Coherence of Charge Oscillation and Emittance Compensation

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High-current low-emittance injectors

- Normalized emittance < ~1 mm·mrad
- Peak current > 100 A
- Needed for:
  - X-ray FELs
  - Linear colliders
  - 4th generation SR sources (ERLs)
  - Compton X-ray sources

Emittance compensation

- Bruce Carlsten NIM **A 285** (1989)
  - The term
  - Qualitative explanation of the phenomenon
- All the record emittance injectors use this technique
- Experimental demonstration (states of slices in a bunch): X. Qiu, K. Batchelor, I. Ben-Zvi, and X-J. Wang. Phys. Rev. Lett., **76** (1996)
- No justified analytical picture of the phenomenon
- No practically suitable analytical estimations
- A number of popular misconceptions

Space charge or emittance?

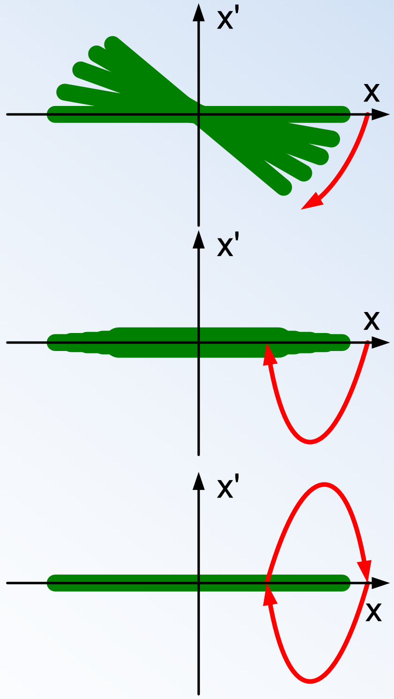
- What dominates: space charge or emittance?

$$\frac{I}{2I_0\beta\gamma} \leftrightarrow \frac{\varepsilon_n^2}{\sigma^2}$$

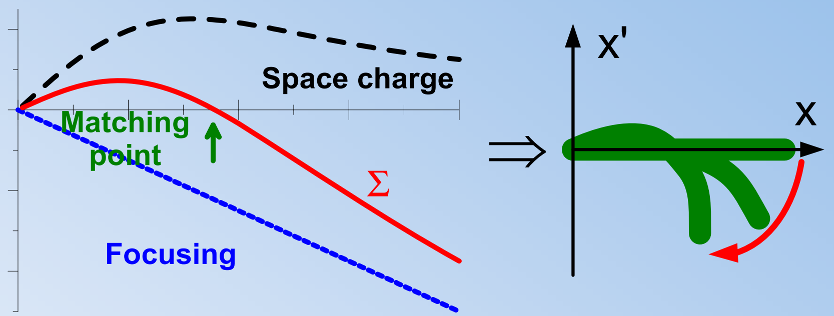
- Locally cold long bunch
  - $\varepsilon_T = 0$
  - γ = const
  - $l\gamma \gg \sigma$
  - ρ = const in a slice
  - Circular symmetry
- Locally cold laminar slice
  - The same assumptions except of ρ = const
  - + perfect laminarity of motion

Basic phenomena

- Longitudinal charge inhomogeneity



- Transverse charge inhomogeneity



Principal trajectory

- Rms emittance

$$\varepsilon_x = \sqrt{<x^2><x'^2> - <xx'>^2}$$

- Kapchinsky – Vladimirsky equation without emittance in rms values for circular symmetrical systems

$$x'' + \frac{(\beta\gamma)'}{\beta\gamma} x' = \frac{j}{2x} - gx$$

$$j = I / I_0 (\beta\gamma)^3 \qquad g = eG / p$$

- The **principal trajectory** is a solution of this equation with given initial conditions
- The principal trajectory for another slice

$$\propto \sqrt{j}$$

- The linearized equation for a small dimensionless deviation from the principal trajectory → **charge oscillation**

$$\delta'' + \left(2\frac{x'}{x} + \frac{(\beta\gamma)'}{\beta\gamma} \right) \delta' = -\frac{j}{x^2} \delta$$

$$\delta = \delta x / x$$

- **Charge oscillation phase**

$$\varphi = \arctan\left(\frac{-C' x}{C\sqrt{j}} \right)$$

- Conditions of emittance minima

$$\delta' = 0 \qquad \text{or} \qquad \varphi = n\pi$$

Nonlinear phase advance

- A nonlinear equation for a small dimensionless deviation from the principal trajectory

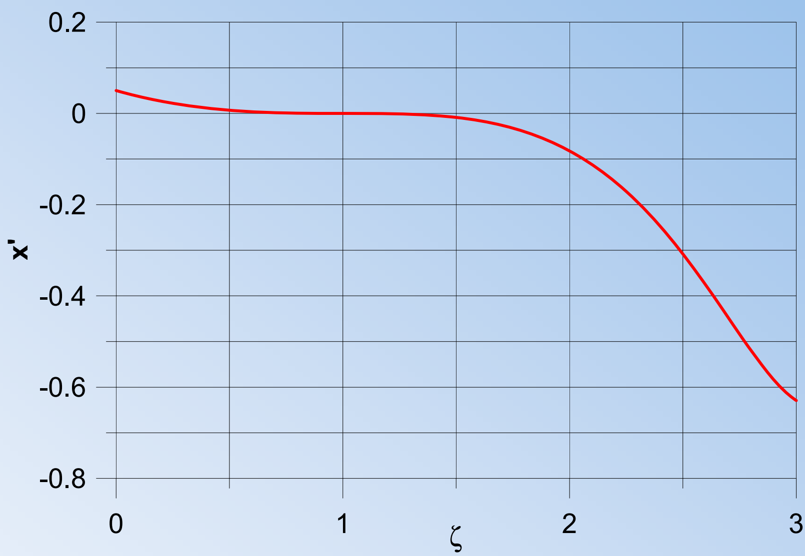
$$\delta'' + \left(2\frac{x'}{x} + \frac{(\beta\gamma)'}{\beta\gamma}\right)\delta' + \frac{j}{x^2}\delta = \frac{j}{2x^2}\left(\delta^2 - \delta^3\right)$$

