

DESY-Cockcroft computer algorithms for estimating depolarisation in the DR, Linac & BDS in the ILC

Desmond P. Barber

**DESY
University of Liverpool
Cockcroft Institute**

Topics

0. The Why and How of simulation.
1. Recap: T-BMT and stability of spin motion.
2. A local coordinate system for spin.
3. Digression: definition of polarisation.
4. Modelling depolarisation due to synchrotron radiation, noise, scattering....
5. Results of an ILC dampin ring.
7. Future plans.

Literature

1. <http://www.desy.de/~mpybar> (lectures, talks, papers)
2. A.W. Chao, 1981 US Acc. School SLAC-PUB-2781. 1981.
3. DESY Reports by: Barber, Heinemann, H. Mais, G. Ripken.
4. M. Berglund, DESY-THESIS-2001-044
5. S-I Tomonaga, "The Story of Spin" .
6.

Together with Larisa Malysheva

Help and advice from Ian Bailey, Kai Meng Hock, Andy Wolski.....

Upgrades of SLICKTACK with Kai Meng Hock and Ian Bailey.

The Why and How of simulation

Accelerators need quadrupole fields as well as the simple magnetic fields of the bending magnets (dipoles).

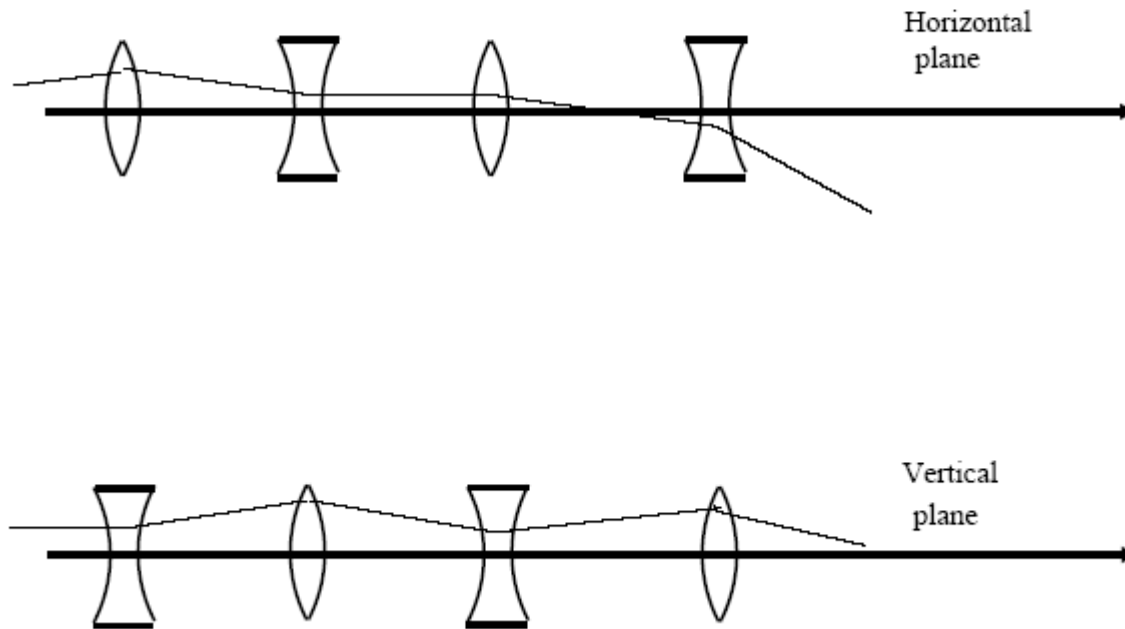
Can these fields influence the polarisation?

If so, how? and by how much?

So simulate:

A strong focussing channel

To confine the particles and transport the beam over long distances:
Alternate focusing and defocusing magnets: quadrupoles (symbolic)



To get a basic picture of feasibility without building hardware.

To learn about the behaviour of the system.

To learn how to avoid mistakes.

To learn how to optimise

etc.

Simulation covers both simple analytical calculations and numerical calcs.

Simulation is a basic, well used technique for designing accelerators.

NO MODERN FACILITY has been designed and built without massive amounts of simulation of the basic particle dynamics.

BUT: simulation will not tell if the system will definitely work.

BUT: it will tell if the system will not work.

Need:

Basic equations of of particle motion in magnetic and electric fields

Basic equation of spin motion in magnetic and electric fields

Usually neglect the S-G effects --> spins are spectators

Spin expectation values obey the T-BMT eqn: classical.

Recap: T-BMT and stability of spin motion

Spin precession: the T-BMT equation

$$\frac{d \vec{S}}{d t} = \frac{e}{m \gamma} \vec{S} \times \left((1 + a \gamma) \vec{B} - a(\gamma - 1) (\hat{t} \cdot \vec{B}) \hat{t} - \left(a \gamma + \frac{\gamma}{1 + \gamma} \right) \frac{\beta \hat{t} \times \vec{E}}{c} \right)$$

$$\frac{d \vec{S}}{d t} = \vec{\tilde{\Omega}}_s \times \vec{S} \quad \text{with}$$

$$\vec{\tilde{\Omega}}_s = -\vec{\tilde{\Omega}} = -\frac{e}{m \gamma} \left((1 + a \gamma) \vec{B} - a(\gamma - 1) (\hat{t} \cdot \vec{B}) \hat{t} - \left(a \gamma + \frac{\gamma}{1 + \gamma} \right) \frac{\beta \hat{t} \times \vec{E}}{c} \right)$$

Stability of spin motion

Normally it suffices to calculate w.r.t. the **design** orbit by defining

$$\vec{\Omega} = -\frac{e}{m\gamma} \left((a\gamma + 1)\vec{B} - \vec{B}_{\text{guide}} - \frac{a\gamma^2}{1+\gamma}(\vec{\beta} \cdot \vec{B})\vec{\beta} - (a\gamma + \frac{\gamma}{1+\gamma})\frac{\vec{\beta} \times \vec{E}}{c} \right) \frac{dt}{ds}$$

where s is the distance around the design orbit and $\vec{B}_{\text{guide}}(s)$ is the magnetic field in the dipoles defining the design orbit, and writing:

$$\frac{d\vec{S}}{ds} = \vec{\Omega} \times \vec{S}$$

Now clean up the notation again and write: $\vec{\Omega} \Rightarrow \vec{\Omega}$

So in "machine coordinates": i.e., w.r.t. the design orbit:

$$\frac{d\vec{S}}{ds} = \vec{\Omega} \times \vec{S}$$

The magnets and cavities are fixed in space --- so this is the description that we need.

Orbit coordinates: level 1:

For particle optics/motion we calculate w.r.t. the design orbit or if there are distortions, w.r.t. the **1-turn periodic closed orbit**.

Orbit coordinates: level 2:

And for insights and analysis we express particle coordinates

$$u \equiv (x, x', y, y', \sigma, \delta = \Delta E / E)$$

in terms of Courant-Snyder parameters and/or 1-turn eigenvectors.

$$u(s_2) = M_{6 \times 6}(s_2, s_1)u(s_1)$$

$$M_{6 \times 6}(s_1 + C, s_1)v_k(s_1) = e^{2\pi i Q_k} v_k(s_1) \quad k = I, II, III, -I, -II, -III$$

$$Q_{-I} = -Q_I ; v_{-I} = v_I^* \quad \text{etc.}$$

$$u(s) = \sum_k A_k v_k(s) \quad \text{Treat the } A\text{'s as } \mathbf{coordinates} \text{ in the basis } v$$

The motion is **quasiperiodic**.

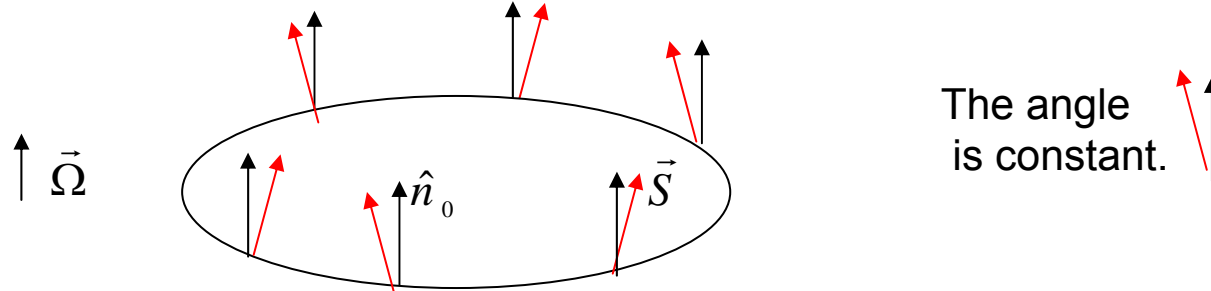
Orbital motion is sensitive/unstable w.r.t. to small perturbations (imperfections, nonlinear terms....) at resonances

$$(m_1 Q_1 + m_2 Q_2 + m_3 Q_3 = 0)$$

The analogue of the orbital closed orbit for spin?:

In a perfectly aligned flat ring with no solenoids the design orbit is flat and the particles only see a vertical $\vec{\Omega} \propto \vec{B}$

A vertical spin remains vertical : it is **1-turn periodic**. It is an example of the vector $\hat{n}_0(s)$ which is the 1-turn periodic solution of the T-BMT eqn. on the design or closed orbit. Any other spin precesses around the vertical $a\gamma$ times. If $a\gamma$ is an integer, all spins are 1-turn periodic and $\hat{n}_0(s)$ is non-unique!



$a\gamma$ is called the **spin tune**.

The vector $\hat{n}_0(s)$ lies at the centre of all serious calculations of polarisation in rings.

How do we generalise to complicated field configurations?

