Generalized Long-Range Transverse Wakefields

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Generalized Transverse Wakefields:

Required because of cavity design and fabrication tolerances:

- Asymmetries may lead to anisotropic dipole modes (different $R/Q, \omega$ for different planes)
- Principal axes may deviate from accelerator x, y system $\rightarrow$ coupling
- Asymmetry + Asymmetric lossy boundary conditions: Rotating modes, $\rightarrow$ coupling
- Beam Dynamics effects need to be estimated

Generalization of Wakefield Kicks

ACD's *Omega3p* solves the double-curl eigenvalue equation for E

$$\nabla \times \frac{1}{\mu} \nabla \times \mathbf{E}_n(\mathbf{x}) - \omega_n^2 \varepsilon \mathbf{E}_n(\mathbf{x}) = 0 + f(\omega)\mathbf{E}_n \tag{1}$$

FEM discretization turns it into a non-linear eigenvalue equation:

$$M\mathbf{e}_n - \omega_n^2 S\mathbf{e}_n = f(\omega) R\mathbf{e}_n \tag{2}$$

M, S, R: *Mass, Stiffness, Damping* matrices
R removes self-adjointness: complex eigenvalues + solutions

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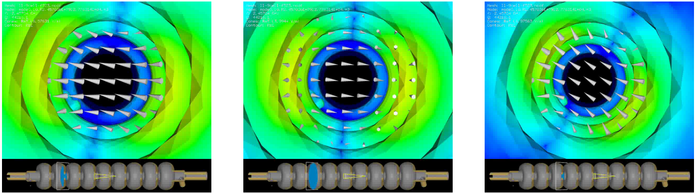
Rotating Modes

$R \neq 0$, simplest case: Damped dipole mode

With damped and sufficiently asymmetric cavities: degeneracies are lifted, mode splits into two elliptically polarized modes:



Rotating Modes: Movie

http://www.slac.stanford.edu/~akabel/SlidingCones4.avi

Normal Mode Expansion

Non-selfadjoint problem, *biorthogonality* relations:

$$\langle m | n \rangle = \int_\Omega \mathbf{E}_m^{adj}(\mathbf{x}) \cdot \mathbf{E}_n(\mathbf{x}) \mathrm{d}^3 x = \delta_{nm} + O(1/Q)$$

Decompose:

$$\mathbf{E}(\mathbf{x}, \omega) = \sum_n a_n(\omega) \mathbf{E}_n(\mathbf{x})$$

$$a_n^{\mathbf{m}}(\omega) = \frac{-i\omega}{{\omega_n^*}^2 - \omega^2} \int_\Omega \mathbf{j}^{\mathbf{m}}(\mathbf{x}, \omega) \cdot \mathbf{E}_n^\star(\mathbf{x}) \mathrm{d}^3 x$$

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Kick Calculation:

- Expand **j** in transverse multipoles
- Transform to time domain
- Apply Panofsky-Wenzel
- We are interested in the *coupling matrix*

$$K_{ik} = -\frac{\partial \Delta x'_i}{\partial x_k}$$

The coupling matrix:

Get it by fourier-transforming in angle and time:

$$\varphi_m = \frac{1}{2\pi R^m}\int_0^{2\pi}\int_0^L E^{\|}_{n,z}(R,\zeta,z)e^{i(m\zeta-\omega_n z)}\mathrm{d}z\mathrm{d}\zeta$$

With complex sine- and cosine coefficients:

$$\varphi = a_r - b_i + i(a_i - b_r)$$
$$\bar{\varphi} = a_r + b_i + i(a_i + b_r)$$

Looks like kick factor, but four free parameters/eigenmode:

$$K = -\frac{q_l q_t}{2\omega_n\gamma m}\Im e^{-i\omega_n t}\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}^+ =$$
$$\frac{q_l q_t}{2\omega_n\gamma m}\left[\sin\omega_n t\begin{pmatrix} |a|^2 & \Re a^* b \\ \Re ab^* & |b|^2 \end{pmatrix} - \cos\omega_n t\begin{pmatrix} 0 & \Im a^* b \\ \Im ab^* & 0 \end{pmatrix}\right]$$

Coupling:

- Kick factor has to be generalized to a hermitian *kick matrix*
- Real-valued off-diagonal elements can be removed by rotating by δ
- Imaginary ones remain:

$$H = \frac{\sin\omega_t}{2}(p_{\bar{x}}^2 + \beta^2 p_{\bar{y}}^2) - \frac{\alpha}{2}(\bar{x}^2 + \bar{y}^2(p_{\bar{x}}\bar{y} - p_{\bar{y}}\bar{x})\sin\Delta)$$

- (β: amplitude ratio, Δ: phase angle (a, ib^*), $\alpha = \frac{|a|^2 q_l q_t}{2\omega_n \gamma m}$)
- Mode can be characterized by strength α, excentricity $1-\beta^2$, coupling angles δ and Δ.