- gives ramp phase correction

$$\Delta\varphi \cong \frac{1}{12}\varphi a^2$$

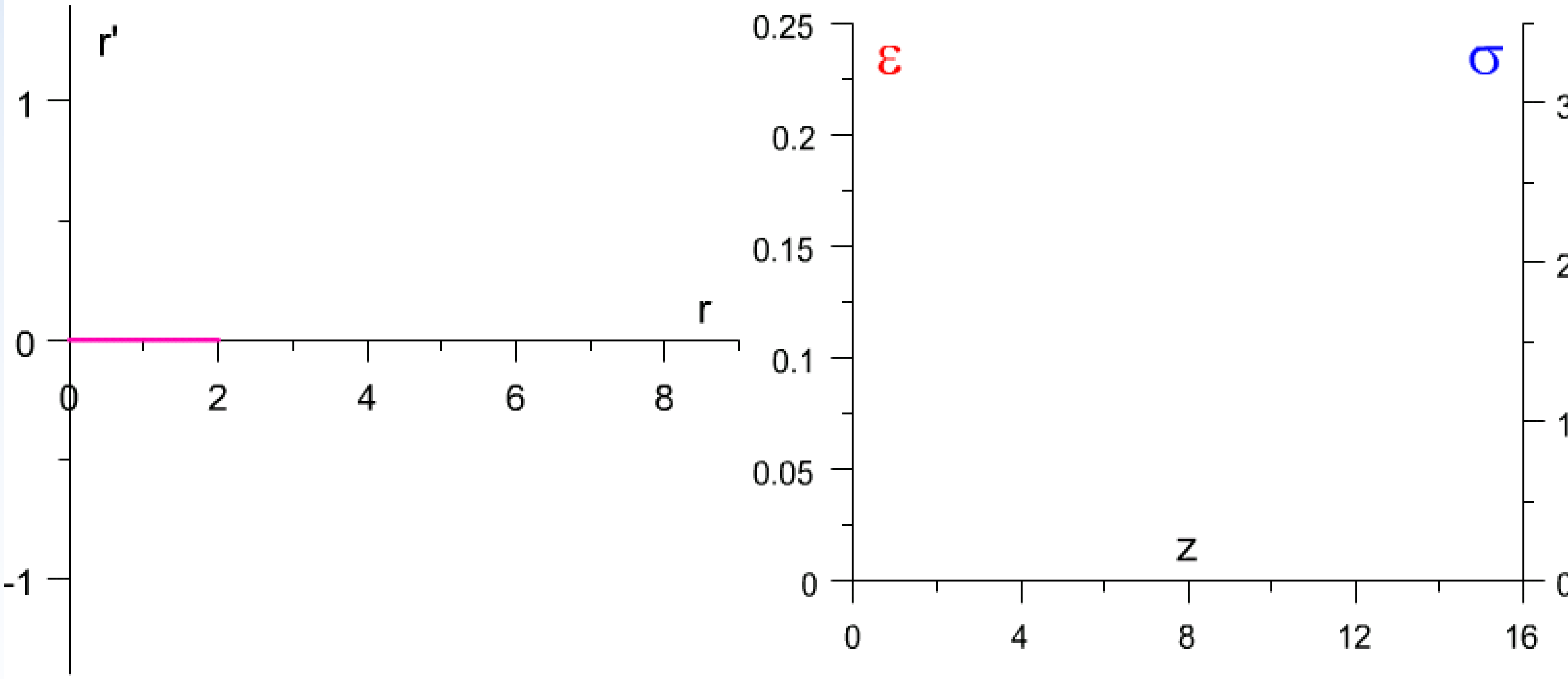
- where a is the relative amplitude of charge oscillation.
- This permits to find the residual emittance and the parameters of the optimal beamline.

- If the phase portraits of adjacent slices are aligned, than more distant ones are somewhat spread due to nonlinear phase advances.



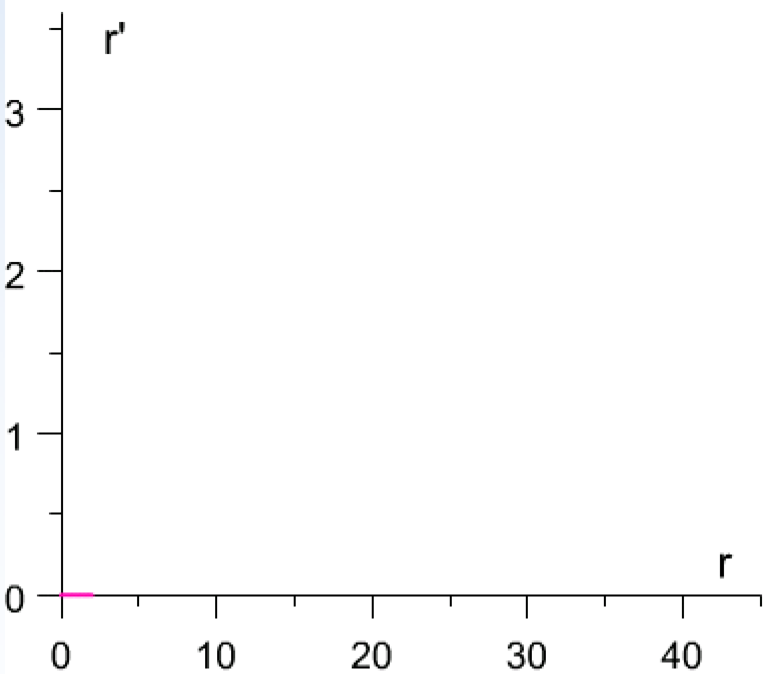
Tilts of slices in a bunch at the end of a beamline. ζ is the longitudinal coordinate in a bunch.

