# Emittance Compensation in Non-Circular-Symmetrical Beamlines

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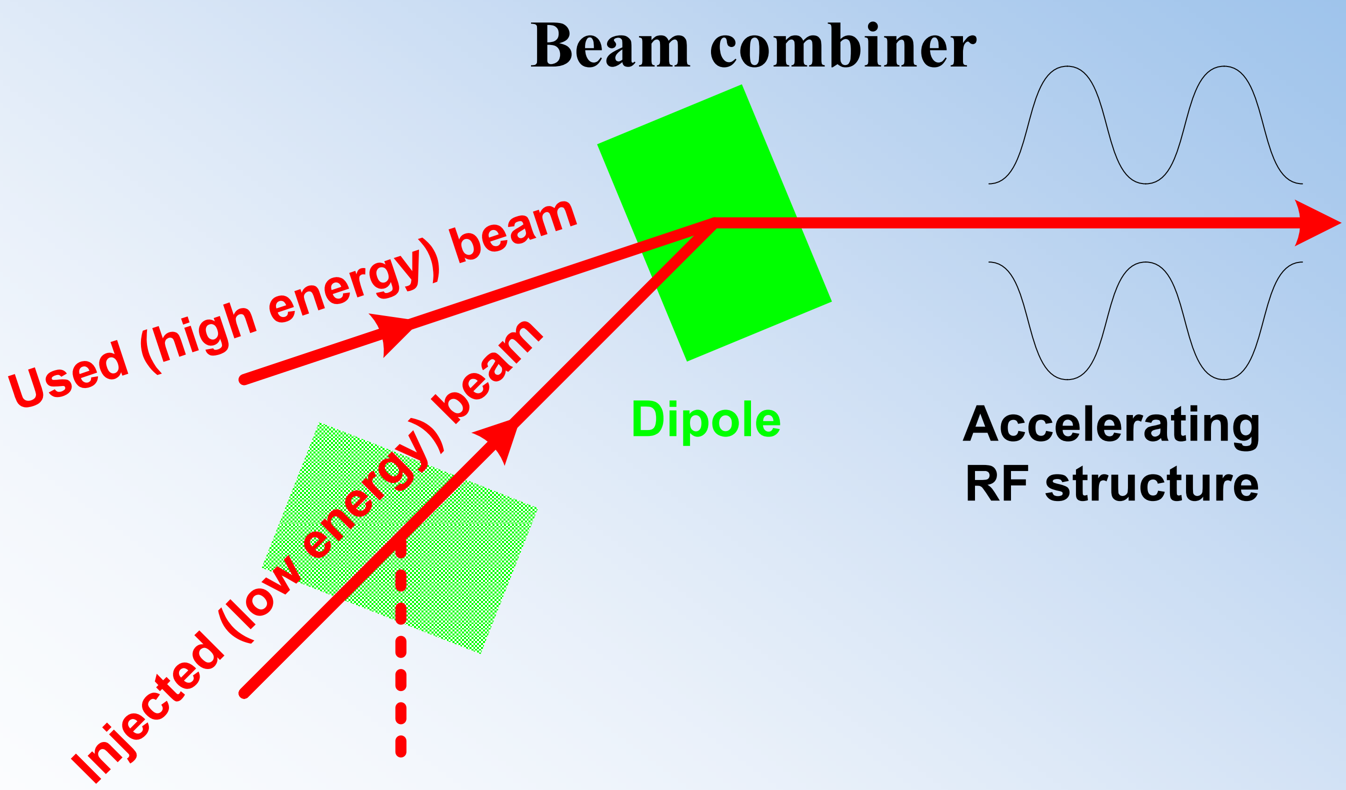
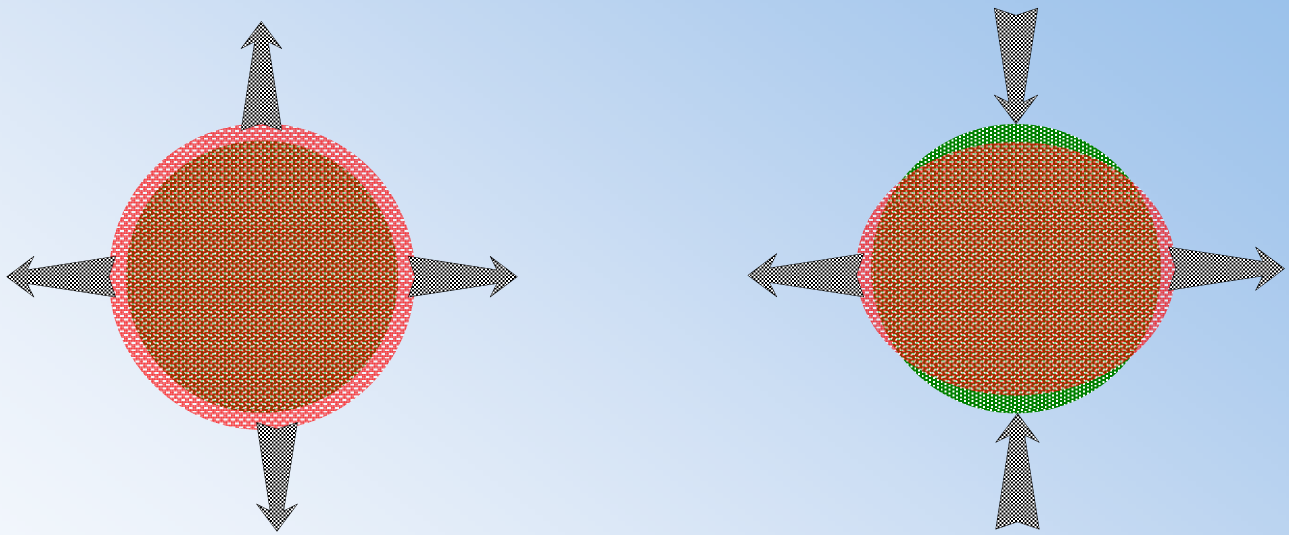
# Demand

