

Answers for the Exercises for the lecture “Superconducting RF-I and II”

7 Oct 2007, K.Saito

Exercise I

$$\xi = \sqrt{\frac{2\pi}{\phi_0} \cdot H_{c2}},$$

$$\frac{H_c}{H_{c2}} = \frac{2\pi\xi^2}{2\pi\sqrt{2}\lambda\xi} = \frac{\xi}{\sqrt{2}\lambda}$$

$$\lambda = \frac{\xi}{\sqrt{2}} \cdot \frac{H_{c2}}{H_c} = \sqrt{\frac{\phi_0}{4\pi} \cdot \frac{H_{c2}}{H_c^2}}$$

$$\lambda(T) = \frac{\lambda(0)}{\sqrt{1 - \left(\frac{T}{T_c}\right)^4}}, \quad H_c(T) = H_c(0) \cdot \left[1 - \left(\frac{T}{T_c}\right)^2\right]$$

$$H_{c2}(T) = \frac{4\pi\lambda^2(0)H_c^2(0)}{\phi_0} \cdot \frac{1 - \left(\frac{T}{T_c}\right)^2}{1 + \left(\frac{T}{T_c}\right)^2} = H_{c2}(0) \cdot \frac{1 - \left(\frac{T}{T_c}\right)^2}{1 + \left(\frac{T}{T_c}\right)^2}$$

$$\xi(T) = \sqrt{\frac{\phi_0}{2\pi H_{c2}(0)}} \cdot \frac{\sqrt{1 + \left(\frac{T}{T_c}\right)^2}}{\sqrt{1 - \left(\frac{T}{T_c}\right)^2}} = \xi(0) \cdot \frac{\sqrt{1 + \left(\frac{T}{T_c}\right)^2}}{\sqrt{1 - \left(\frac{T}{T_c}\right)^2}}$$

$$\kappa(T) = \frac{\lambda(T)}{\xi(T)} = \frac{1}{\sqrt{2}} \cdot \frac{H_{c2}(T)}{H_c(T)} = \frac{H_{c2}(0)}{\sqrt{2}H_c(0)} \cdot \frac{1}{1 + \left(\frac{T}{T_c}\right)^2} = \frac{\kappa(0)}{1 + \left(\frac{T}{T_c}\right)^2}$$

$$H_c^{RF}(T) = \sqrt{2} \frac{H_c(T)}{\kappa(T)} = \frac{\sqrt{2}H_c(0)}{\kappa(0)} \cdot \left[1 + \left(\frac{T}{T_c}\right)^4\right] = H_c^{RF}(0) \cdot \left[1 + \left(\frac{T}{T_c}\right)^4\right]$$

## Exercise II

1)

$$k = \alpha + i\beta, \quad \alpha = \omega\sqrt{\mu\epsilon} \left[ \frac{\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1}{2} \right]^{1/2}, \quad \beta = \omega\sqrt{\mu\epsilon} \left[ \frac{\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1}{2} \right]^{1/2}$$

when  $\sigma/\omega\epsilon \gg 1$  (very good electric conductor case),

$$\alpha = \beta \approx \sqrt{\mu\epsilon} \left[ \frac{\sqrt{\left(\frac{\sigma}{\omega\epsilon}\right)^2}}{2} \right]^{1/2} = \sqrt{\frac{\mu\sigma\omega}{2}}$$

$$\therefore k \approx (1+i) \cdot \sqrt{\frac{\mu\sigma\omega}{2}}$$

$$\delta(\text{skin depth}) \equiv \frac{1}{\beta} = \sqrt{\frac{2}{\mu\sigma\omega}}$$

$$R_s(\text{surface resistance}) \equiv \text{Re}(Z) = \text{Re}\left(\frac{\mu\omega}{k}\right),$$

$$Z = \mu\omega \cdot \sqrt{\frac{2}{\mu\sigma\omega}} \cdot \frac{1}{1+i} = \mu\omega \cdot \sqrt{\frac{2}{\mu\sigma\omega}} \cdot \frac{1-i}{2} = (1-i) \cdot \sqrt{\frac{\mu\omega}{2\sigma}}$$

$$\text{Re}(Z) = R_s = \sqrt{\frac{\mu\omega}{2\sigma}} = \frac{1}{\sigma} \sqrt{\frac{\mu\sigma\omega}{2}} = \frac{1}{\sigma\delta}$$

2)