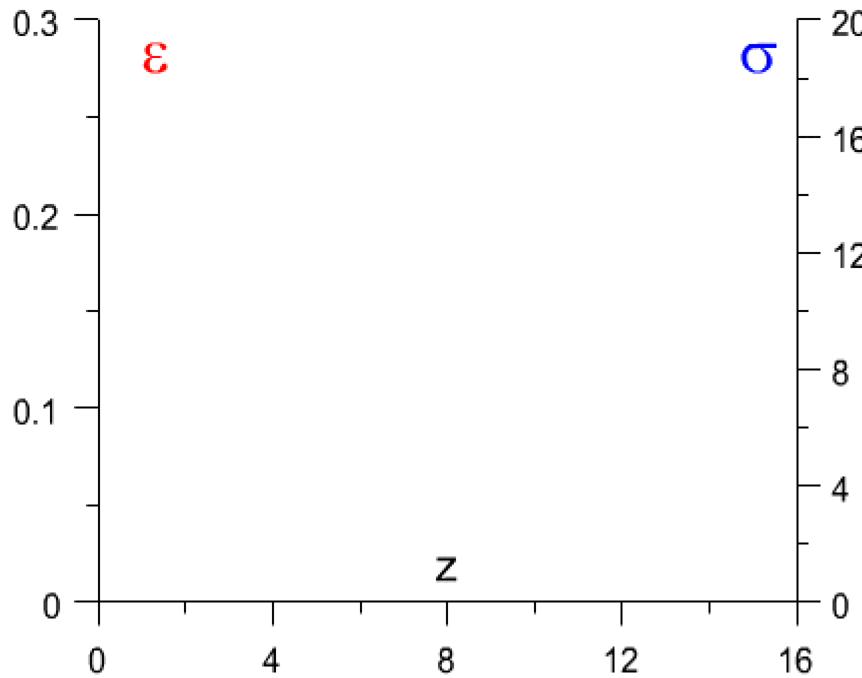
Bunch motion in an optimal homogeneous beamline



r_0 = 2, r_0' = 0, g = 0.09, j = 0.032, 0.065, 0.13, 0.18, 0.25, 0.5, 1. The emittance and the rms size of a Gaussian bunch.

Bunch motion without focusing





r_0 = 2, r_0' = 0, g = 0, j = 0.032, 0.065, 0.13, 0.18, 0.25, 0.5, 1. The emittance and the rms size of a Gaussian bunch.

Transverse charge inhomogeneity

An equation for particle motion in a slice

$$x'' + \frac{(\beta\gamma)'}{\beta\gamma} x' = \frac{2\tilde{j}}{x} - gx$$

$\tilde{j}$ is the current within the radius *x*

Current $\tilde{j}$ is preserved if slice motion is perfectly laminar. The condition of laminarity

$$\left(1 - \frac{x}{2\tilde{j}} \Big/ \frac{dx}{d\tilde{j}}\right)^2 + \frac{x^2}{4\tilde{j}} \left(\frac{dx'}{d\tilde{j}} \Big/ \frac{dx}{d\tilde{j}} - \frac{x'}{x}\right)^2 < \frac{1}{2}$$

In a Gaussian slice, it is violated only for 12% of particles in the halo.

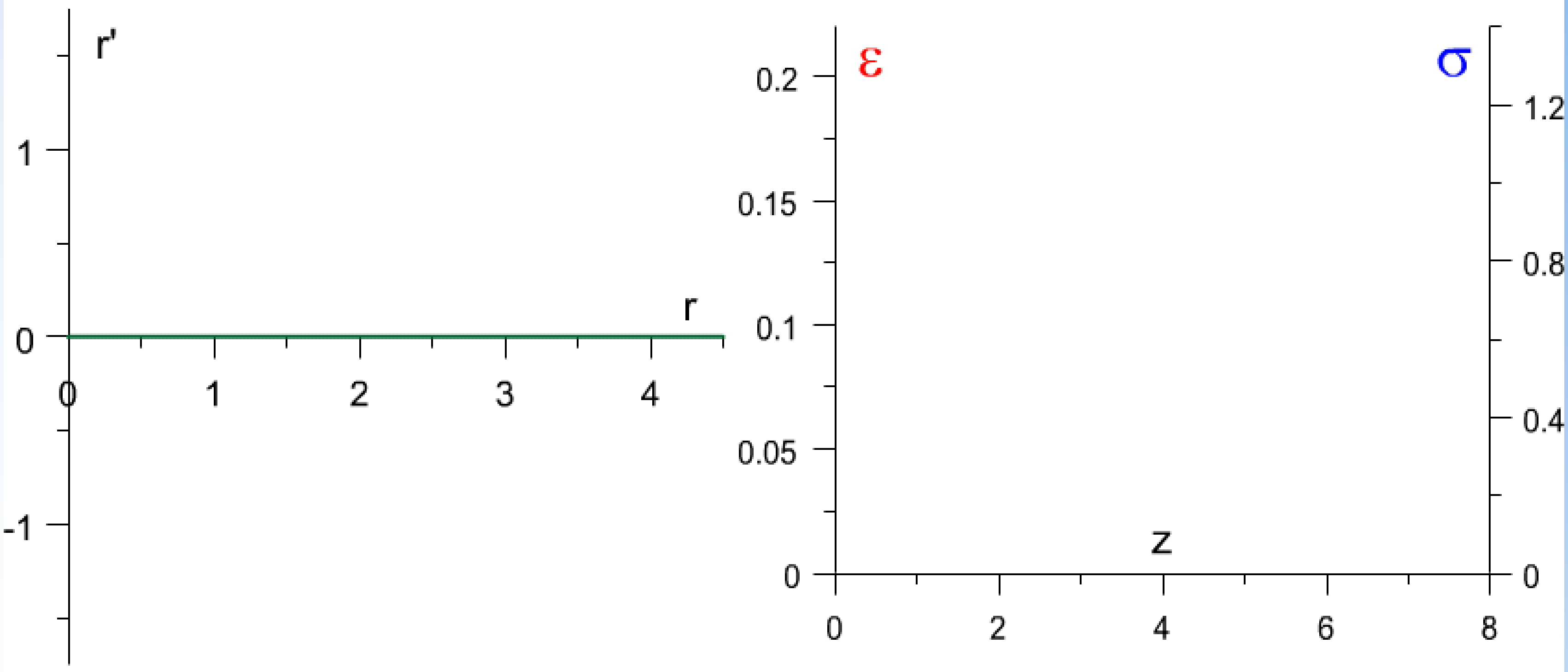
The linearized equation for a small dimensionless deviation from the principal trajectory