A local coordinate system for spin

In general:

$$\frac{d \vec{S}_1}{d s} = \vec{\Omega} \times \vec{S}_1 \text{ and } \frac{d \vec{S}_2}{d s} = \vec{\Omega} \times \vec{S}_2 \text{ then } \vec{S}_1 \cdot \vec{S}_2 = \text{const} .$$

so that the length of a spin is constant. Then we write

$$\vec{S}(s_1) = R_{3 \times 3}(s_1, s_0) S(s_0) \text{ with } \vec{S} =: \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} \quad R(s_n, s_0) = R(s_n, s_{n-1}) \dots R(s_1, s_0)$$

where R is an orthogonal matrix: $R^T = R^{-1} \Rightarrow R^T R = I$

To transport a spin for 1 turn along the design orbit: we use $R(s_0 + C, s_0)$

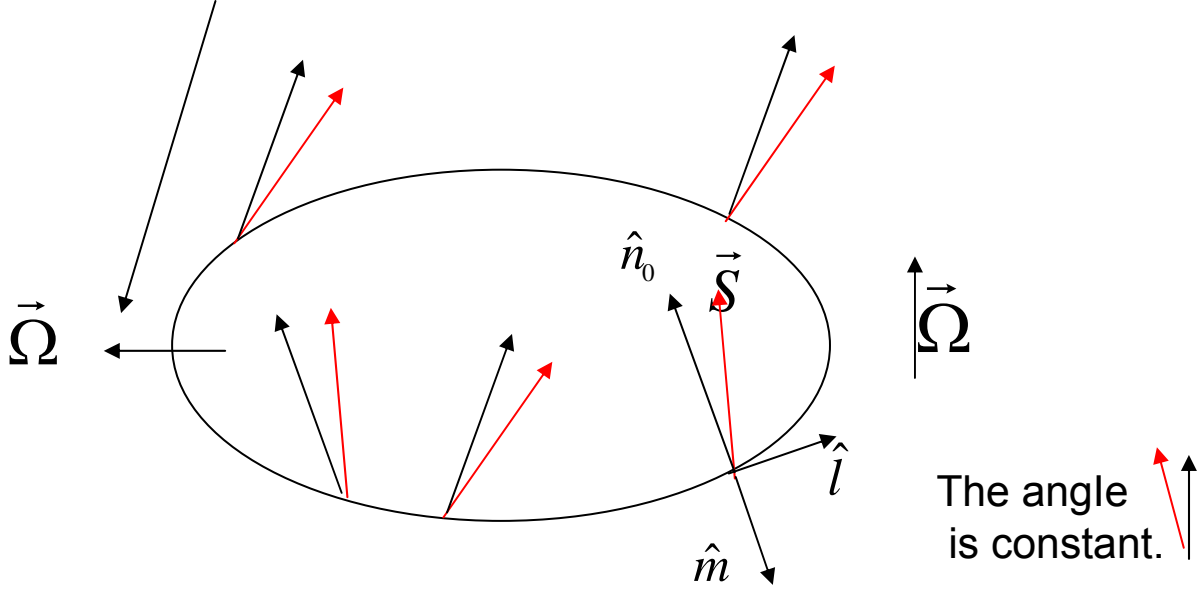
The eigenvalues of an orthogonal matrix take the form $1, e^{2\pi i \nu_0}, e^{-2\pi i \nu_0}$

There is always at least 1 real eigenvector $\hat{n}_0(s_0)$:

$$\hat{n}_0(s_0 + C) = R(s_0 + C, s_0) \hat{n}_0(s_0) = \hat{n}_0(s_0)$$

It corresponds to the eigenvalue of 1. It is the direction of the effective axis of rotation over 1 turn.

Consider a local perturbation – a piece of radial field. Then $\hat{n}_0(s)$ cannot be vertical everywhere



$\hat{n}_0(s)$ is periodic by definition and construction but \vec{S} is usually non-periodic.

An arbitrary spin moving on the central orbit rotates around $\hat{n}_0(s)$ by $2\pi\nu_0$ per turn

This ν_0 is the generalisation of the concept of **spin tune** to non-flat rings – such as HERA with its spin rotators and/or rings with solenoids.

If ν_0 not an integer the other 2 eigenvectors are complex and conjugate pairs.

Write them as $\hat{m}_0(s) \pm i\hat{l}_0(s)$ for $e^{\pm 2\pi i \nu_0}$ Can prove : $\hat{l}_0 \perp \hat{n}_0$, $\hat{m}_0 \perp \hat{n}_0$, $\hat{l}_0 \perp \hat{m}_0$

The set $(\hat{n}_0(s), \hat{m}_0(s), \hat{l}_0(s))$ with $\hat{n}_0(s) = \hat{m}_0(s) \times \hat{l}_0(s)$ gives a right-handed local coordinate system (**dreibein**)

If ν_0 not an integer \hat{m}_0 and \hat{l}_0 are not periodic.

Construct the **periodic** vectors \hat{m} and \hat{l} by winding back \hat{m}_0 and \hat{l}_0 uniformly..

$$\hat{m}(s) + i\hat{l}(s) = e^{-i2\pi\nu_0(s-s_0)/C} \left(\hat{m}_0(0) + i\hat{l}_0(0) \right)$$

$$\Rightarrow \frac{d(\hat{m}(s) + i\hat{l}(s))}{ds} = \left(\vec{\Omega} + \frac{2\pi\nu_0}{C} \hat{n}_0 \right) \times (\hat{m}(s) + i\hat{l}(s)) \quad \text{Also recall : } \frac{d}{dt} \Rightarrow \frac{d}{dt} + \vec{\omega} \times$$

We now have a **periodic** r.h. coordinate system $(\hat{n}_0(s), \hat{m}(s), \hat{l}(s))$ for spin.

That there is periodic coordinate system in which spins precess uniformly is a manifestation of the Floquet theorem for solutions of differential equations with periodic coefficients.

$\hat{n}_0(s)$ is 1-turn periodic – if viewed stroboscopically turn-by-turn it is a constant vector. Any other solution of the T-BMT equation on the closed orbit would exhibit a harmonic at ν_0 under a discrete Fourier transform.

See the printed extract from PRSTAB for details (Sect 3).

Many other periodic coordinate systems $\hat{n}_0, \hat{m}, \hat{l}$ can be constructed with

E.g. with $\hat{m} = \frac{\hat{x} \times \hat{n}_0}{|\hat{x} \times \hat{n}_0|}$, $\hat{l} = \hat{n}_0 \times \hat{m}$ but the precession is not uniform.

An arbitrary spin moving on the design rotates around $\hat{n}_0(s)$ by $2\pi\nu_0$ per turn

The subscript 0 on $\hat{n}_0(s)$ is **essential**.

If ν_0 is an integer, $R(s_0 + C, s_0)$ is a unit matrix and $\hat{n}_0(s)$ is arbitrary!

In a perfectly aligned flat ring with no solenoids $\nu_0 = a\gamma$

For electrons we have $\nu_0 = a\gamma = \frac{E}{0.440652} \frac{GeV}{GeV}$

For electrons in a perfectly aligned flat ring with no solenoids, spin motion on the design orbit tends to become unstable at intervals of 440.652 MeV where small perturbations have an overwhelming effect in defining the eigenvectors of an originally unit matrix!



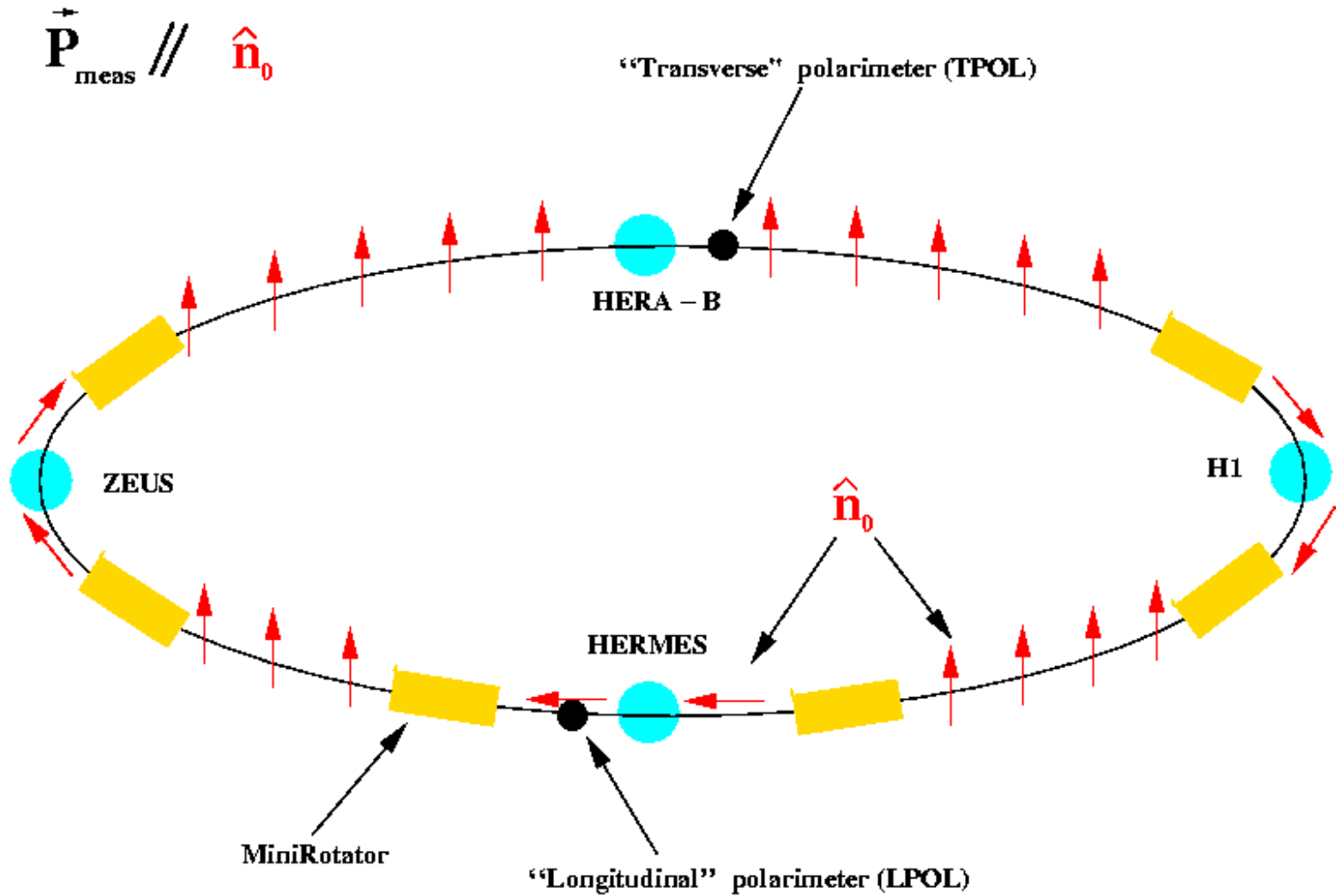
“Integer resonances”

If ν_0 is very near to an integer, a spin is almost in resonance with a perturbing field and the kicks to the spin tend to build up from turn to turn so that the spin motion appears to be very complicated. However, it is still simple, being a precession around $\hat{n}_0(s)$

Instead of doing **multi-turn** tracking, note that all the basic information is contained in $\hat{n}_0(s)$ which is obtained from a **1-turn** map and which will be strongly tilted from the vertical while the angle between the spin and $\hat{n}_0(s)$ is constant.

Recall use of eigenvectors and C-S parameters for orbital motion.

HERA electron/positron ring 2001 --



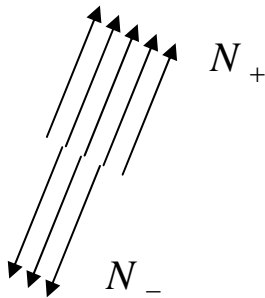
Definition of polarisation for spin-1/2

$$\vec{P} = \frac{1}{|\langle \vec{S} \rangle|} \frac{\sum_{i=1}^{1=N} \langle \vec{S}_i \rangle}{N}$$

Just an average of normalised expectation values – applicable both to pure and mixed states.