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} = \sqrt{\frac{2\rho}{\mu\omega}} = \sqrt{\frac{2 \times 1.72 \cdot 10^{-8}}{4\pi \cdot 10^{-7} \cdot 2\pi \cdot 1.3 \cdot 10^9}} = 1.83 \cdot 10^{-6}$$

$$\delta = 1.83 \mu\text{m}$$

$$R_s = \frac{1}{\sigma\delta} = \frac{\rho}{\delta} = \frac{1.72 \cdot 10^{-8}}{1.83 \cdot 10^{-6}} = 9.40 \cdot 10^{-3}$$

$$R_s = 9.40 \text{ m}\Omega$$

3)

$$R_s(4.2\text{K}) = \sqrt{\frac{\mu\omega}{2\sigma(4.2\text{K})}} = \sqrt{\frac{\mu\omega}{2\sigma(300\text{K})} \cdot \frac{\sigma(300\text{K})}{\sigma(4.2\text{K})}} = R_s(300\text{K}) \cdot \sqrt{\frac{1}{RRR}} = \frac{9.40}{\sqrt{40}} = 1.49 \text{ m}\Omega$$

### Exercise III

By the Two fluid mode,

$$\begin{aligned}
 J &= \left( \frac{n_n e^2}{\nu m_e} - i \frac{n_s q_s^2}{\omega m_s} \right) E = (\sigma_n - i\sigma_s) E = \sigma E \\
 Z &= (1-i) \sqrt{\frac{\mu\omega}{2\sigma}} = (1-i) \sqrt{\frac{\mu\omega}{2(\sigma_n - i\sigma_s)}} \\
 &= (1-i) \sqrt{\frac{\mu\omega}{2}} \cdot \frac{1}{\sqrt{-i\sigma_s \left(1 - \frac{\sigma_n}{i\sigma_s}\right)}} \\
 &\approx (1-i) \cdot \sqrt{\frac{\mu\omega}{2}} \cdot \frac{(1 + \frac{\sigma_n}{2i\sigma_s})}{\sqrt{-i\sigma_s}}, \quad \because \frac{\sigma_n}{\sigma_s} \ll 1 \\
 &= \sqrt{\mu\omega} \cdot \left( \frac{\sigma_n}{2\sigma_s^{3/2}} + i \frac{1}{\sigma_s^{1/2}} \right) \quad \because \sqrt{i} = \sqrt{e^{-\pi/2}} = \frac{1-i}{\sqrt{2}}
 \end{aligned}$$

If define the  $\lambda$ (London's penetration depth) as following,

$$\lambda_L \equiv \sqrt{\frac{m_s}{n_s q_s^2 \mu}} = c \sqrt{\frac{2m}{4n_s \mu e^2}} = c \sqrt{\frac{m}{2n_s \mu e^2}}$$

One can get the impedance Z as:  $Z = \frac{\mu\omega\lambda_L^3}{\delta^2} + i\mu\omega\lambda_L = \frac{\mu^2\omega^2\lambda_L^3\sigma_n}{2} + i\mu\omega\lambda_L$

$Z = R_S + iX$ ,

then

$$R_S = \frac{1}{2} \sigma_n \omega^2 \mu^2 \lambda_L^3, \quad X = \omega \mu \lambda_L, \quad \sigma_n \equiv \frac{n_n e^2}{\nu m_e}$$

$$R_S =$$

$$R_S = \frac{1}{2} \sigma_n \omega^2 \mu^2$$

At T=0K, all electrons become superconducting electrons (ground state).

At a finite temperature T[K], some electrons are excited from the ground state (T=0K, E=0) to the state (T=TK, E=Δ), which breaks superconducting state (normal electron). The exciting probability exciting (T, Δ) is estimated by Boltzman statistics as:

$$n_n(T, \Delta) = n_s(0, 0) \exp\left(-\frac{\Delta}{k_B T}\right).$$

So one can be written as:

$$\sigma_n = \frac{e^2}{mV} n_s(0) e^{-\frac{\Delta}{k_B T}}$$

Thus the Rs of RF superconductivity is given as:

$$R_s(T, f) = A \cdot f^2 \exp\left(-\frac{\Delta}{k_B T}\right).$$

#### Exercise IV

This will take a time. Please try it after this summer school.

#### Exercise V

$$\omega_{0,1,0} = \frac{c}{\sqrt{\mu\epsilon}} \sqrt{\frac{\rho_{0,1}^2}{a^2}} = \frac{2.405 \cdot c}{\sqrt{\mu\epsilon} \cdot a}$$

$$a = \frac{2.405 \cdot c}{\sqrt{\mu\epsilon} \cdot 2\pi f} = \frac{2.405 \cdot 3.00 \times 10^{10}}{2\pi \cdot 1.3 \times 10^9} = 8.83 \text{ cm}, \text{ Damtr} = 17.66 \text{ cm}$$

$$\mu = \mu_0 = \epsilon = \epsilon_0 = 1 \text{ in Gauss unit,}$$

$$d = \frac{\lambda}{2} = \frac{c/f}{2} = \frac{3.00 \times 10^{10} / 1.30 \times 10^9}{2} = 11.54 \text{ cm}$$