$$\delta'' + \left(2\frac{x'}{x} + \frac{(\beta\gamma)'}{\beta\gamma}\right)\delta' = -\frac{4\tilde{j}}{x^2}\delta$$

The definition of phase is the same.

Although phase portraits in this case are bent and straightened, but not spread and aligned, the conditions of emittance minima are the same.

Slice motion in an optimal homogeneous beamline



$\sigma_0 = 1$, $\sigma_0' = 0$, $g = 0.38$, $j = 1$. The emittance and the rms size of a Gaussian slice.

Effects and beamlines

- Effect of longitudinal inhomogeneity.
- Effect of transverse inhomogeneity.
- Combined effect.

- Uniform beamline: the focusing is matched to one of the slices.
- Simplest nonuniform beamline: empty space + thin lens + empty space.
- Matched focusing beamline with bunching. $g \propto j$
- Matched focusing beamline with accelerating. $g \propto (\beta\gamma)^{-3}$

- Always: $x_0 = 1$, $x_0' = 0$, $j = 1$

Parameters of optimal beamlines

Parameter	Uniform beamline		
	Longitudinal inhomogeneity	Transverse inhomogeneity	Combined effect
ε^c	0.023	0.0079	0.037
g^c	0.09	0.38	0.13
L^c	14.2	7.15	11.85
	Simplest nonuniform beamline		
ε^c	0.030	0.0144	0.0461
D^c	0.381	0.688	0.445
L^c	14.0	8.0	12.0
	Distributed focusing: bunching		
ε^c	$0.0215 \cdot \sqrt[3]{\upsilon}$		$0.0349 \cdot \upsilon^{0.28}$
g^c	0.08…0.11		0.10…0.14
L^c	$15.7 / \sqrt[3]{\upsilon}$		$12.7 \cdot \upsilon^{-0.28}$
	Distributed focusing: accelerationg		
ε^c	$0.0220 \cdot \alpha^{-0.136}$		0.035
g^c	0.1…0.16		$0.115 \cdot \alpha^{0.227}$
L^c	$11.96 + 6.05\alpha$		$10.89 + 5.03\alpha$