Obviously $|\vec{P}| \leq 1$

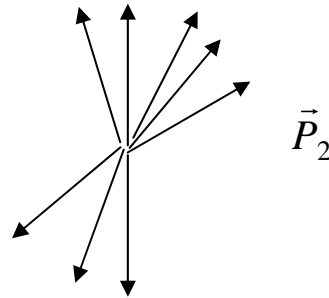
Example 1: a mixed fermion state consisting of N_+ (N_-) spins pointing up (down) along some direction:



$$P_1 = \frac{N_+ - N_-}{N_+ + N_-}$$

along the common direction.

Example 2: a mixed state consisting of spins pointing in many directions.



For fermions: if $\vec{P}_1 = \vec{P}_2$ the two states are **completely indistinguishable** w.r.t. any observations.

The polarisation and the T-BMT equation are linear in the spins.
So the polarisation for spins in an infinitesimal volume of phase space around an orbit obeys the T-BMT equation.

For fermions: if $\vec{P}_1 = \vec{P}_2$ the two states are **completely indistinguishable** w.r.t. any observations.

The spin density matrix:
$$\rho_{1/2} = \frac{1}{2} \{ I_{2 \times 2} + \vec{P} \cdot \vec{\sigma} \}$$

For a particles on the CO polarised along $\hat{n}_0(s)$, $\rho_{1/2}$ is diagonal w.r.t. $\hat{n}_0(s)$

→ model the ensemble with spins (anti)parallel to $\hat{n}_0(s)$

Modelling depolarisation due to synchrotron radiation, scattering, noise....

E.g.: synchrotron radiation (as a motivation).
-- it works

Reminder: synchrotron radiation causes spin flip at asymmetric rates so that the polarisation can build up.

Sokolov and Ternov around 1962: uniform magnetic field, Dirac solutions

Baier, Katkov, Strakhovenko, 1967..... : semiclassical, arbitrary geometries.

On the C.O., the polarisation builds up parallel to $\hat{n}_0(s)$!

Times scales! \longrightarrow Can forget the Sokolov-Ternov effect for the ILC!

Characteristic times of processes

M.Berglund
Thesis 2000.
After Montague '84

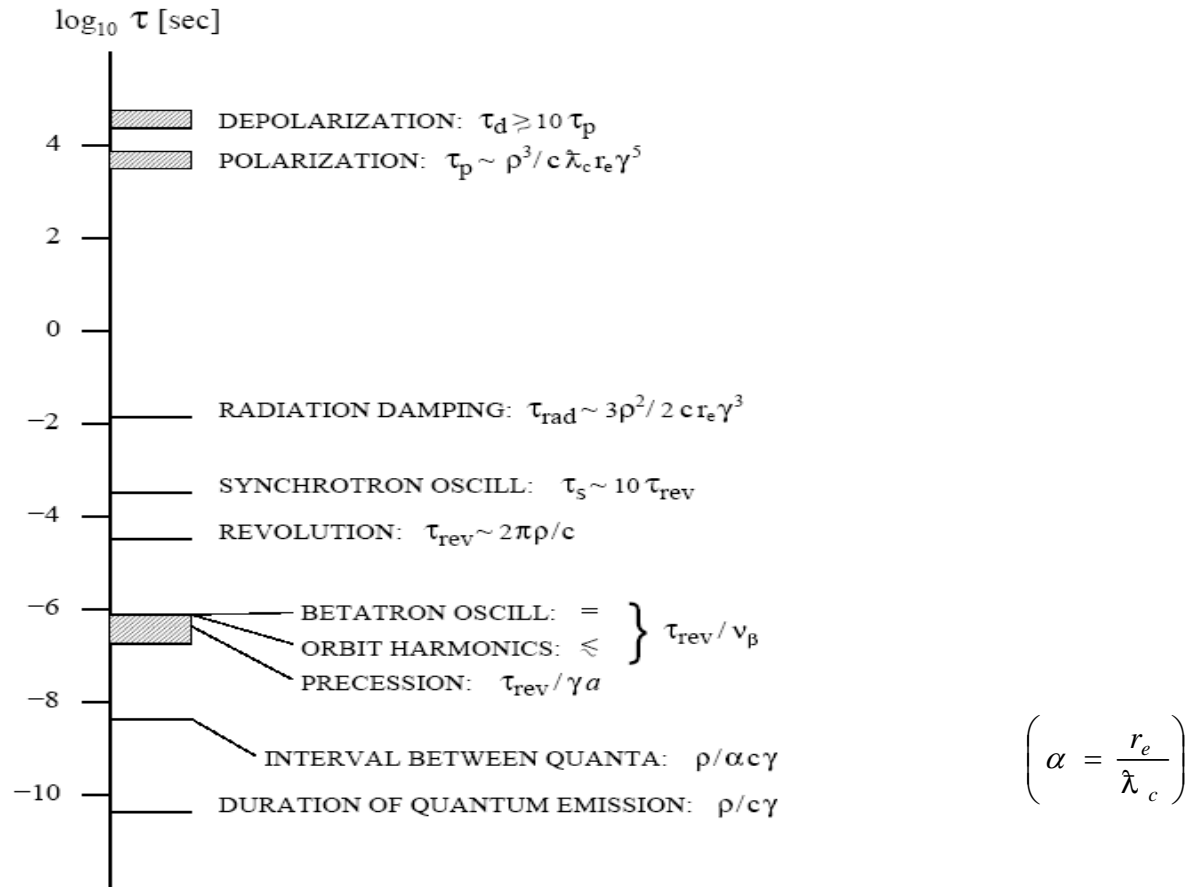


Figure B.1: Characteristic time scales in a typical 25 GeV electron storage ring [Mo84]. Legend: ρ = bending radius, λ_c = Compton wavelength, r_e = classical electron radius, ν_β = betatron tune, α = fine structure constant, a = gyromagnetic anomaly. Although it is desirable that $\tau_d \geq 10 \tau_p$, this is difficult to achieve in practice.

Why depolarisation due to synchrotron radiation?

--- for the same reason that electron (positron) beams have no memory!

namely that the energy loss caused by synchrotron radiation leads to damping effects and the discrete content (photons) of the radiation leads to **irreversible excitation of the orbits.**

The joint effect is that electron motion is that the particle distribution reaches an equilibrium state which depends **only of the geometrical and optical state of the ring.**

If the beam is disturbed, it returns to its equilibrium over a few **damping times and forgets the disturbance.**

A damping time usually corresponds to a several 100 turns: an electron beam is a weakly dissipative system.

Time scales!

Depolarisation has three origins:

Spin –orbit coupling (T-BMT equation)

Inhomogeneous fields (e.g. quadrupoles)

Noise due to synchrotron radiation.

A single photon emission results in only a tiny disturbance to a spin but there is a very large number of photons!

Real depolarisation is due to irreversibility.

Protons can exhibit reversible, temporary drops in polarisation provided that the system parameters change slowly:
manifestation of an adiabatic invariance.

A proton beam emits essentially no synchrotron radiation and in the absence of perturbations like wake fields, noise,the proton trajectories can be predicted using (reversible) Hamiltonian mechanics. Then the dimensions of proton bunches depend only on the volume of phase space occupied at the source, the gain in energy by acceleration and on the Courant-Snyder parameters.

Depolarisation 1: the SLIM algorithm (A.W. Chao NIM, 1981).

Orbit coordinates: level 2 (w.r.t. the closed orbit.)

$$u \equiv (x, x', y, y', \sigma, \delta = \Delta E / E) \quad u(s_2) = M_{6 \times 6}(s_2, s_1)u(s_1)$$

$$M_{6 \times 6}(s_1 + C, s_1)v_k(s_1) = e^{-2\pi i Q_k} v_k(s_1) \quad k = I, II, III, -I, -II, -III$$

$$Q_{-I} = -Q_I ; v_{-I} = v_I^* \quad \text{etc.} \quad u(s) = \sum_k A_k v_k(s)$$

$$M(s_2, s_1) \text{ represents a canonical transformation.} \Rightarrow M^T J M = J$$

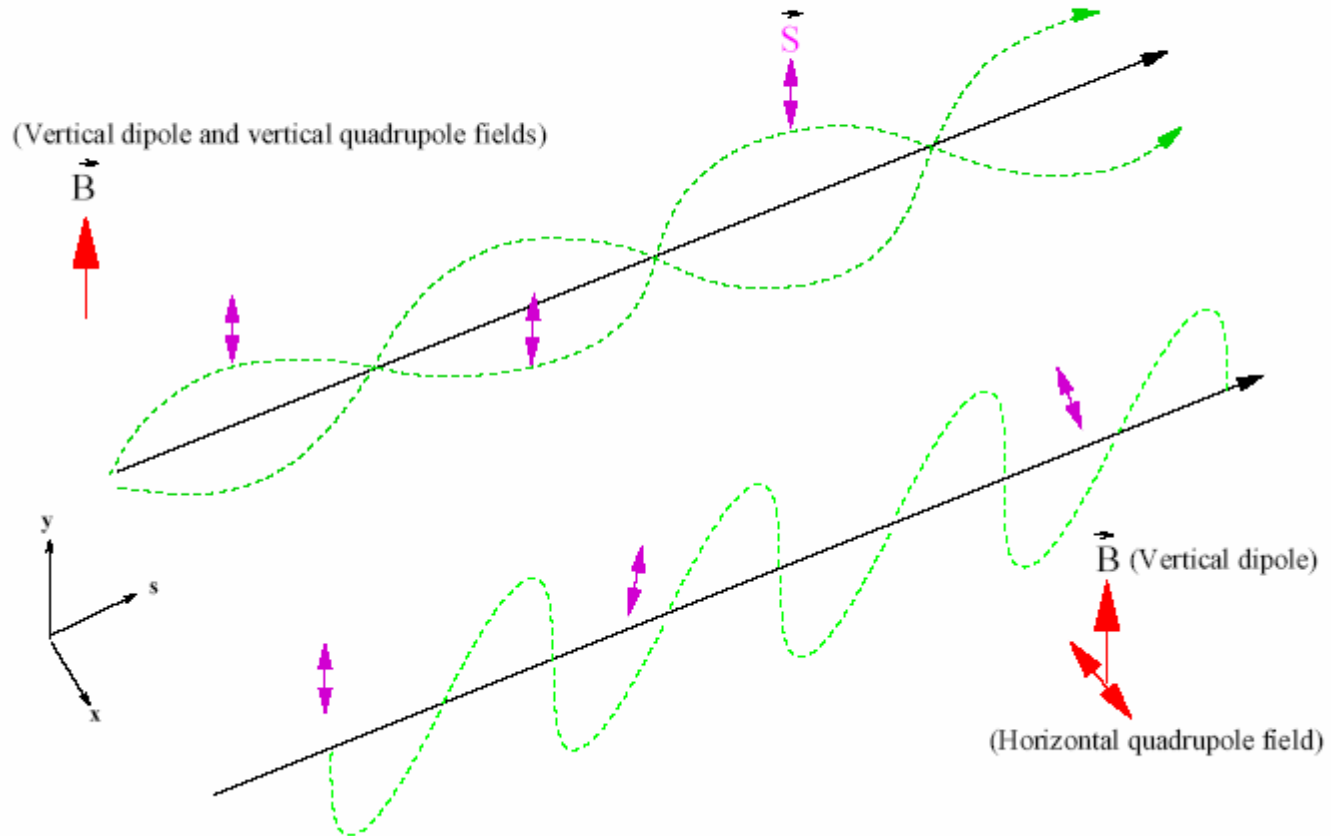
$$\text{with } J = \begin{bmatrix} j_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & j_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & 0_{2 \times 2} & j_{2 \times 2} \end{bmatrix}; \quad j_{2 \times 2} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad (\text{Recall } R^T R = I \text{ for } SO(3))$$

Orthogonality of eigenvectors:: e.g., $(\vec{v}_k^*)^T J \vec{v}_k = i$ etc.