- Needed for energy-recovery accelerators.



# Elliptical symmetry

- Two modes of charge oscillation:



- Monopole and quadrupole
- However, the phenomena and the equations are quite similar to those in circular symmetrical systems

# Elliptical symmetry

- <u>2D motion equation:</u>

$$x'' + \frac{(\beta\gamma)'}{\beta\gamma} x' = \frac{j}{2x} - gx.$$

$$\begin{cases} x'' + \dfrac{(\beta\gamma)'}{\beta\gamma} x' = \dfrac{j}{x+y} - gx, \\ y'' + \dfrac{(\beta\gamma)'}{\beta\gamma} y' = \dfrac{j}{x+y} - hy. \end{cases}$$

- Circular symmetry
- Elliptical symmetry

- Two coordinates or two modes are to be "compensated" in this case.

# Linearized equation

$$\begin{cases} \delta_x'' + \left( \dfrac{2x'}{x} + \dfrac{(\beta\gamma)'}{\beta\gamma} \right) \delta_x' = -j\left( \dfrac{2x+y}{x(x+y)^2} \delta_x + \dfrac{y}{x(x+y)^2} \delta_y \right), \\ \delta_y'' + \left( \dfrac{2y'}{y} + \dfrac{(\beta\gamma)'}{\beta\gamma} \right) \delta_y' = -j\left( \dfrac{2y+x}{y(x+y)^2} \delta_y + \dfrac{x}{y(x+y)^2} \delta_x \right). \end{cases}$$

where $\delta_x = \delta x / x,\ \delta_y = \delta y / y.$

- Conditions of emittance minima are still $\delta_x' = \delta_y' = 0.$
- Let's divide an injector into two parts:
  1. Circular symmetrical one;
  2. Elliptically symmetrical one.
- At the beginning of the second part