Emittance

$$\varepsilon_n \cong \varepsilon^c x \sqrt{\frac{|I|}{I_0 \beta\gamma}}$$

Length of beamline

$$L \cong L^c x \Big/ \sqrt{\frac{|I|}{I_0 (\beta\gamma)^3}}$$

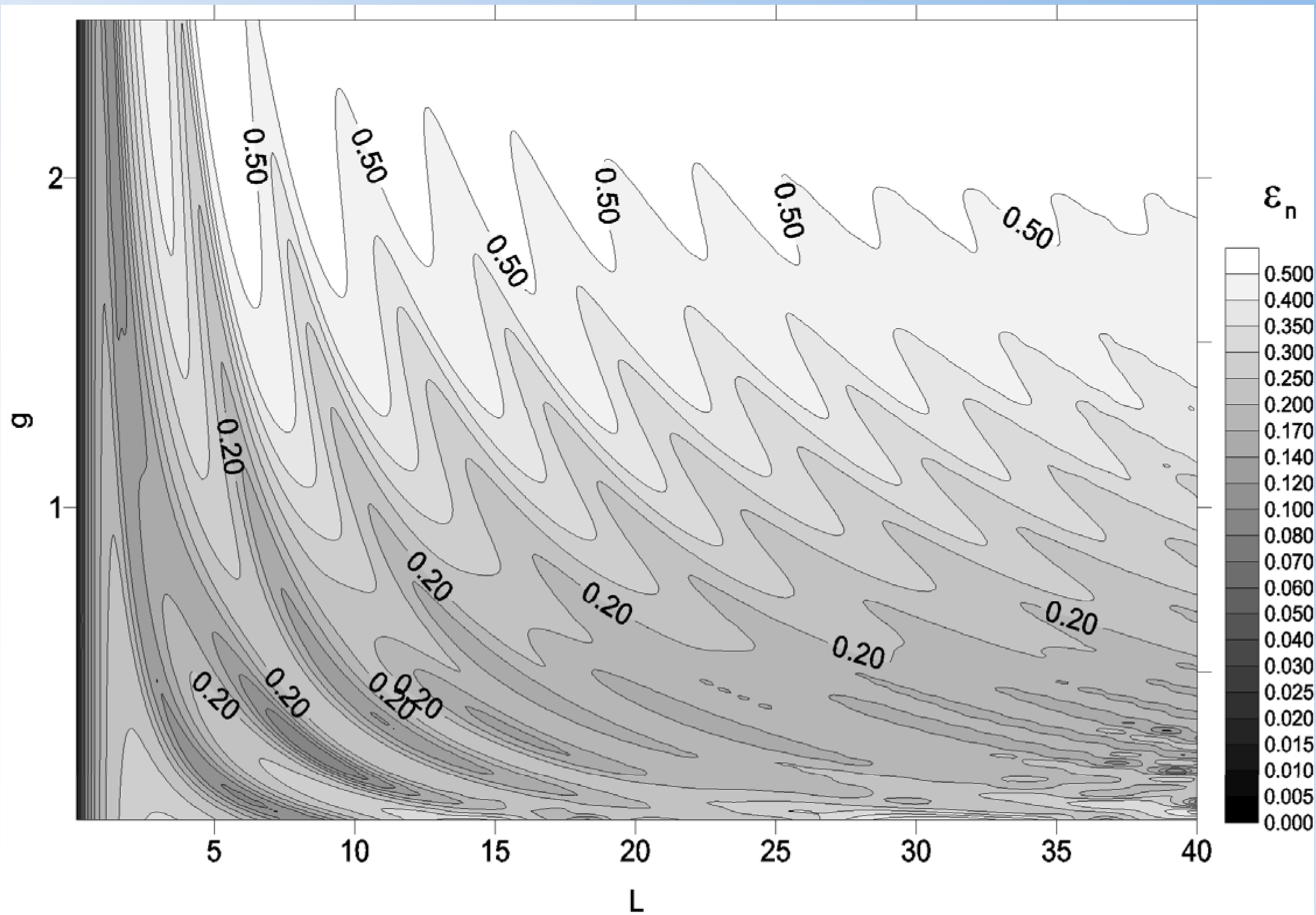
Focusing

$$g \cong \frac{g^c}{x^2} \frac{|I|}{I_0 (\beta\gamma)^3}$$

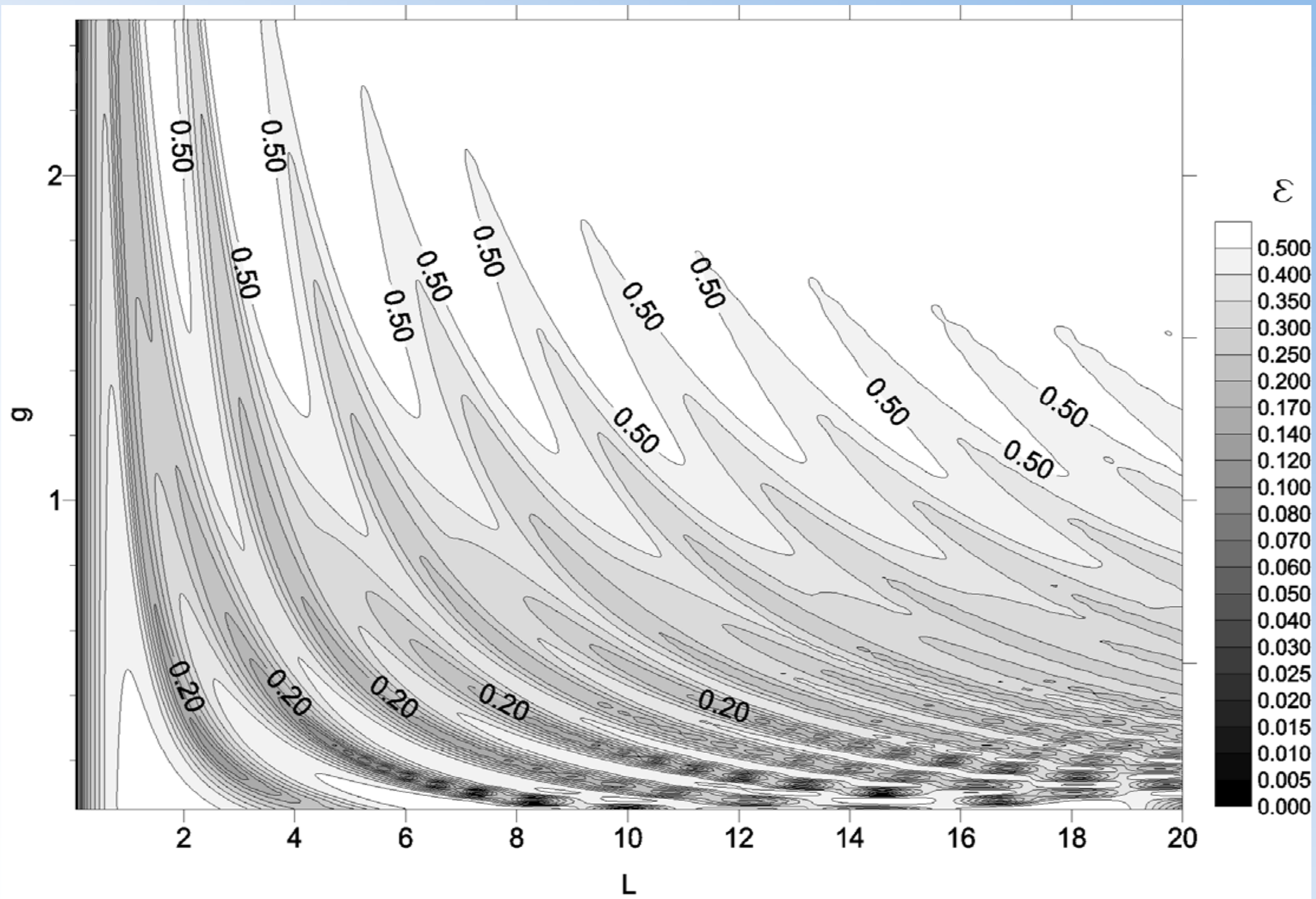
Lens strength

$$D \cong \frac{D^c}{x} \sqrt{\frac{|I|}{I_0 (\beta\gamma)^3}}$$

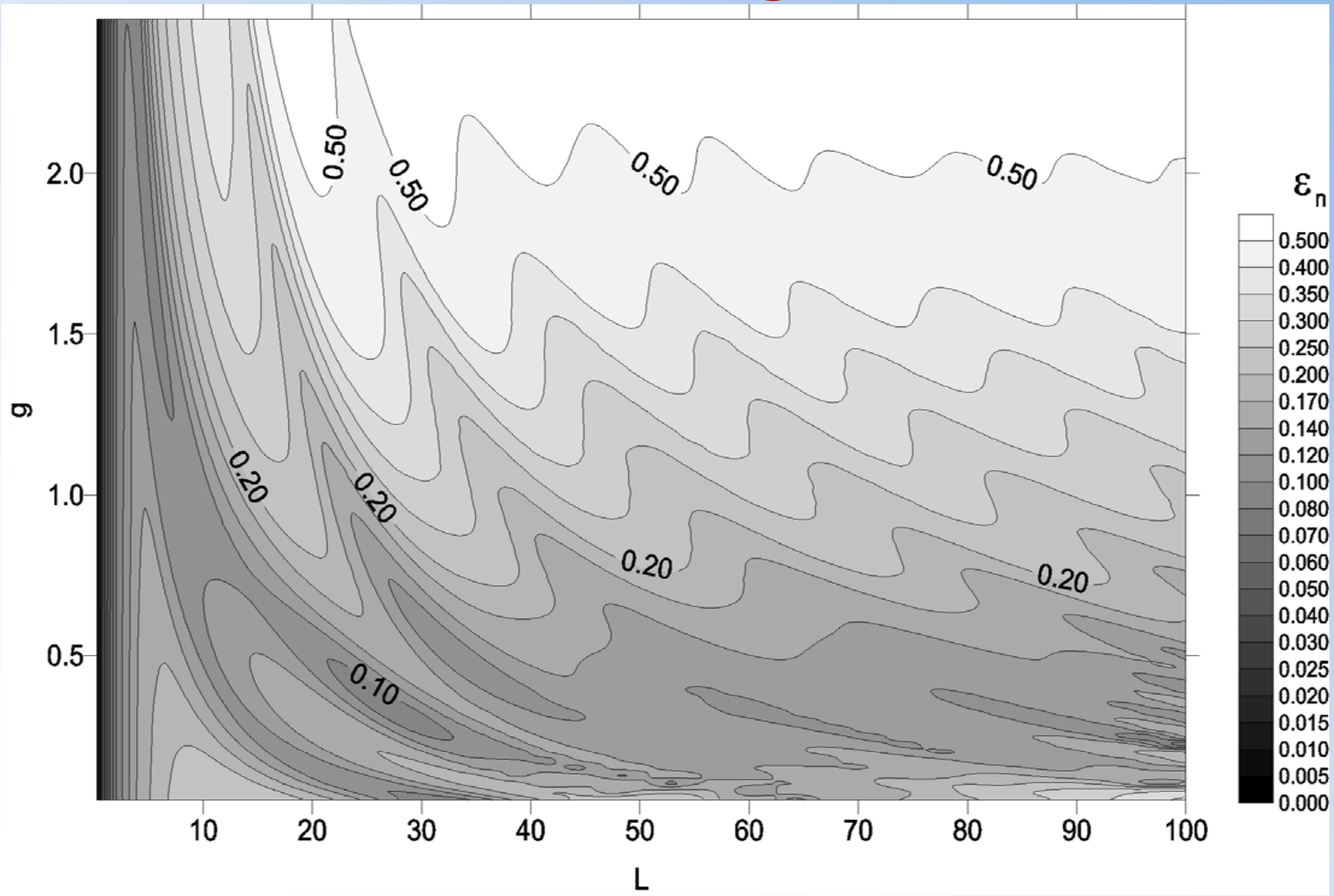
Emittance at the end of beamline vs its length L and focusing g



Emittance at the end of beamline vs its length L and focusing g: bunching 10/1



Normalized emittance at the end of beamline vs its length L and focusing g: accelerating 5/1



Electron guns

- Strongest space charge effect
- Violation of the model:
  - Metallic electrodes near the cathode → induced charge
  - An area always exists, where $l\gamma < \sigma$
  - The head and the tail are in different conditions → the gained transverse momentum depends not only on the current, but also on the position in the bunch
- At the same time, in nonrelativistic approximation

$$\frac{\varepsilon}{r_e\sqrt{j}} = const$$

- The coefficient is to be found numerically

Differential parameters of a beam

- A bunch is an **ordered** set of slices.
- Smooth dependences of the current $I(\zeta)$, the size $x(\zeta)$ and the tilt $x'(\zeta)$ of a slice on the longitudinal coordinate in a bunch ζ.
- Similar for particles in a slice.
- → It is possible to define the differential phase and the differential relative amplitude of charge oscillation.

$$\varphi = \arctan \frac{\dfrac{dx'}{d\zeta} - \dfrac{x'}{x}\dfrac{dx}{d\zeta}}{\dfrac{1}{2\sqrt{j}}\dfrac{dj}{d\zeta} - \dfrac{\sqrt{j}}{x}\dfrac{dx}{d\zeta}}$$

$$A = \frac{\sqrt{\left(\dfrac{2j}{x}\dfrac{dx}{d\zeta} - \dfrac{dj}{d\zeta}\right)^2 + 4j\left(\dfrac{dx'}{d\zeta} - \dfrac{x'}{x}\dfrac{dx}{d\zeta}\right)^2}}{\left|\dfrac{dj}{d\zeta}\right|}$$