The eigenvectors and eigenvalues encode everything about linearised motion

Combined effect of vertical dipole fields and horizontal quadrupole fields:

Non-commutation!



Look at spin motion in the $(\hat{n}_0, \hat{m}_0, \hat{l}_0)$ frame and then in the $(\hat{n}_0, \hat{m}, \hat{l})$ frame

Write $\vec{\Omega}(u; s) = \vec{\Omega}_0(s) + \vec{\omega}(u; s);$ (Define $\vec{\omega} = \vec{\Omega} - \Omega_0$)

$\vec{\omega}(s; u)$ embodies the effect of synchro-betatron motion on spin.

By working in the $(\hat{n}_0, \hat{m}_0, \hat{l}_0)$ frame we, in effect, “subtract out” the trivial spin motion so that we can concentrate on the more tricky stuff.

--- and work with small quantities!

$$\frac{d \vec{S}}{d s} = (\vec{\Omega}_0 + \vec{\omega}) \times \vec{S} = \hat{n}_0 \dot{S}_n + \hat{m}_0 \dot{S}_{m_0} + \hat{l}_0 \dot{S}_{l_0} + S_n \dot{\hat{n}}_0 + S_{m_0} \dot{\hat{m}}_0 + S_{l_0} \dot{\hat{l}}_0$$

$$\frac{d \vec{S}}{d s} = (\vec{\Omega}_0 + \vec{\omega}) \times \vec{S} = \hat{n}_0 \dot{S}_n + \hat{m}_0 \dot{S}_{m_0} + \hat{l}_0 \dot{S}_{l_0} + \vec{\Omega}_0 \times \vec{S} \quad (\text{Also recall } : \frac{d}{d s} \Rightarrow \frac{d}{d s} + \vec{\omega} \times)$$

$$\Rightarrow \hat{n}_0 \dot{S}_n + \hat{m}_0 \dot{S}_{m_0} + \hat{l}_0 \dot{S}_{l_0} = \vec{\omega} \times (\hat{n}_0 \dot{S}_n + \hat{m}_0 \dot{S}_{m_0} + \hat{l}_0 \dot{S}_{l_0})$$

Change notation to fit convention $S_n \Rightarrow \gamma_0$; $S_{m_0} \Rightarrow \alpha_0$; $S_{l_0} \Rightarrow \beta_0$ $\gamma_0 = \sqrt{1 - \alpha_0^2 - \beta_0^2}$

$$\Rightarrow \frac{d}{d s} \begin{pmatrix} \gamma_0 \\ \alpha_0 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} 0 & -\hat{l}_0 \cdot \vec{\omega} & \hat{m}_0 \cdot \vec{\omega} \\ \hat{l}_0 \cdot \vec{\omega} & 0 & -\hat{n}_0 \cdot \vec{\omega} \\ -\hat{m}_0 \cdot \vec{\omega} & \hat{n}_0 \cdot \vec{\omega} & 0 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \alpha_0 \\ \beta_0 \end{pmatrix} \quad \text{i.e., as usual, the coefficient matrix for rotations is antisymmetric} \\ \rightarrow \text{orthogonal transport matrices..}$$

Consider very small α_0 and β_0 : $\alpha_0^2 + \beta_0^2 \ll 1 \Rightarrow \gamma_0 \approx 1$ and treat $\vec{\omega}$ as a small perturbation.

Then $\vec{S} \approx \hat{n}_0 + \alpha \hat{m}_0 + \beta \hat{l}_0$ i.e., α_0 and β_0 are 2 small angles of tilt from : \hat{n}_0

$$\text{Then at 1st order: } \frac{d \alpha_0}{d s} = \vec{\omega} \cdot \hat{l}_0 \quad \frac{d \beta_0}{d s} = -\vec{\omega} \cdot \hat{m}_0$$

We prefer a periodic coordinate system: $\hat{m}(s) + i \hat{l}(s) = e^{-i\psi_0(s)} [\hat{m}_0(s) + i \hat{l}_0(s)]$ $\psi_0(s+C) - \psi_0(s) = 2\pi\nu_0$

$$\Rightarrow \frac{d\alpha}{ds} = \vec{\omega} \cdot \hat{l}_0 + \beta \psi'_0 \quad \frac{d\beta}{ds} = -\vec{\omega} \cdot \hat{m}_0 - \alpha \psi'_0 \quad \frac{d}{ds} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \psi'_0 \\ -\psi'_0 & 0 \end{bmatrix}}_{\text{Homogeneous.}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \underbrace{\vec{\omega} \cdot \begin{bmatrix} \hat{l} \\ -\hat{m} \end{bmatrix}}_{\text{Inhom. (driving)}}$$

We impose **3** linearisations

The orbital motion (use transfer matrices)

The dependence of $\vec{\omega}(u; s)$ on u in the T-BMT eqn. (*)

The spin motion: small $(\alpha, \beta) \Rightarrow$ simple (α', β')

*: e.g., in a quad $\vec{\omega}(u; s) = (a\gamma + 1)k(s)x\hat{y} - (a\gamma + 1)k(s)y\hat{x}$ $k =$ quad strength.

$\Rightarrow \vec{\omega} \cdot \hat{l} = (a\gamma + 1)k(s)xl_y - (a\gamma + 1)k(s)yl_x$ Analogue for $\vec{\omega} \cdot \hat{m}$

In general write

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_s \end{pmatrix} = F_{3 \times 6}(s) \begin{pmatrix} x \\ x' \\ y \\ y' \\ \sigma \\ \delta \end{pmatrix}$$

$\Rightarrow \frac{d \alpha}{d s}$ and $\frac{d \beta}{d s}$ are functions of s and linear functions of u just as , e.g., $x' = -k(s)x$

So now work with the spin-orbit vectors (u, α, β) !

The linearised orbital eqns can be integrated to give 6x6 transport matrices.

Now expand to 8x8 matrices:.

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ \sigma \\ \delta \\ \alpha \\ \beta \end{pmatrix} (s_1) = \begin{bmatrix} M_{6 \times 6} & 0_{6 \times 2} \\ G_{2 \times 6} & D_{2 \times 2} \end{bmatrix} (s_1, s_0) \begin{pmatrix} x \\ x' \\ y \\ y' \\ \sigma \\ \delta \\ \alpha \\ \beta \end{pmatrix} (s_0) \quad D(s_0 + C, s_0) = \begin{bmatrix} \cos 2\pi\nu_0 & \sin 2\pi\nu_0 \\ -\sin 2\pi\nu_0 & \cos 2\pi\nu_0 \end{bmatrix}$$

No Stern-Gerlach

The spin-orbit coupling matrix G is obtained by joint integration of the EOM and EOSM

The matrix D does the book keeping for motion in the periodic reference frame
 In the $(\hat{n}_0, \hat{m}_0, \hat{l}_0)$ frame it would just be $I_{2 \times 2}$

Example of a quad:

$$\left[\begin{array}{cc|cc|cc|cc} C_x & S_x & & & & & & \\ \hat{C}_x & \hat{S}_x & & & & & & \\ \hline & & C_y & S_y & & & & \\ & & \hat{C}_y & \hat{S}_y & & & & \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ \hline (1+a\gamma)\hat{C}_x l_y & (1+a\gamma)(\hat{S}_x-1)l_y & -(1+a\gamma)\hat{C}_y l_x & -(1+a\gamma)(\hat{S}_y-1)l_x & 0 & 0 & 1 & 0 \\ -(1+a\gamma)\hat{C}_x m_y & -(1+a\gamma)(\hat{S}_x-1)m_y & (1+a\gamma)\hat{C}_y m_x & (1+a\gamma)(\hat{S}_y-1)m_x & 0 & 0 & 0 & 1 \end{array} \right]$$

This simply expresses the expectation $\Delta \mathcal{G}_{\text{spin}} = (1+a\gamma)\Delta \mathcal{G}_{\text{orb}}$ in the $(\hat{n}_0, \hat{m}, \hat{l})$ frame!

So we could have written G by using intuition and without integrating!

Barber et al., Particle Accel. 1985.

By using a periodic reference frame $\vec{S} \approx \hat{n}_0 + \alpha \hat{m} + \beta \hat{l}$ we ensure that G is periodic.

Then we can do an eigen-analysis of the whole 8x8 matrix.

NOTE THE FACTORS $(1+a\gamma)$

Again: the matrix G just expresses the standard expectations – but in a well organised form.

Example of a quad:

$$\left[\begin{array}{cc|cc|cc|cc}
 C_x & S_x & & & & & & \\
 \hat{C}_x & \hat{S}_x & & & & & & \\
 \hline
 & & C_y & S_y & & & & \\
 & & \hat{C}_y & \hat{S}_y & & & & \\
 \hline
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 \hline
 (1+a\gamma)\hat{C}_x l_y & (1+a\gamma)(\hat{S}_x-1)l_y & -(1+a\gamma)\hat{C}_y l_x & -(1+a\gamma)(\hat{S}_y-1)l_x & 0 & 0 & 1 & 0 \\
 -(1+a\gamma)\hat{C}_x m_y & -(1+a\gamma)(\hat{S}_x-1)m_y & (1+a\gamma)\hat{C}_y m_x & (1+a\gamma)(\hat{S}_y-1)m_x & 0 & 0 & 0 & 1
 \end{array} \right]$$

Vertical quadrupole fields due to radial betatron motion: cause spins to tilt left/right
 If \hat{n}_0 is vertical (m_y, l_y) vanish. .
 So no effect as expected for vertical spins in vertical fields.

If \hat{n}_0 is horizontal, (m_y, l_y) are $O(1)$.

If \hat{n}_0 is tilted due to misalignments, there is some effect spin precession around the vertical quad fields.

Radial quadrupole fields due to vertical betatron motion cause spins to tilt forward and backward. If \hat{n}_0 is vertical (\hat{m}, \hat{l}) are horizontal $O(1)$.