  $x = y,\ \delta_x = \delta_y,$ and $\delta_x' = \delta_y'.$

# Further linearization

- Preserving generality, let βγ = 1, $j$ = 1, $x = 1 + \xi$, a $y = 1 + \upsilon$

$$\begin{cases} \delta_x'' + 2\xi'\delta_x' = -\left(\left(\frac{3}{4} - \xi - \frac{1}{2}\upsilon\right)\delta_x + \left(\frac{1}{4} - \frac{1}{2}\xi\right)\delta_y\right), \\ \delta_y'' + 2\upsilon'\delta_y' = -\left(\left(\frac{3}{4} - \upsilon - \frac{1}{2}\xi\right)\delta_y + \left(\frac{1}{4} - \frac{1}{2}\upsilon\right)\delta_x\right). \end{cases}$$

- If $\xi = \upsilon = 0$, its solution at the given initial conditions is $\delta_x = \delta_y = \cos(z + \varphi)$
- We need $z + \varphi = n\pi$ at the exit for emittance compensation

# Further linearization

- If $\xi \neq 0$ and $\upsilon \neq 0$, than $\delta_x = \cos(z + \varphi) + v_x$, $\delta_y = \cos(z + \varphi) + v_y$
- A linearized equation for $v_x$ and $v_y$

$$\begin{cases} v''_x + \frac{3}{4} v_x + \frac{1}{4} v_y = 2\xi' \sin(z+\varphi) + \left(\frac{3}{2}\xi + \frac{1}{2}\upsilon\right)\cos(z+\varphi), \\ v''_y + \frac{3}{4} v_y + \frac{1}{4} v_x = 2\upsilon' \sin(z+\varphi) + \left(\frac{3}{2}\upsilon + \frac{1}{2}\xi\right)\cos(z+\varphi). \end{cases}$$

- With initial conditions $v_x = 0$, $v'_x = 0$, $v_y = 0$, $v'_y = 0$

# Linear conditions for emittance minima

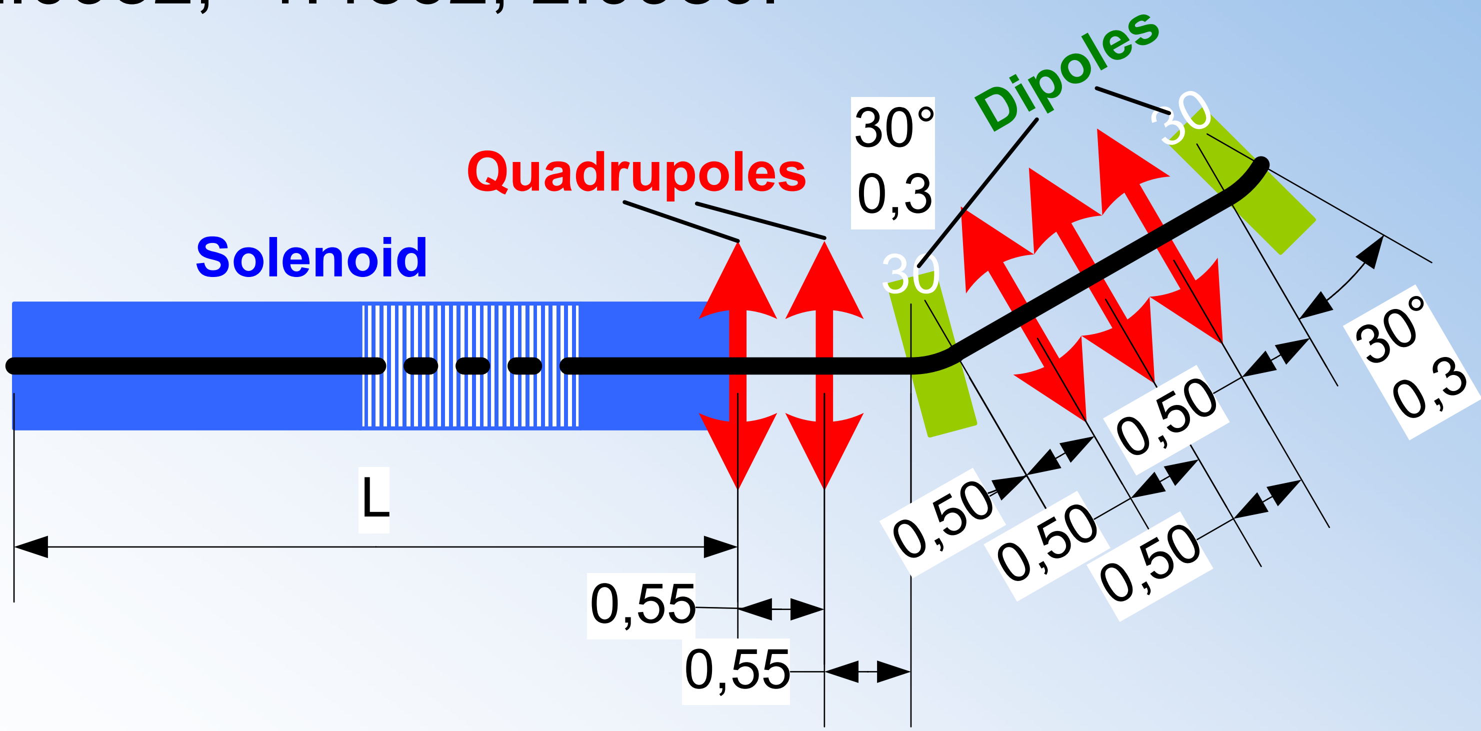
- $\delta_x' = \delta_y' = 0 \rightarrow v_x' = v_y' = 0$:

$$\begin{cases} (\xi+\upsilon)\big|_{z=L} - \dfrac{1}{2}\displaystyle\int_0^L \sin(2(z+\varphi))(\xi+\upsilon)dz = 0, \\ (\xi-\upsilon)\big|_{z=L} - \displaystyle\int_0^L \Big[\sin(z+\varphi)\cos(z/\sqrt{2}+\varphi+L(1-1/\sqrt{2})+ \\ +\dfrac{3}{\sqrt{2}}\cos(z+\varphi)\sin(z/\sqrt{2}+\varphi+L(1-1/\sqrt{2})\Big](\xi-\upsilon)dz = 0. \end{cases}$$

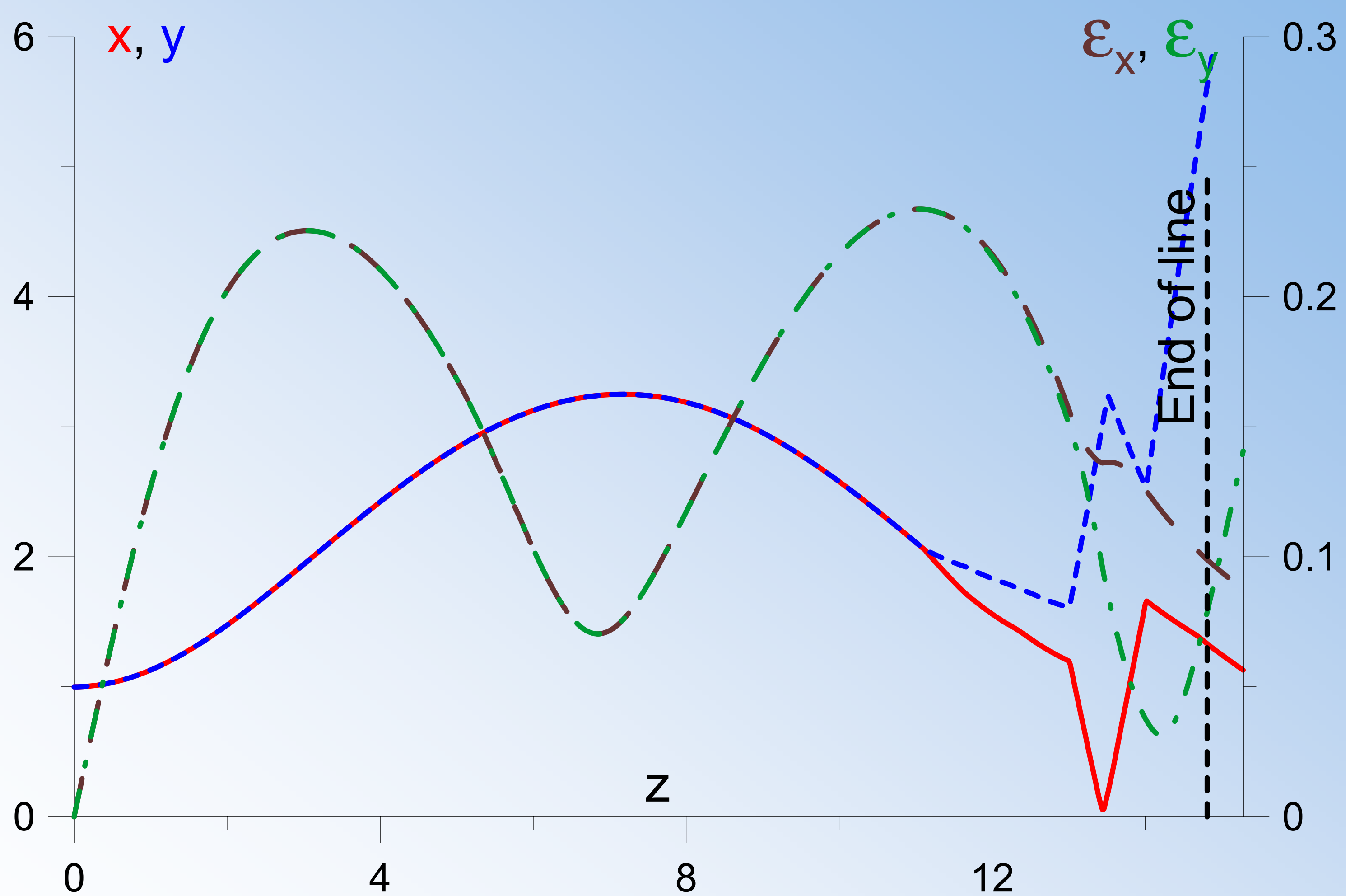
- First condition is valid if $x \approx y \approx$ const.
- The simplest way to meet the second condition is to control (ξ - υ) at the exit.