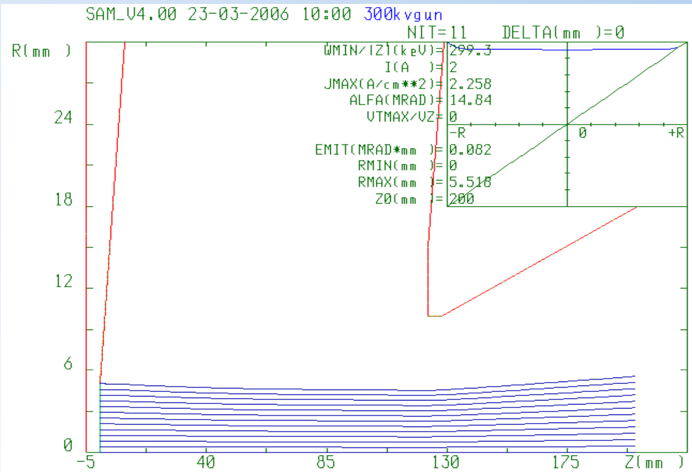
Bunch

$$\varphi = \arctan \frac{\dfrac{x'}{x} - \dfrac{dx'}{dx}}{\dfrac{2\sqrt{\tilde{j}}}{x} - \dfrac{1}{\sqrt{\tilde{j}}}\cdot\dfrac{d\tilde{j}}{dx}}$$

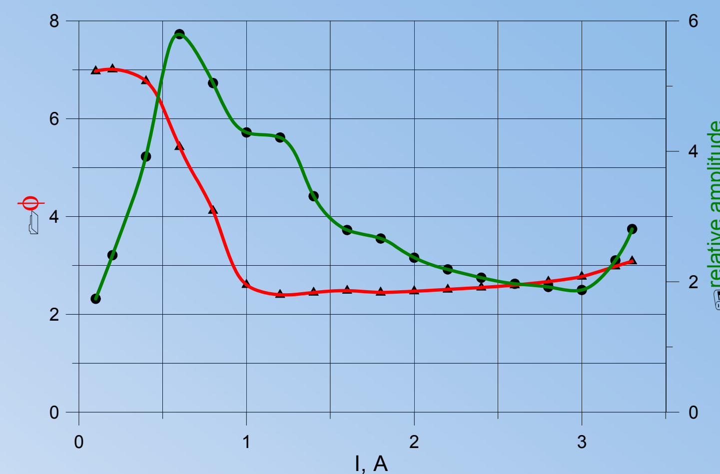
$$A = \frac{\sqrt{\left(\dfrac{dx}{d\tilde{j}}\cdot\dfrac{\tilde{j}}{x} - \dfrac{1}{2}\right)^2 + \dfrac{\tilde{j}}{4}\left(\dfrac{dx'}{d\tilde{j}} - \dfrac{dx}{d\tilde{j}}\cdot\dfrac{x'}{x}\right)^2}}{\dfrac{dx_e}{d\tilde{j}}\cdot\dfrac{\tilde{j}}{x_e} - \dfrac{1}{2}}$$

Slice

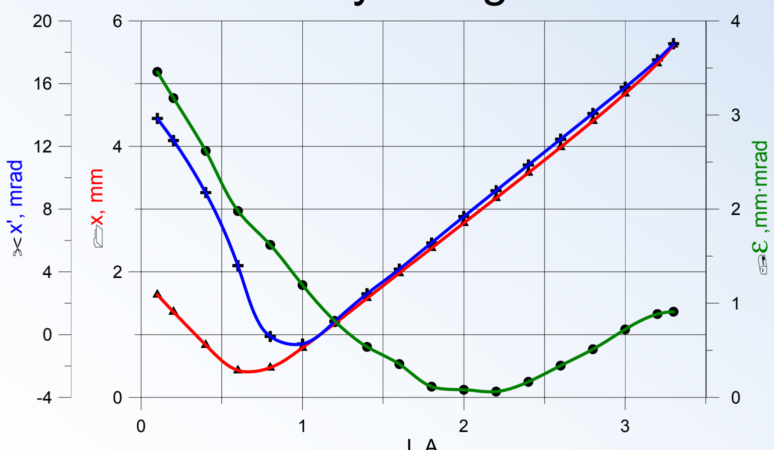
Simulation of guns



Geometry of a gun



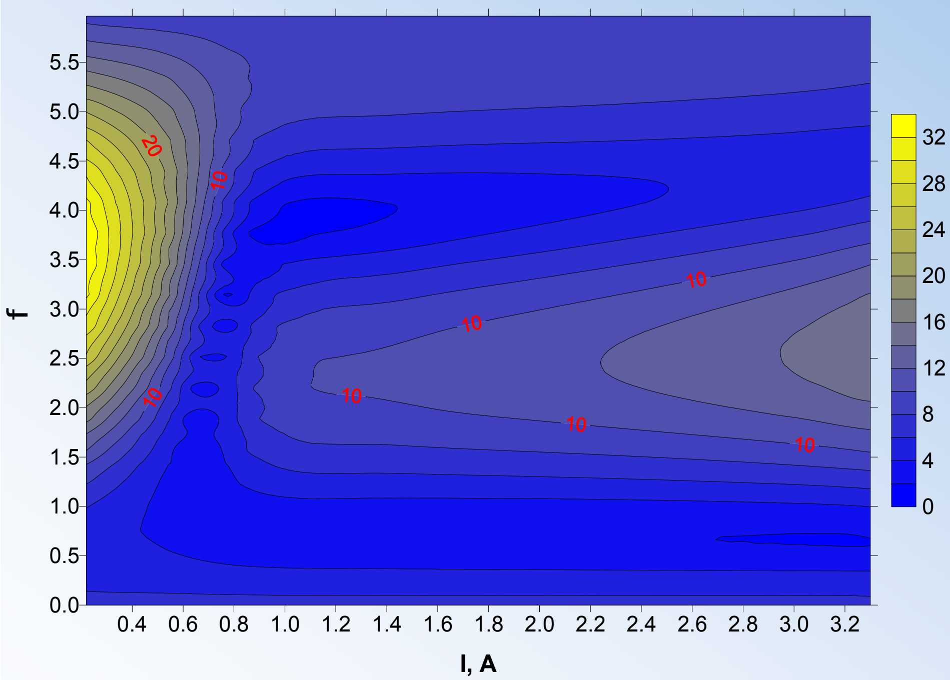
Differential parameters of a slice vs current: the phase and the relative amplitude of charge oscillation.