In quads:
 no sync. terms
 at this order.

Now apply the methods for calculating orbital emittances of electron beams to spin as well

The ‘‘DESY approach’’ (Mais + Ripken): a stochastic calculus which puts text book accounts of the effects of synchrotron radiation on a formal basis and thereby exposes more detail. (See also A.W. Chao J.App.Phys. 1979, Lebedev and Kolomensky, S+T 1968)

Express the energy loss by sync. rad. in the form:

$P(s) = P_{\text{class}}(s) + P_{\text{stoch}}(s)$ where $P_{\text{stoch}}(s)$ results from a series of +ve & -ve kicks which average to zero.

$$\frac{d u}{d s} = \underbrace{(B(s))}_{\text{symplectic motion}} + \underbrace{(\delta B(s))}_{\text{damping with sync. rad.}} u + \Xi(s) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \zeta \end{bmatrix}$$

$$\Xi(s) = \left(\frac{55}{24 \sqrt{3}} \frac{r_e \hat{\lambda}_c}{|\rho^3|} \gamma^5 \right)^{1/2}$$

← Noise from photon emission
δ –function kicks

Model the energy kicks from photon emission as ‘‘Gaussian white noise’’ :

$$\langle \zeta(s_1) \zeta(s_2) \rangle = \delta(s_1 - s_2)$$

For a quad : $B =$

$$\begin{bmatrix} 1 & 0 & & & & \\ -k & 1 & & & & \\ & & 1 & 0 & & \\ & & +k & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

A solution for u depends on the history of the kicks.

The only definite objects are averages over all possible histories of kicks
i.e. "stochastic averages"

We take these averages to represent averages over the ensemble of electrons:
each electron has its own stochastic history (Wiener process).

For the origin of "stochastic" see:

<http://smccd.net/accounts/goth/MainPages/wordphys.htm>

-- and then wonder why we use the word.

Recall: in the absence of sync. rad. $u(s) = \sum_k A_k v_k(s)$

If an electron gets an energy kick $\Delta\delta$ at s_0 , the A_k change: $\Delta A_k = i v_{k5}^*(s) \Delta\delta \quad \forall k$
 $\Rightarrow \frac{\partial A_k}{\partial \delta} = i v_{k5}^*(s)$

Include damping too:
 and avoid orb. res.
 (needed averaging!).

$$\frac{d A_k}{d s} = A_k (-\alpha_k - i 2\pi (\delta Q_k)) + i \Xi(s) v_{k5}^*(s) \zeta(s)$$

↑ damping constant for mode k
↑ real tiny tune shift

Finally after some foot work :the emittances: $\langle |A_k(s)|^2 \rangle = \text{constant} = \frac{55}{24\sqrt{3}} r_e \tilde{\lambda}_c \gamma^5 \frac{1}{2\alpha_k} \oint_s d\tilde{s} \frac{1}{|\rho^3(\tilde{s})|} |v_{k5}(\tilde{s})|^2$
 If α_k is very small.

The system can be fully coupled. $|v_{k5}(s)|^2$ Is the analogue of function "H" usually seen when using beta functions and dispersions.

Beam sizes: $\langle u_m(s) u_n(s) \rangle = 2 \sum_{k=I,II,III} \langle |A_k|^2 \rangle \text{Re}[v_{km}(s) v_{kn}^*(s)]$

The damping times $\tau_k = \frac{C}{c \alpha_k}$ usually corresponds to many 100's of turns. Time scales!

To keep an emittance $\langle |A_k(s)|^2 \rangle$ small, keep its $\nu_{k5}^*(s)$ small where $1/\rho^3$ is large.
E.g. damping rings.

$\frac{\partial A_k}{\partial \delta} = i v_{5k}^*(s) \Rightarrow v_{5k}^*(s)$ encodes the sensitivity of the amplitude k to energy kicks

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ \sigma \\ \delta \\ \alpha \\ \beta \end{pmatrix} (s_1) = \begin{bmatrix} M_{6 \times 6} & 0_{6 \times 2} \\ G_{2 \times 6} & D_{2 \times 2} \end{bmatrix} (s_1, s_0) \begin{pmatrix} x \\ x' \\ y \\ y' \\ \sigma \\ \delta \\ \alpha \\ \beta \end{pmatrix} (s_0)$$

The spin $\vec{S} \approx \hat{n}_0 + \alpha \hat{m} + \beta \hat{l}$ is coupled to the orbit via the matrix G and we expect that as a result of the noise in the orbital motion, α and β execute some kind of a **random walk**.

Then with $\sqrt{1 - \alpha^2 - \beta^2} \approx 1 - \frac{1}{2}(\alpha^2 + \beta^2)$ we want to find

$$\tau_{\text{dep}}^{-1} = -\frac{1}{P_n} \frac{dP_n}{dt} \approx \frac{d}{dt} \left\{ \frac{1}{2} (\sigma_\alpha^2 + \sigma_\beta^2) \right\} = \frac{d}{dt} \frac{1}{2} \{ \langle \alpha^2 \rangle + \langle \beta^2 \rangle \}$$

Extend the orbital eigenvectors to include spin

$$\vec{q}_k(s_0) = \begin{pmatrix} \vec{v}_k(s_0) \\ \vec{w}_k(s_0) \end{pmatrix}, \quad \vec{q}_{-k}(s_0) = [\vec{q}_k(s_0)]^* \quad \vec{q}_k(s_0) = \begin{pmatrix} \vec{0}_6(s_0) \\ \vec{w}_k(s_0) \end{pmatrix}, \quad \vec{q}_{-k}(s_0) = [\vec{q}_k(s_0)]^*$$

for $k = I, II, III$; for $k = IV$

No back reaction of the spin on the orbit (no S-G) so:

$$\vec{w}_k(s_0) = - \left[\mathbf{D}(s_0 + C, s_0) - \hat{\lambda}_k \right]^{-1} \mathbf{G}(s_0 + C, s_0) \vec{v}_k(s_0)$$

for $k = I, II, III$;

$$\vec{w}_{IV}(s_0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\psi_{\text{spin}}(s_0)}$$

for $k = IV$

Resonance denominators!
Trouble if for any mode k :

$$\nu_0 \approx \text{integer} \pm Q_k$$

unless $G \nu_k = 0$

➡ SPIN MATCHING!

Extend the stochastic calculus used for the orbit to the spin.

Introduce the periodic "diffusion vector" $\vec{d}(s) \equiv \underbrace{d_\alpha(s)\hat{m}(s) + d_\beta(s)\hat{l}(s)}_{\text{periodic}} \text{ perp.to } \hat{n}_0$

Where
$$\begin{bmatrix} d_\alpha \\ d_\beta \end{bmatrix}(s) = -2 \operatorname{Im} \sum_{k=I,II,III} v_{k5}^*(s) \begin{bmatrix} w_{k\alpha} \\ w_{k\beta} \end{bmatrix}$$

This characterises the strength of **random walk** of α and β :

Note that the linearisation allows a decomposition into the effects of each mode.

$$\tau_{\text{dep}}^{-1} = -\frac{1}{P_n} \frac{dP_n}{dt} = \frac{d}{dt} \frac{1}{2} \{ \langle \alpha^2 \rangle + \langle \beta^2 \rangle \} = \frac{c}{C} \frac{55 \sqrt{3}}{144} r_e \tilde{\lambda}_c \gamma^5 \oint ds \underbrace{\frac{(d_\alpha^2(s) + d_\beta^2(s))}{|\rho^3(s)|}}_{\text{orbit excitation}}$$

Although there is a damping mechanism for electrons, it is this mechanism that contributes to depolarisation!

Resonances

Spins are passengers/spectators subject to interleaved kicks from quadrupoles and precessions in the dipoles.

If their natural precession motion is coherent with the kicks from the synchro-betatron motion, there can be a resonant and very strong accumulation of the disturbance to the spins.

If α and β diffuse due to noise, there can be resonant enhancement of the diffusion:
This is automatically contained in the

$$\begin{bmatrix} d_\alpha \\ d_\beta \end{bmatrix}(s) = -2 \operatorname{Im} \sum_{k=I,II,III} v_{k5}^*(s) \begin{bmatrix} w_{k\alpha} \\ w_{k\beta} \end{bmatrix}$$

Only first order resonances $\nu_0 = m_0 \pm Q_I, \nu_0 = m_0 \pm Q_{II}, \nu_0 = m_0 \pm Q_{III}$

Add the diffusion in incoherently with the S-T (B-K)

$$\tau_{\text{tot}}^{-1} = \tau_{\text{ST}}^{-1} + \tau_{\text{dep}}^{-1}$$

$$\Rightarrow P_{n_0}(s = \infty) = P_{\text{BK}} \frac{\tau_{\text{tot}}}{\tau_{\text{ST}}}$$

Skipping the full Derbenev-Kondratenko treatment and the Invariant Spin Field

This formalism provides a fast practical algorithm for getting a first look at the potential (de)polarisation in a ring. It can handle arbitrary coupling and misalignments

Although some pieces were glued together, things can be put on a more rigorous basis.

Note that the damping constants do not appear in the depolarisation rate and that the particle distribution reaches equilibrium whereas, in the absence of the S-T effect, the spin distribution does not reach equilibrium: there is no damping effect for spin. Moreover, the damping mechanism for electrons **contributes** to depolarisation!

This is a consequence of the adiabatic invariance of a special spin gadget coming later.

The original thin lens code **SLIM** of Chao was upgraded to thick lenses in 1982 (by me): **SLICK** using the formalism in various Mais-Ripken papers.

This **semi-analytical** formalism is the main work horse for a first look at the possibilities for polarisation in an electron STORAGE ring.

For damping rings we need a Monte-Carlo simulation!

Monte Carlo!

The 8x8 matrices can be used to carry out spin-orbit tracking within a **Monte-Carlo** setting: **SLICKTRACK**

Break the ring up into sections and radiate “big photons” at each dipole.