# First shot

- An optimal uniform beamline $\Delta\varphi = 3\pi/2$: $x = 1$, $x' = 0$, $j = 1$, g = 0.09, $L = 11.11$.
- An achromatic bend: $D_i = 0.0925, -0.0530, 2.0982, -1.4862, 2.0980$.



# First shot

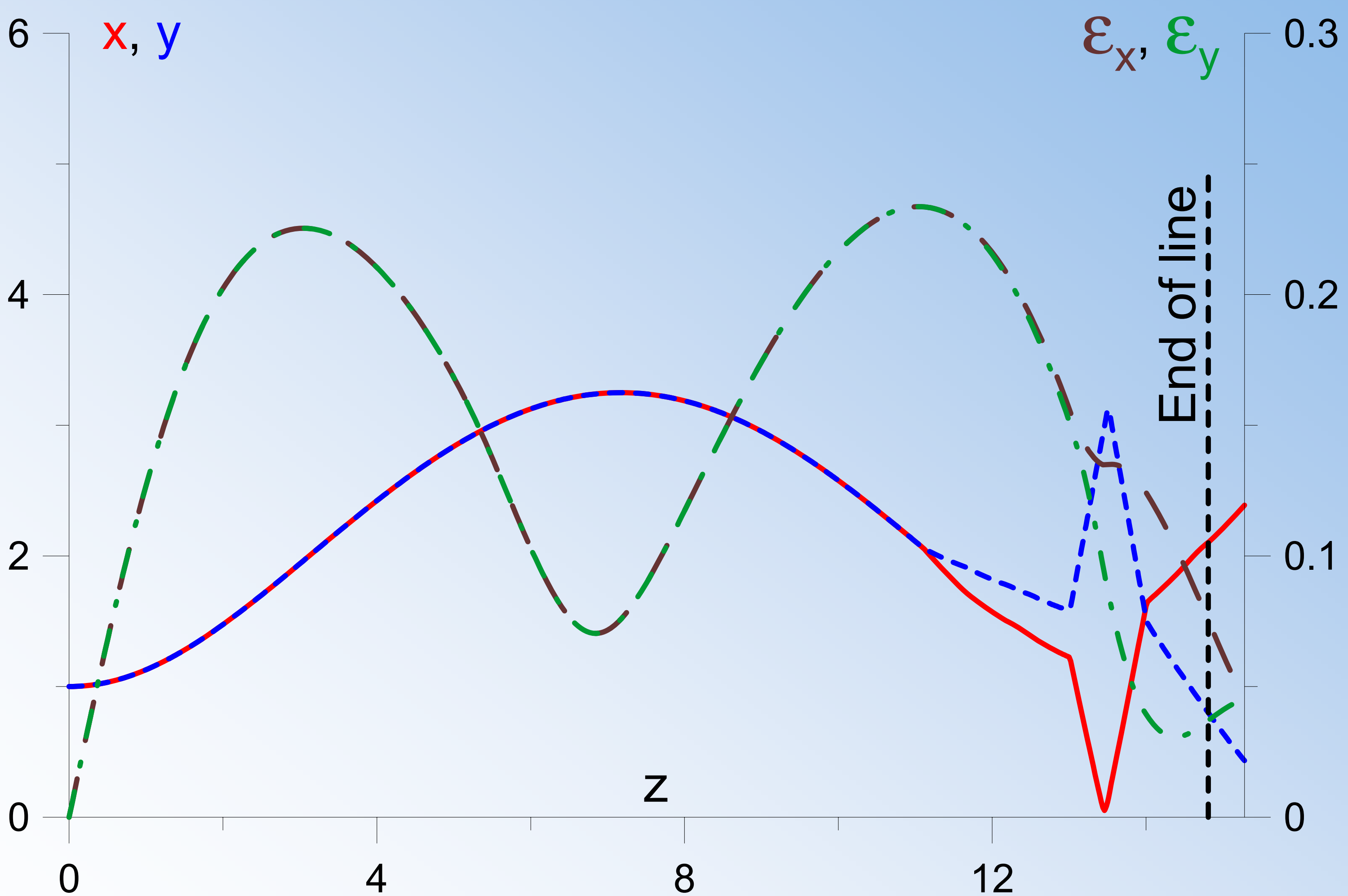



$x_0 = y_0 = 1$, $x'_0 = y'_0 = 0$, $j = 1 \rightarrow \varepsilon_x = 0.099$, $\varepsilon_y = 0.080$.

# Linear optimization

- The same uniform beamline.
- All the lenses are optimized to meet the linear conditions of emittance minimum **and** $\eta = \eta' = 0$ at the exit.
- Lenses became: $D_i$ = 0.0862, -0.0544, 2.0560, -2.0860, 1.5583.

