}{\epsilon} \sum_{j=1}^n \left[l_j \left(\prod_{k \neq j} \frac{a_k}{a_k - a_j} \right) + (p \Leftrightarrow \bar{p}) \right] \right\}$$

related expression for one-loop integrals derived
by S. Dittmaier in terms of C0 functions

IR finite cross section

step I: $$\int_m d\sigma^V \rightarrow \int_{m+q} d\sigma^{DUAL}$$

$$\int_{m+1} d\sigma^{DUAL} = \int d\Phi^{(m+1)}(\{p_i\},q) \quad \tilde{M}^{(m+1)}(\{p_i\},q) \quad F^{(m)}(\{p_i\})$$

$$\Updownarrow$$

$$\int_{m+1} d\sigma^{R} = \int d\Phi^{(m+1)}(\{p_i\},q) \quad M^{(m+1)}(\{p_i\}) \quad F^{(m+1)}(\{p_i\})$$

step II: apply to dual tems the same general strategies/methods as for real emission

IR behaviour simpler than for real emission

("q" dependence fully decoupled form kinemtics, observable dependence)

Recurrence relations for dual integrals

alternative to overall substraction

soft integral (n≥2)

$$
\begin{aligned}
I_s^{(n)}(p,\bar{p};k_1,\ldots,k_n) &= \frac{1}{k_1^2} I_s^{(n-1)}(p,\bar{p};k_2,\ldots,k_n) \\
&- \left[\frac{2\,p\cdot k_1}{k_1^2} I_c^{(n)}(p;k_1,\ldots,k_n) + (p \Leftrightarrow \bar{p}) \right] \\
&- \frac{1}{k_1^2} I_{s,fin}^{(n)}(p,\bar{p};k_1,\ldots,k_n)
\end{aligned}
$$

residual IR finite term (no soft/coll singularities)

$$
I_{s,fin}^{(n)} = \mu^{4-d} \int \frac{d^d q}{(2\pi)^{d-1}} \delta_+(q^2) \, \frac{2\,k_{1\perp}\cdot q_\perp}{q_\perp^2} \left[\prod_{j=1}^{N-2} \frac{1}{2\,q\cdot k_j + k_j^2} \right]
$$

$\perp$ wrt p and $\bar{p}$

analogous recurrence relations for **collinear integrals**

- start from analytic expression for $I_s^{(n=1)}$, $I_c^{(n=2)}$ (~ Boxes)
- iterate recurrence relations

$$I_s^{(n)} = \ldots\ldots\ldots + \int d^d q \;\; \delta_+(q^2) \;\; \{\ldots. + \ldots. + \ldots. + \ldots.\}$$

Analytic:
includes all ε poles

one single dual integral

integrand: sum of terms separately finite

limited number: Pentagons $\longrightarrow 1$

Hexagons $\longrightarrow 4$

Heptagons $\longrightarrow 10$

No Gram determinants

Summary/Outlook

- proposal of a method (analytical+numerical) to compute one-loop virtual contributions to **NLO** cross-sections based on
  relating loops integrals to phase-space integrals (**DUAL INTEGRALS**)

- **Dual integrals**
  - basic setup ✔
  - IR singularities ✔
  - check with well known results (boxes, pentagons) (almost completed)

- **Work in progress**
  - decomposition IR div. (analytical) + IR fin. (numerical)
  - tensor integrals
  - massive case

- **future work**
  - implementation at the cross-section level
  - automation ?
  - 2-loops ?