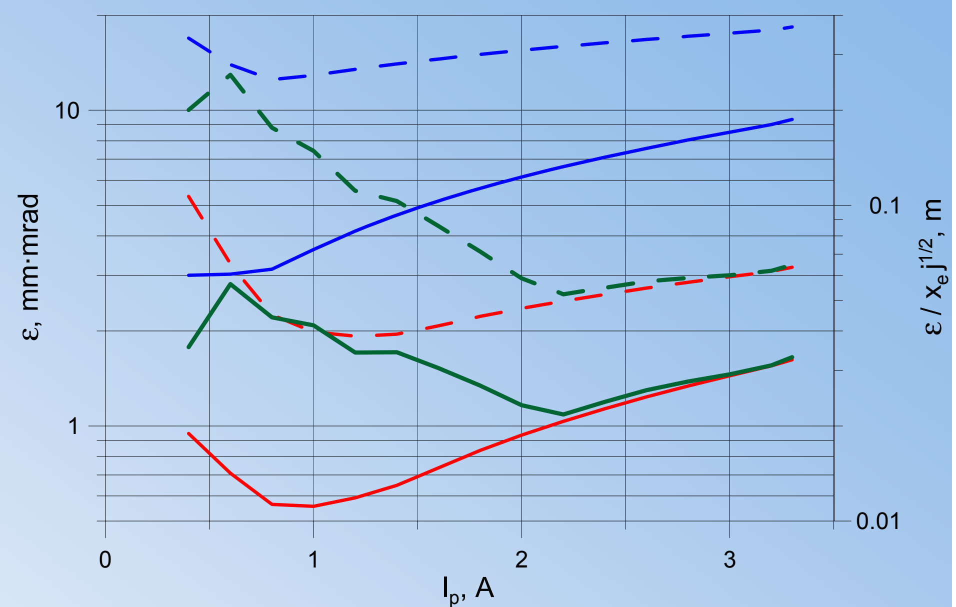
The rms size, the tilt, and the emittance of a slice vs current.

- **The phase of charge oscillation in a slice has no pronounced plateau.**
- **Its relative amplitude is small enough.**
- **Simultaneous compensation of the effect of longitudinal inhomogeneity and the effect of transverse one is impossible.**
- **It is not necessary, as the latter is much more weak.**

Simulation of guns



The emittance, mm·mrad, of a 2.2 A peak current bunch vs charge phase advance in the beamline and the current of the matched slice.



The emittance (solid) and the quality factor (dashed) vs the peak current of a bunch: blue for the exit of the gun, red for the optimal ideal beamline, and green for the optimal nonuniform beamline.

Parameters of model guns

Gun	ε_0^c	ε_h^c	I_m / I_p	φ	ε_r^c
300kvGun	0.25…0.35	0.04…0.06	0.4…0.5	2.4…2.5	0.05…0.08
150kvGun	0.25…0.35	0.03…0.05	0.37…045	2.0…2.1	0.04…0.06
300kvShortGun	0.3…0.4	0.04…0.07	0.56…0.67	1.9	0.1…0.2
850kvGun	0.35…0.5	0.07…0.1	0.49…0.59	2.5…2.7	0.11…0.17
300kvLongGun	0.26	0.045…0.055	0.3…0.45	3.1…3.4	0.05…0.08
300kvLongGunPl	0.3…0.5	0.06…0.08	0.11…0.13	1.6…1.8	0.02…0.025
RFGun1MV	0.2…0.35	0.037…0.06	0.022…0.067	0.6…1.0	0.017…0.023
RFGun2MV	0.2…0.35	0.037…0.065	0.02…0.08	0.6…1.1	0.018…0.022

- The emittance estimation is the same as for beamlines:

$$\varepsilon_n \cong \varepsilon^c x \sqrt{\frac{|I|}{I_0 \beta\gamma}}$$

- Adding of an optimal beamline decreases the emittance by 2…15 times.
- A planar cathode electrode gives better emittance for a pulsed gun than quasi-Pierce geometry.
- The parameters of best existing guns approach the obtained estimation.

Grid effects

- Scattering on wires.
- Focusing/defocusing in cells at non-optimal current.
- Thinning out of current by cells.
- The second effect is the strongest. It exceeds the macroscopic space charge effect with emittance compensation and can be neglected without the latter.

$$\varepsilon_n \approx 0.01 \frac{(r_e I)^{1/3} d}{\sqrt{D}}$$

d is the cell size,
D is the cathode-to-grid distance,
0.01 is the dimensional coefficient, all the values are in meters and Amps.

Conclusions

- A principal trajectory is an arbitrary solution of the motion equation.
- The linearized equation for dimensionless deviation from the principal trajectory reveals all the basic properties of the model.
- Charge oscillation is oscillation of deviation from the principal trajectory.
- Its coherence leads to emittance oscillation.
- Conditions of emittance minima: (1) ™’ = 0 or (2) $\varphi = n\pi$.
- Nonlinearity of charge oscillation → violation of coherence → residual emittance.
- The scaling formulae for residual emittance and parameters of optimal beamlines are universal, for guns too.
- The parameters of best existing injectors approach the obtained estimation.


Thank you for attention!