# Linear optimization

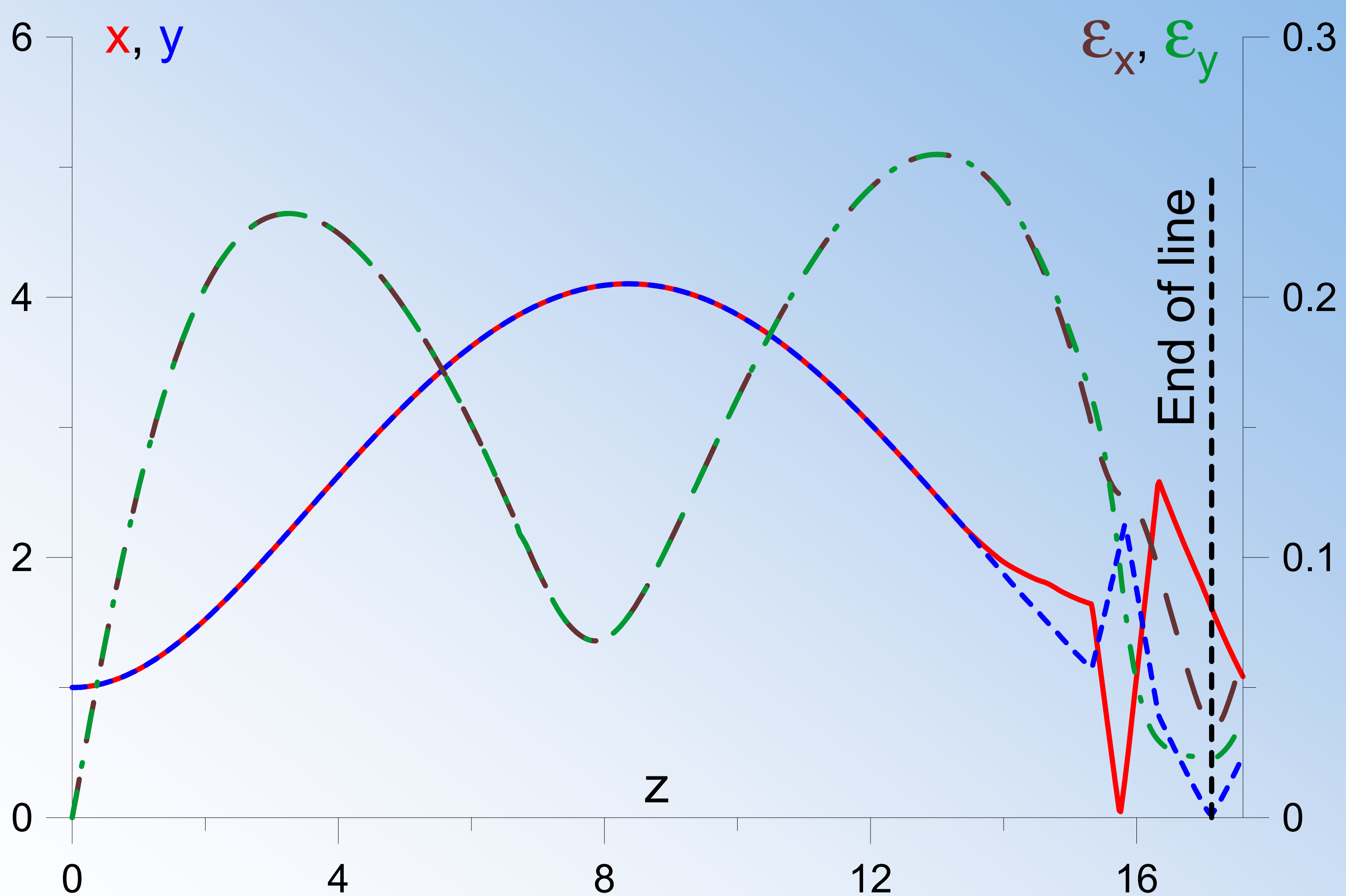



Linear conditions met → $\varepsilon_x = 0.074$, $\varepsilon_y = 0.037$.

# Full optimization

- *L*, *g* and all the lenses are optimized to minimize the emittance **and** meet $\eta = \eta' = 0$ at the exit.
- The uniform beamline became: $L$ =12.718, $g$ = 0.07181.
- Lenses became: $D_i$ = 0.0291, -0.0755, 2.2989, -2.3700, 2.2745.

# Full optimization




Nonlinear optimization → $\varepsilon_x = 0.034$, $\varepsilon_y = 0.022$.

# Optimization results

| Beamline: | $\varepsilon_x$ | $\varepsilon_y$ |
|---|---|---|
| Elliptically symmetrical, not optimized | 0.099 | 0.080 |
| Linear optimization | 0.074 | 0.037 |
| Full optimization | 0.034 | 0.022 |
| Uniform circular symmetrical | 0.023 | |
| Simplest nonuniform | 0.030 | |

$x_0 = y_0 = 1,\ x'_0 = y'_0 = 0,\ j = 1;\ \Delta\varphi \approx 2\pi.$

$$\varepsilon_n \cong \varepsilon^c r \sqrt{\frac{|I|}{I_0 \beta\gamma}}$$

# Conclusions

- Emittance compensation is possible also in elliptically symmetrical systems.
- The conditions of compensation are similar to ones in circular symmetrical systems, but significantly more complicated.
- Linear conditions of compensation can be used as the initial estimate for full numerical optimization.
- The qualities of elliptically symmetrical beamlines and circular symmetrical ones are similar.


Thank you for attention!