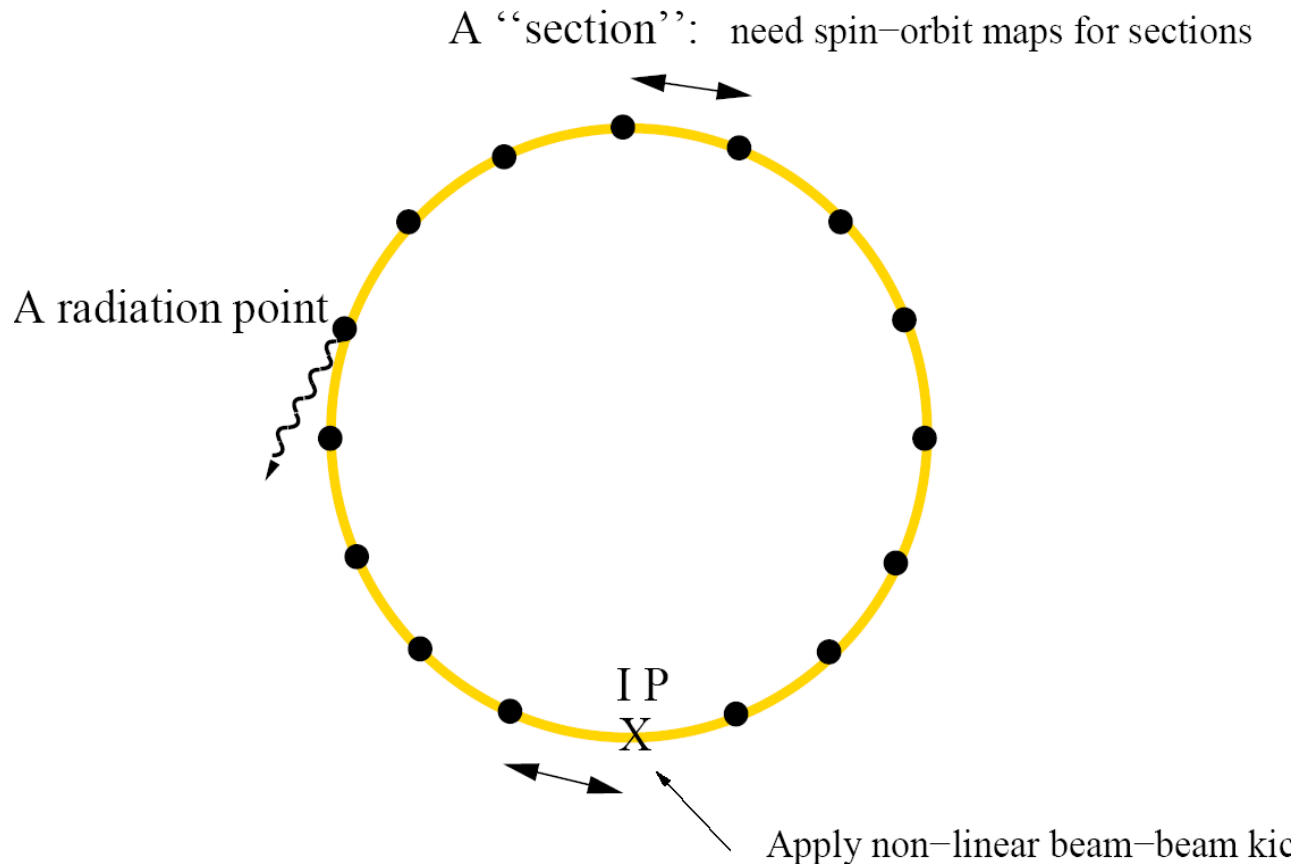
By extension to 9x9 matrices, the linearisation of the spin motion is avoided and full 3-D spin motion is treated \longrightarrow extra depolarising effects exposed, manifested as higher order resonances

$$v_0 = m_0 + m_I Q_I + m_{II} Q_{II} + m_{III} Q_{III}$$

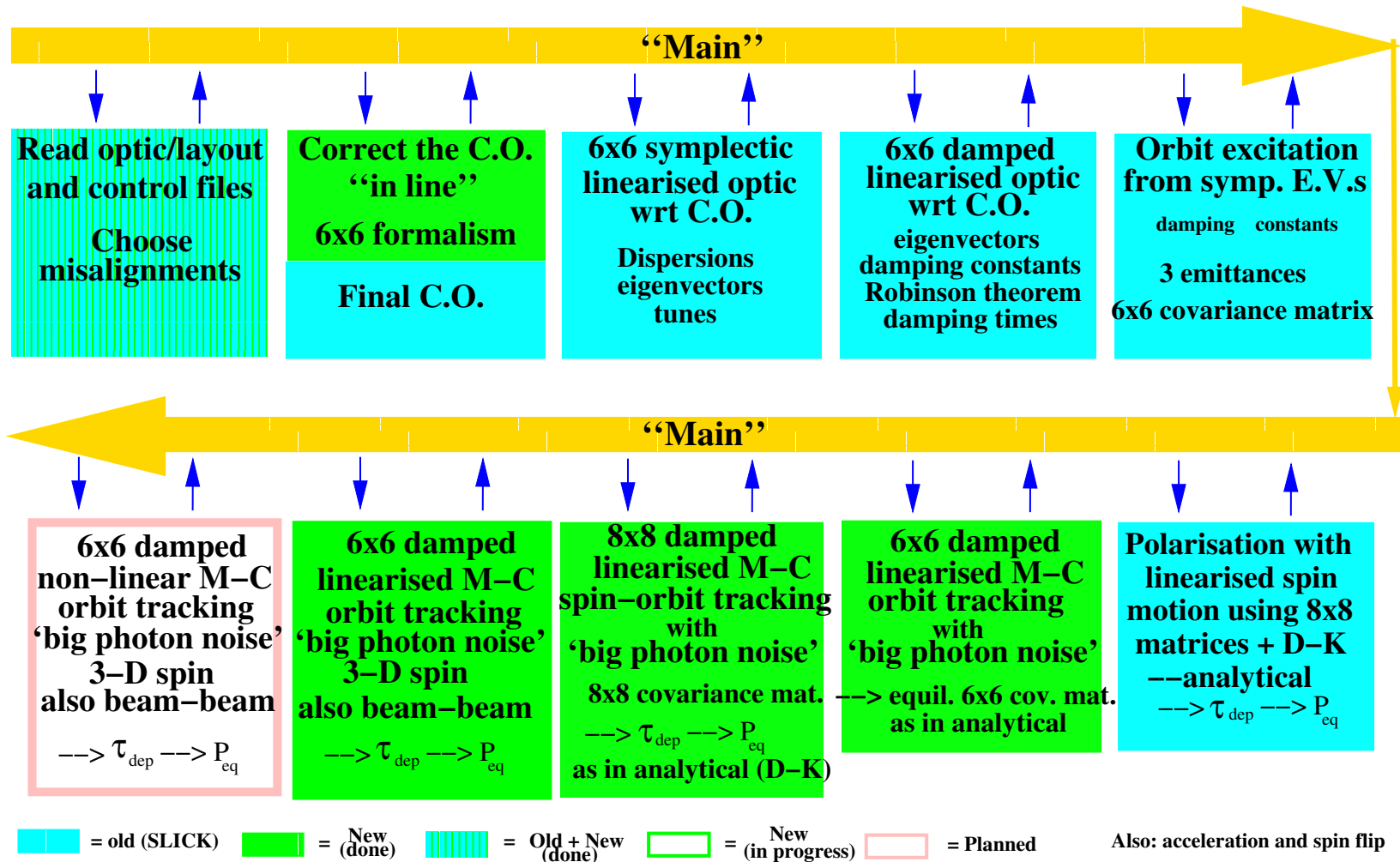
Even with linear orbital motion.!!!

Can also include non-linear optical elements like sextupoles.

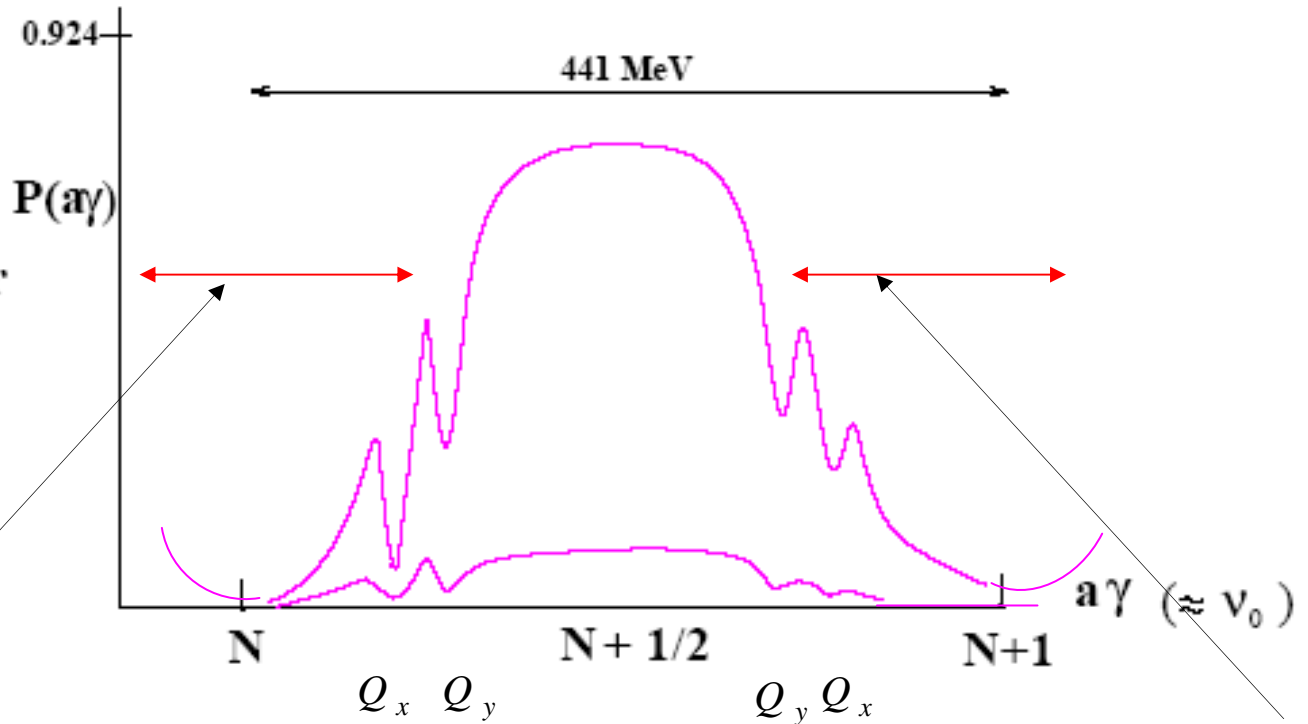
Note: $\dot{\vec{d}}^2 \Rightarrow \frac{\tau_{\text{dep}}^{-1}}{\tau_{\text{BK}}^{-1}}$ scales at least like γ^2 . So life gets difficult at very high energy.