Adapted long-range transverse part in *Lucretia* WF tracking algorithm, generalized for offset kicks $X = (1, \mathbf{x})$, K complex 2×3 coupling matrix:

$$\Delta \mathbf{p}_i = \Re \sum_{n,k<i} q_i K_n e^{i\omega(T_i - T_k)} q_k X_k$$

Bunch-by-bunch algorithm:

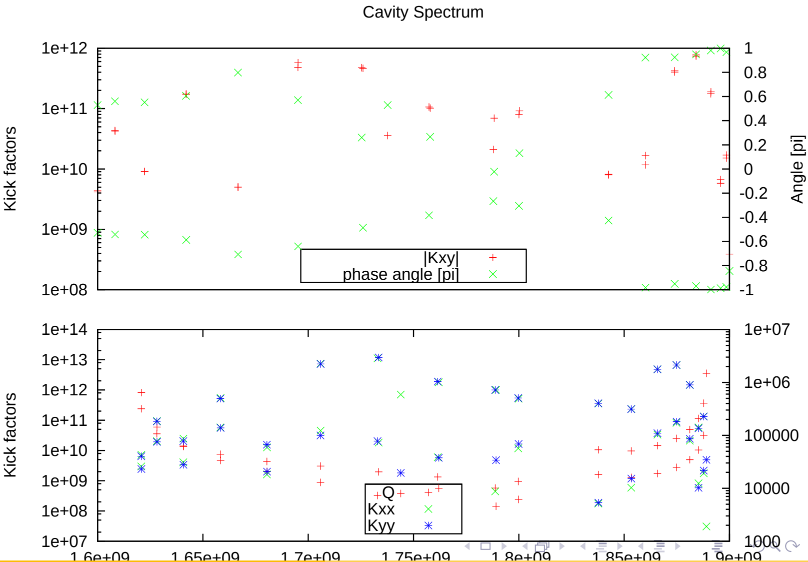
$$\Delta \mathbf{p}_i = \Re \sum_n e^{i\omega_n t_i} q_i K_n \Phi_{n,i-1}$$
$$\Phi_{i,n} = e^{-i\omega_n T}(\Phi_{i,n} + e^{-i\omega_n t_{i-1}} q_{i-1} X_{i-1})$$

Kick matrix K is obtained by post-processing Omega3p output

Sanity Check

Check different combinations of initial offsets, wakefield configuration. All quantities normalized to their maximum values.

Δx	Δy	K_{xx}	K_{yy}	$\Re K_{xy}$	$\Im K_{xy}$	ϵ_x	ϵ_y	$\hat{\epsilon}_x$	$\hat{\epsilon}_y$
1	0	0	1	0	0	10^{-10}	0		
1	0	1	0	0	0	1.0	0		
1	0	1	1	0	0	1.0	0		
1	0	1	1	.05	0	1.0	.034		
1	0	1	1	0	.05	1.0	.033		
0	1	0	1	0	0	1.0	0		
0	1	1	0	0	0	10^{-8}	10^{-12}		
0	1	1	1	0	0	10^{-8}	1.0		
0	1	1	1	.05	0	.034	1.0	10^{-11}	1.03
0	1	1	1	0	.05	.032	1.0	0.023	0.70