#### Exercise VI

$$R_{sh}[\Omega] = 3.087 M\Omega$$

$$V[MV] = \sqrt{R_{sh} \cdot P_{loss}} = \sqrt{3.0868 \times 10^6 \times 1118.1551} = 0.05875 \text{ MV}$$

$$\lambda[m] = 3.0 \times 10^8 / 1.2937743 \times 10^9 = 0.23188 \text{ m}$$

$$E_{acc}[MV/m] = \frac{V}{\lambda/2} = \frac{0.05875}{0.23188/2} = 0.5067 \text{ MV/m}$$

$$H_p / E_{acc} = 1753.44 \times 4\pi \times 10^{-3} / 0.5067 = 43.46 \text{ Oe}/[MV/m]$$

$$E_p / E_{acc} = 1.867$$

$$E_{acc}[V/m] = \frac{\sqrt{R_{sh}/Q_0}}{L_{eff}} \sqrt{P_{loss} \cdot Q_0} = \frac{\sqrt{3.087 \times 10^6}}{0.23188/2} \cdot \sqrt{P_{loss} \cdot Q_0} = 90.15 \cdot \sqrt{P_{loss} \cdot Q_0}$$

$$\Gamma[\Omega] = Q_0 \cdot R_s = 265.171 [\Omega]$$

### Exercise VII

$$C_{in} = \left( \frac{P_0}{P_{in}} \right) \cdot \sqrt{\frac{P'_{in}}{P_0}} = \left( \frac{50.0 \times 10^{-3}}{55.5 \times 10^{-6}} \right) \cdot \sqrt{\frac{22.6 \times 10^{-3}}{39.0 \times 10^{-3}}} = 685.8$$

$$C_r = \left( \frac{P_0}{P_r} \right) \cdot \sqrt{\frac{P'_0}{P'_{in}}} = \frac{50.0 \times 10^{-3}}{10.72 \times 10^{-6}} \cdot \sqrt{\frac{39.0 \times 10^{-3}}{22.6 \times 10^{-3}}} = 6127.1$$

$$C_t = \left( \frac{P_0}{P_t} \right) \cdot \sqrt{\frac{P'_0}{P'_t}} = \frac{50.0 \times 10^{-3}}{3.04 \times 10^{-3}} \cdot \sqrt{\frac{39.0 \times 10^{-3}}{27.9 \times 10^{-3}}} = 19.45$$

### Exercise VIII

$$P_{in} = 685.8 \times 3.11 \times 10^{-3} = 2.133 \text{ W}$$

$$P_r = 6127.1 \times 192 \times 10^{-9} = 1.176 \text{ mW}$$

$$P_t = 19.45 \times 0.142 \times 10^{-3} = 2.762 \text{ mW}$$

$$P_{loss} = 2.133 - 1.176 \times 10^{-3} - 2.762 \times 10^{-3} = 2.129 \text{ W}$$

$$\beta^* = \frac{1 + \sqrt{\frac{P_r}{P_{in}}}}{1 - \sqrt{\frac{P_r}{P_{in}}}} = \frac{1 + \sqrt{5.5157 \times 10^{-4}}}{1 - \sqrt{5.5157 \times 10^{-4}}} = \frac{1 + 0.02349}{1 - 0.02349} = 1.048 \beta_t = \frac{P_t}{P_{loss}} = \frac{2.762 \times 10^{-3}}{2.129} = 1.297 \times 10^{-3}$$

$$\beta_{in} = (1 + \beta_t) \cdot \beta_{in}^* = (1 + 1.297 \times 10^{-3}) \times 1.048 = 1.049$$

$$Q_L = \frac{2\pi f \tau_{1/2}}{\ln(2)} = \frac{2 \times 3.1416 \times 1.303590529 \times 10^9 \times 23.6 \times 10^{-3}}{\ln(2)} = \frac{193.30 \times 10^6}{0.69315} = 2.789 \times 10^8$$

$$Q_0 = (1 + \beta_{in} + \beta_t) \cdot Q_L = (1 + 1.049 + 1.297 \times 10^{-3}) \times 2.789 \times 10^8 = 2.0503 \times 2.789 \times 10^8 = 5.718 \times 10^8$$

$$Q_{in} = (1 + \beta_t) \cdot \beta_{in}^* = (1 + 1.297 \times 10^{-3}) \times 1.048 = 1.049$$

$$Q_t = \frac{Q_0}{\beta_t} = \frac{5.718 \times 10^8}{1.297 \times 10^{-3}} = 4.408 \times 10^{11}$$

$$R_s = \frac{\Gamma}{Q_0} = \frac{265.17}{5.718 \times 10^8} = 463.75 \times 10^{-9} = 463.75 \text{ n}\Omega$$

$$E_{acc} = Z \cdot \sqrt{P_{loss} \cdot Q_0} = 90.148 \times \sqrt{2.129 \times 5.718 \times 10^8} = 3.145 \times 10^6 \text{ V/m} = 3.145 \text{ MV/m}$$

$$E_p = (E_p / E_{acc}) \times E_{acc} = 1.867 \times 3.145 = 5.872 \text{ MV/m}$$

$$H_p = (H_p / E_{acc}) \times E_{acc} = 43.46 \times 3.145 = 136.68 \text{ Oe}$$