The structure of SLICKTRACK



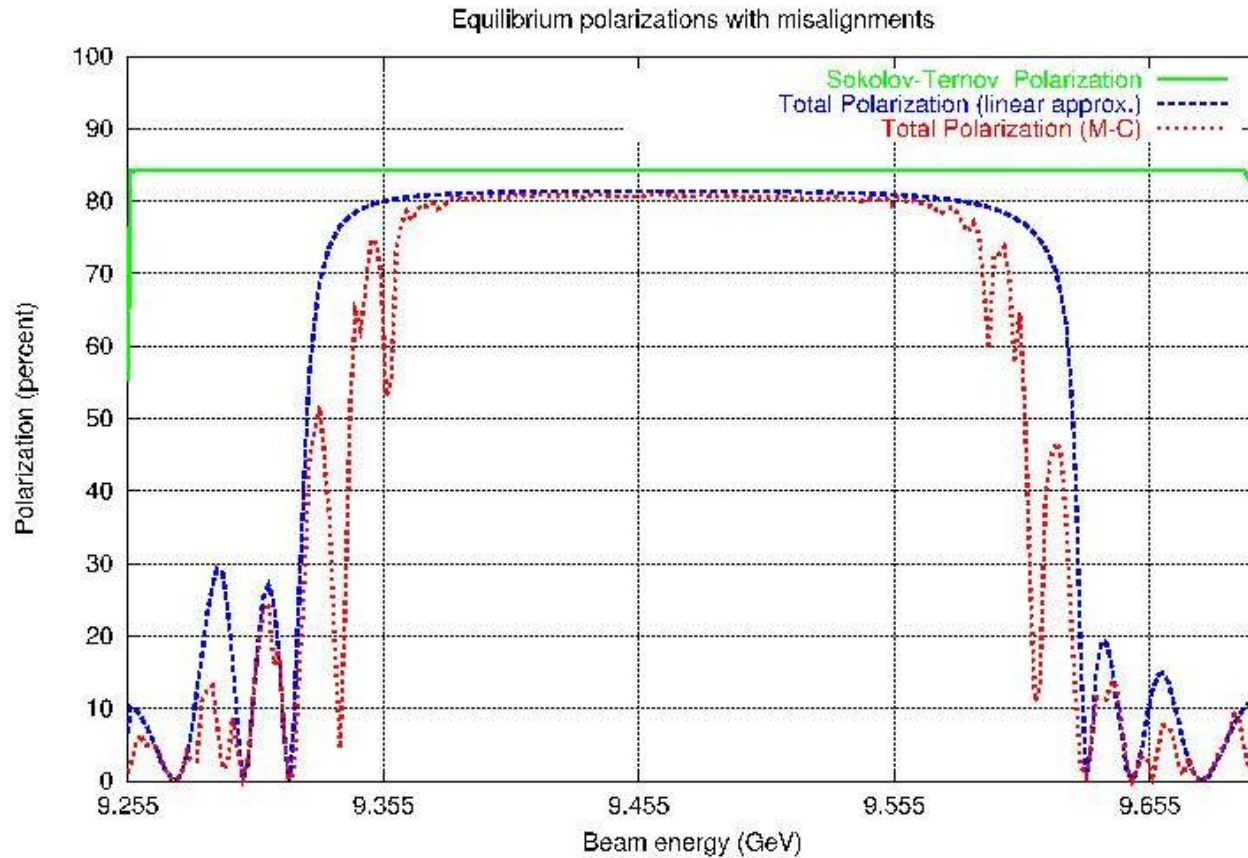
SLIM formalism:
 → only first order resonances



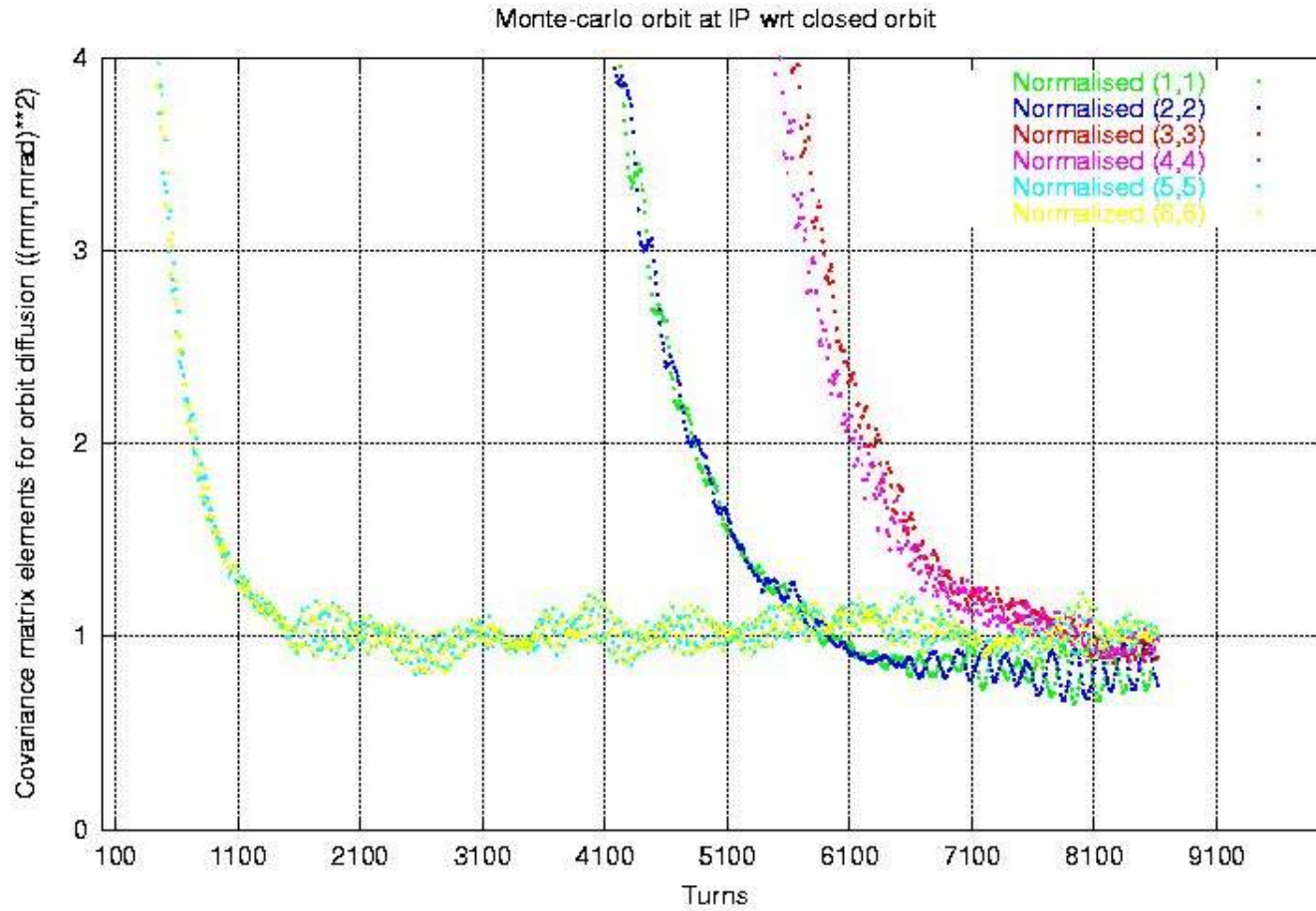
Broad (strong) overlapping
 $\nu_0 = N \pm Q_s$

Broad (strong) overlapping
 $\nu_0 = N + 1 \pm Q_s$

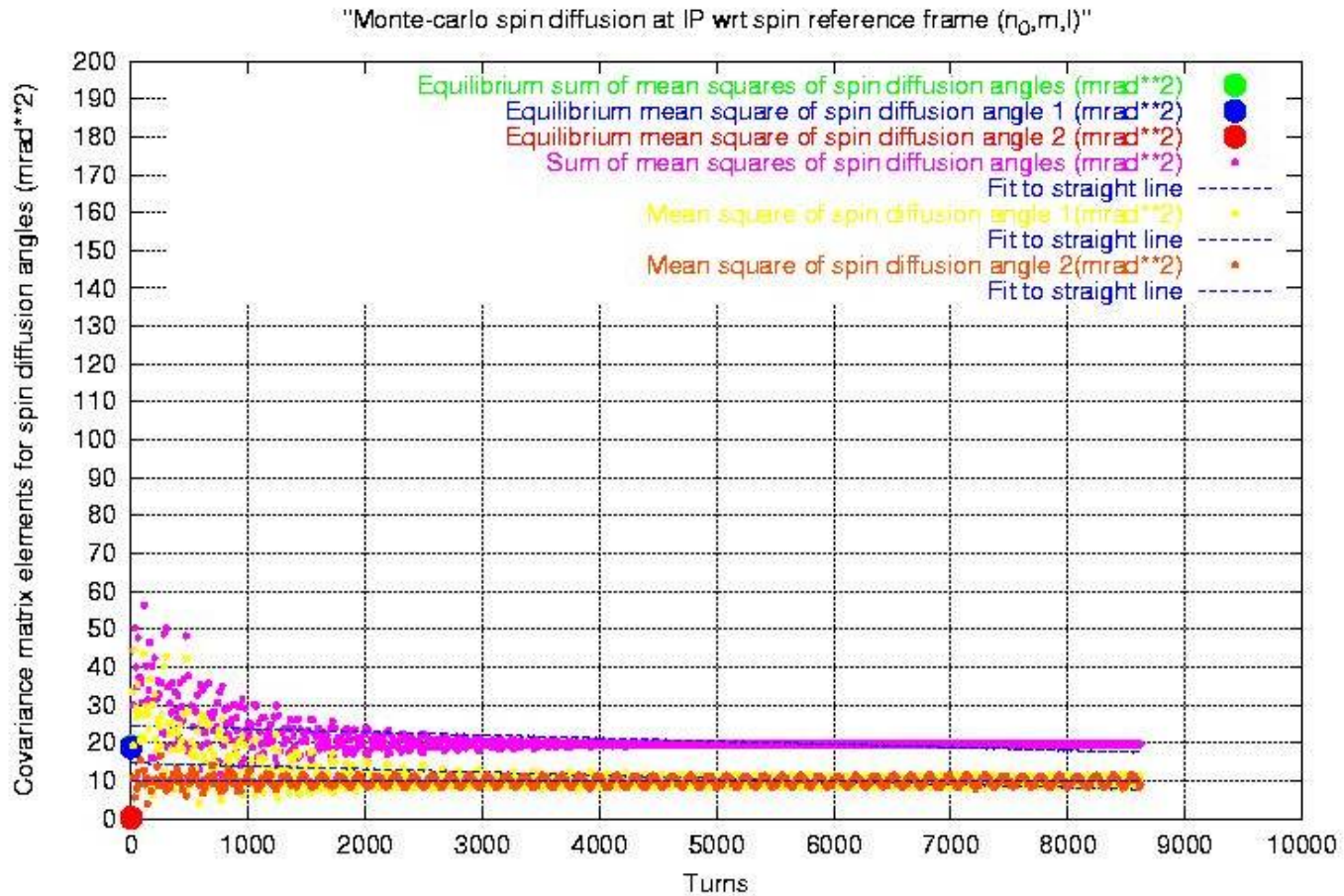
eRHIC: check the M-C calculations against the SLIM results.