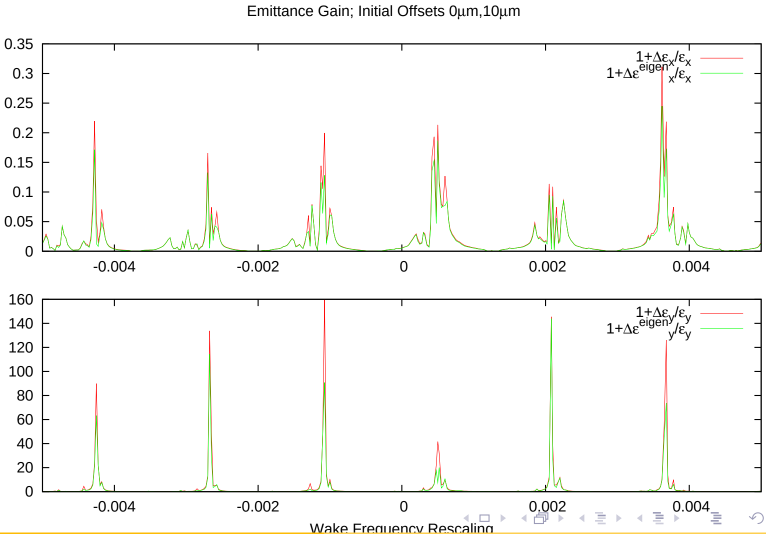

Cavity Spectrum
Kick factors
Angle [pi]
|Kxy|
phase angle [pi]
1e+12
1e+11
1e+10
1e+09
1e+08
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
Kick factors
Q
Kxx
Kyy
1e+14
1e+13
1e+12
1e+11
1e+10
1e+09
1e+08
1e+07
1e+07
1e+06
100000
10000
1000
1.6e+09
1.65e+09
1.7e+09
1.75e+09
1.8e+09
1.85e+09
1.9e+09

Tracking Results

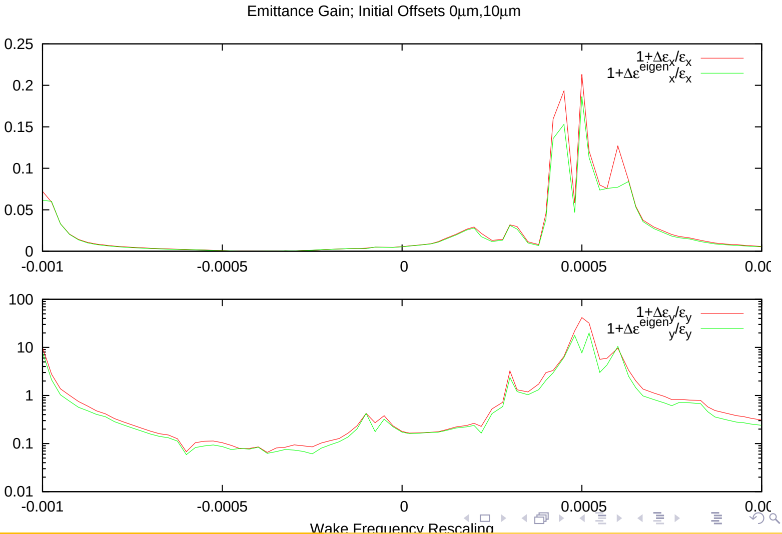
Drastically simplified model:

- No errors, misalignments (exception below)
- 200 bunches only
- Centroids only
- (no product distribution centroids/real transverse)
- *No 3rd band, no imperfections*

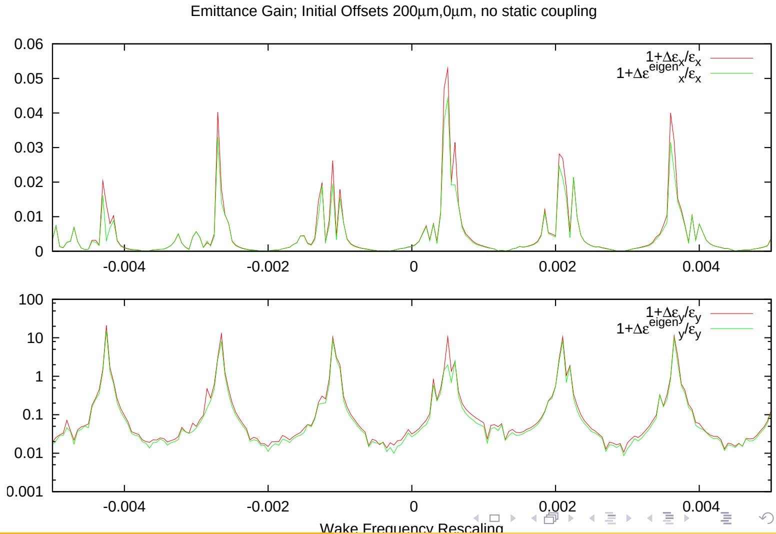
Coupling 1


Emittance Gain; Initial Offsets 0µm,10µm
$1+\Delta\varepsilon_x/\varepsilon_x$
$1+\Delta\varepsilon^{eigen}_x/\varepsilon_x$
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.004
-0.002
0
0.002
0.004
$1+\Delta\varepsilon_y/\varepsilon_y$
$1+\Delta\varepsilon^{eigen}_y/\varepsilon_y$
160
140
120
100
80
60
40
20
0
-0.004
-0.002
0
0.002
0.004
Wake Frequency Rescaling

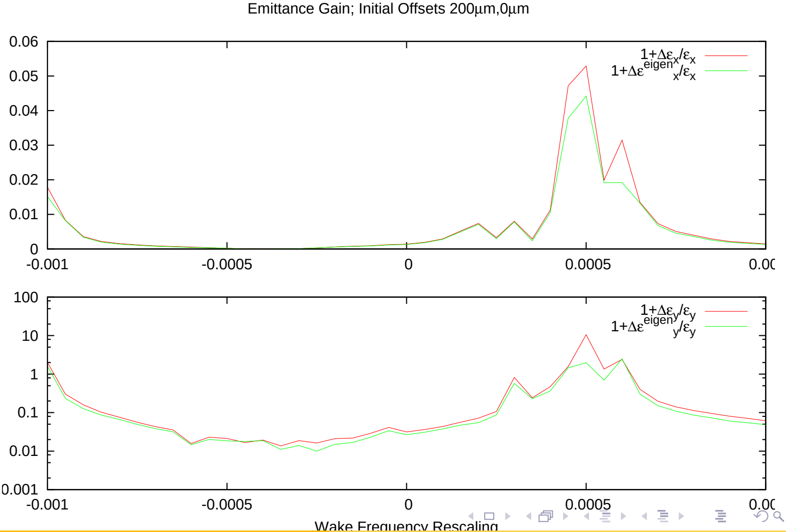
Closeup


Emittance Gain; Initial Offsets 0µm,10µm
$1+\Delta\varepsilon_x/\varepsilon_x$
$1+\Delta\varepsilon^{eigen}_x/\varepsilon_x$
0.25
0.2
0.15
0.1
0.05
0
-0.001
-0.0005
0
0.0005
0.00
$1+\Delta\varepsilon_y/\varepsilon_y$
$1+\Delta\varepsilon^{eigen}_y/\varepsilon_y$
100
10
1
0.1
0.01
Wake Frequency Rescaling

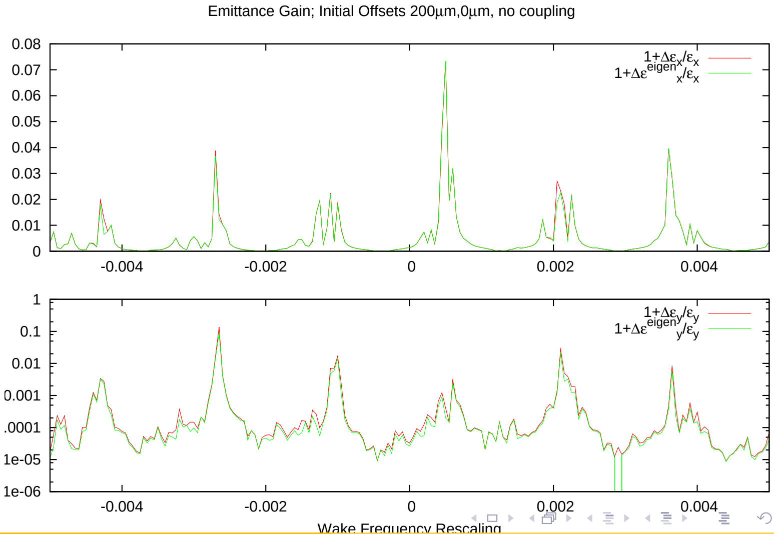
Coupling 2


Emittance Gain; Initial Offsets 200µm,0µm, no static coupling
0.06
0.05
0.04
0.03
0.02
0.01
0
-0.004
-0.002
0
0.002
0.004
$1+\Delta\epsilon_x/\epsilon_x$
$1+\Delta\epsilon^{eigen}_x/\epsilon_x$
100
10
1
0.1
0.01
0.001
$1+\Delta\epsilon_y/\epsilon_y$
$1+\Delta\epsilon^{eigen}_y/\epsilon_y$
Wake Frequency Rescaling

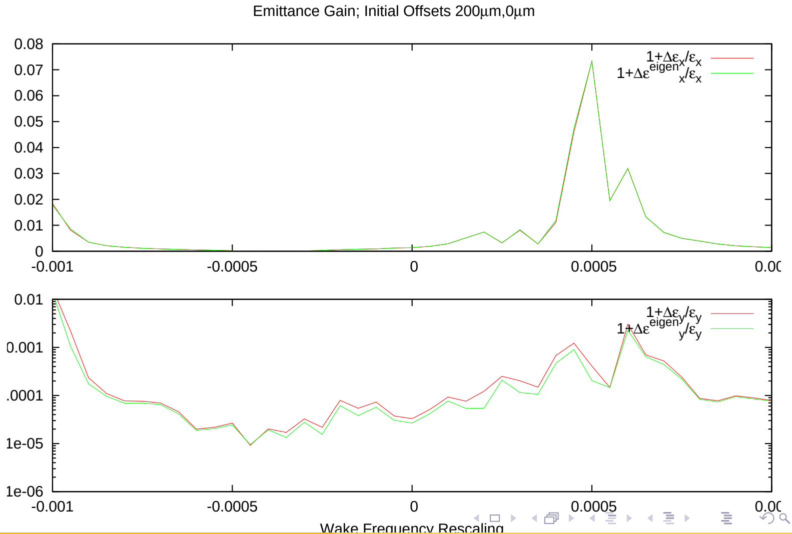
Closeup


Emittance Gain; Initial Offsets 200µm,0µm
0.06
0.05
0.04
0.03
0.02
0.01
0
-0.001
-0.0005
0
0.0005
$1+\Delta\varepsilon_x/\varepsilon_x$
$1+\Delta\varepsilon^{eigen}_x/\varepsilon_x$
100
10
1
0.1
0.01
0.001
$1+\Delta\varepsilon_y/\varepsilon_y$
$1+\Delta\varepsilon^{eigen}_y/\varepsilon_y$
Wake Frequency Rescaling

Coupling removed



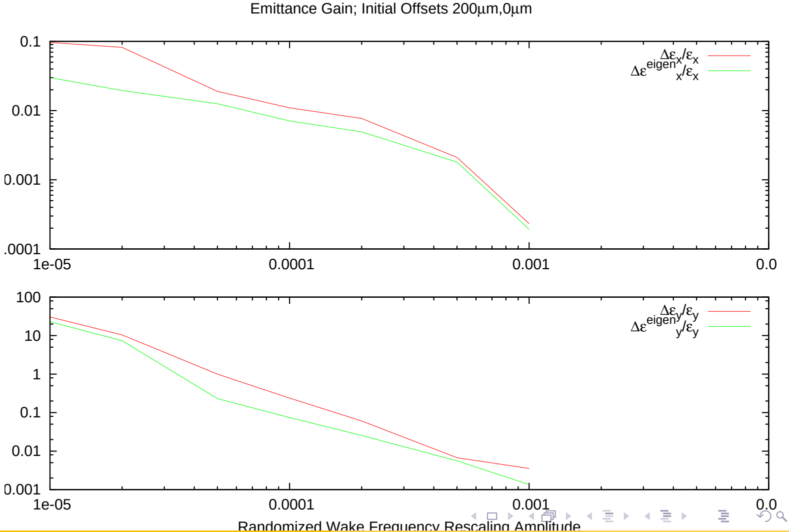
Coupling removed



Randomization of Cavities

- Previous results: *worst case*
- More likely: cavities have scattered fabrication errors
- Add per-cavity random error
- Look at 1.0005 scaling point, increasingly randomize

Randomized Spectra



Conclusion

- Wakefields of realistic cavities may have fewer symmetries: coupling, rotating modes
- Generalized normal-mode expansion formalism: Kick factors need to be replaced by hermitian kick matrices
- We have a toolchain to extract these matrices from high-fidelity frequency-domain calculations and plug them into tracking studies
- Proof-of principle tracking studies with *Lucretia*
- Future studies: 3rd bands, imperfections, misalignments, feedback, ...