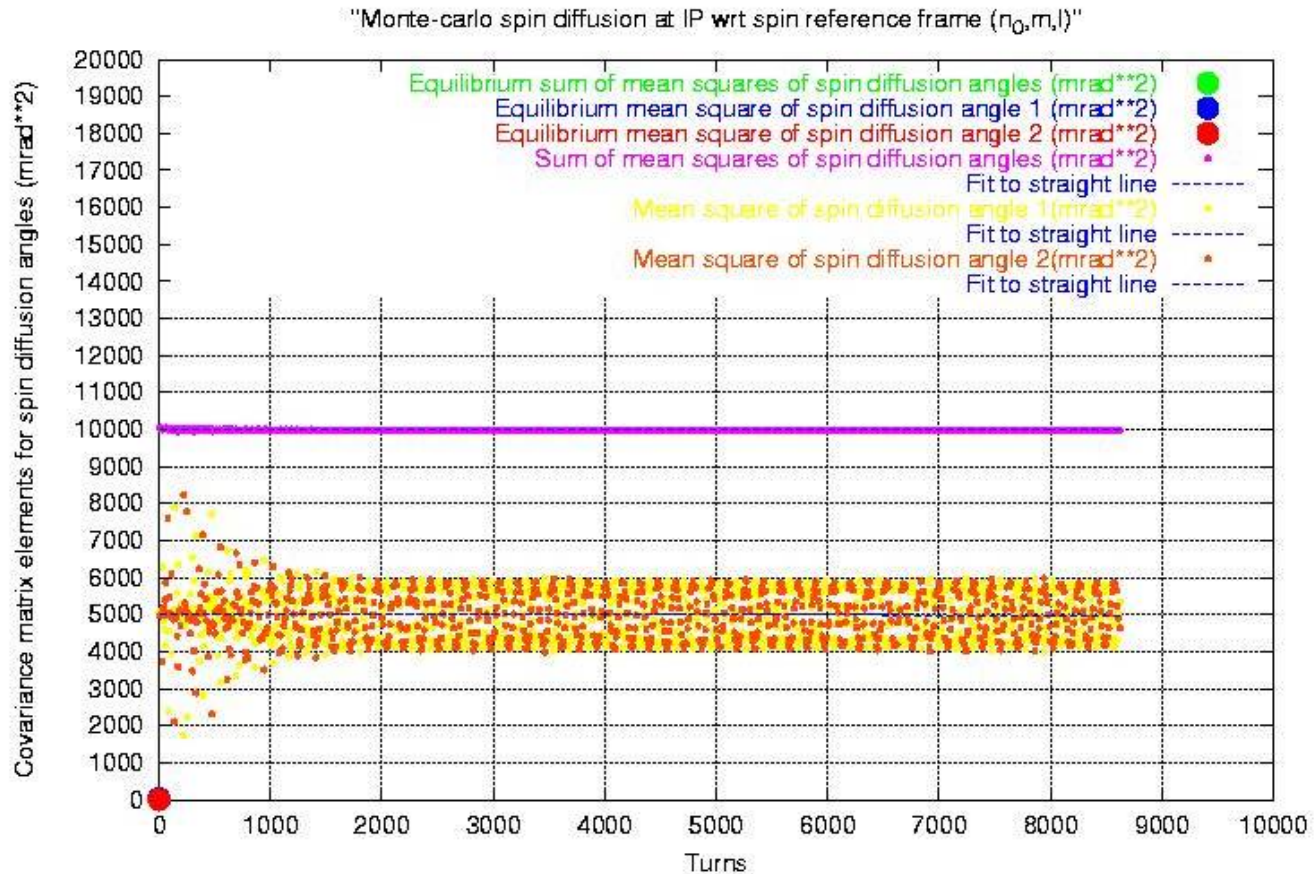
OCS6



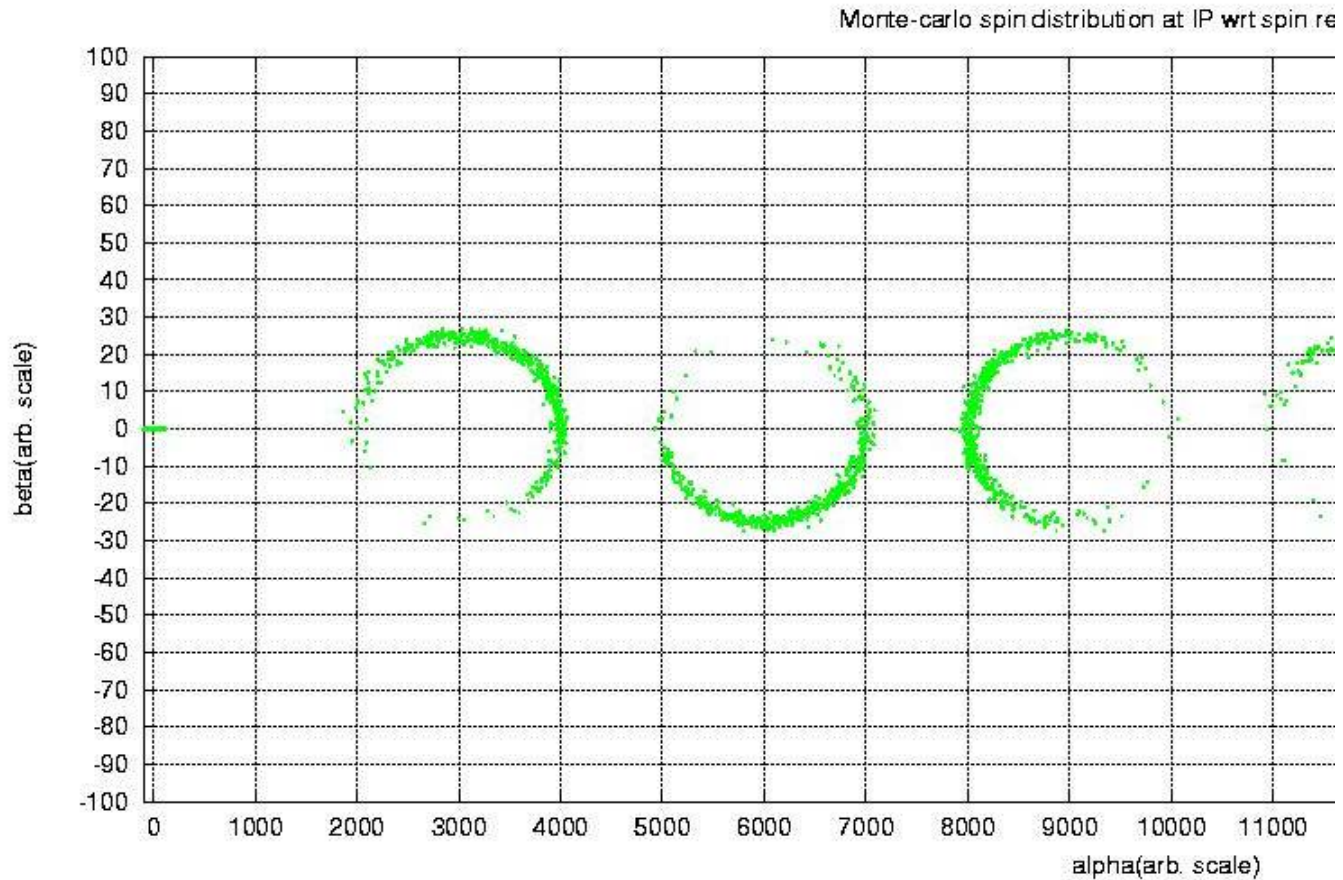
OCS6



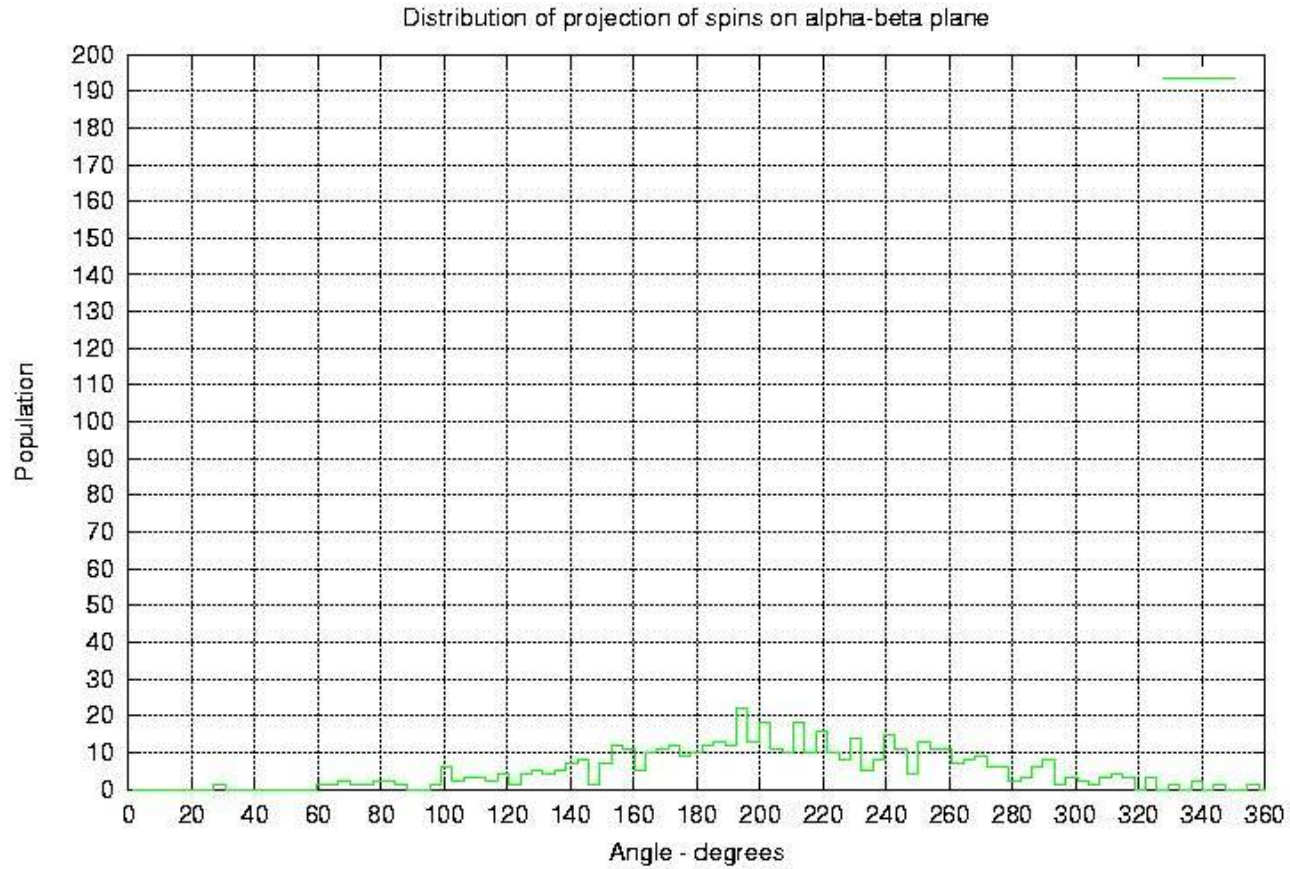
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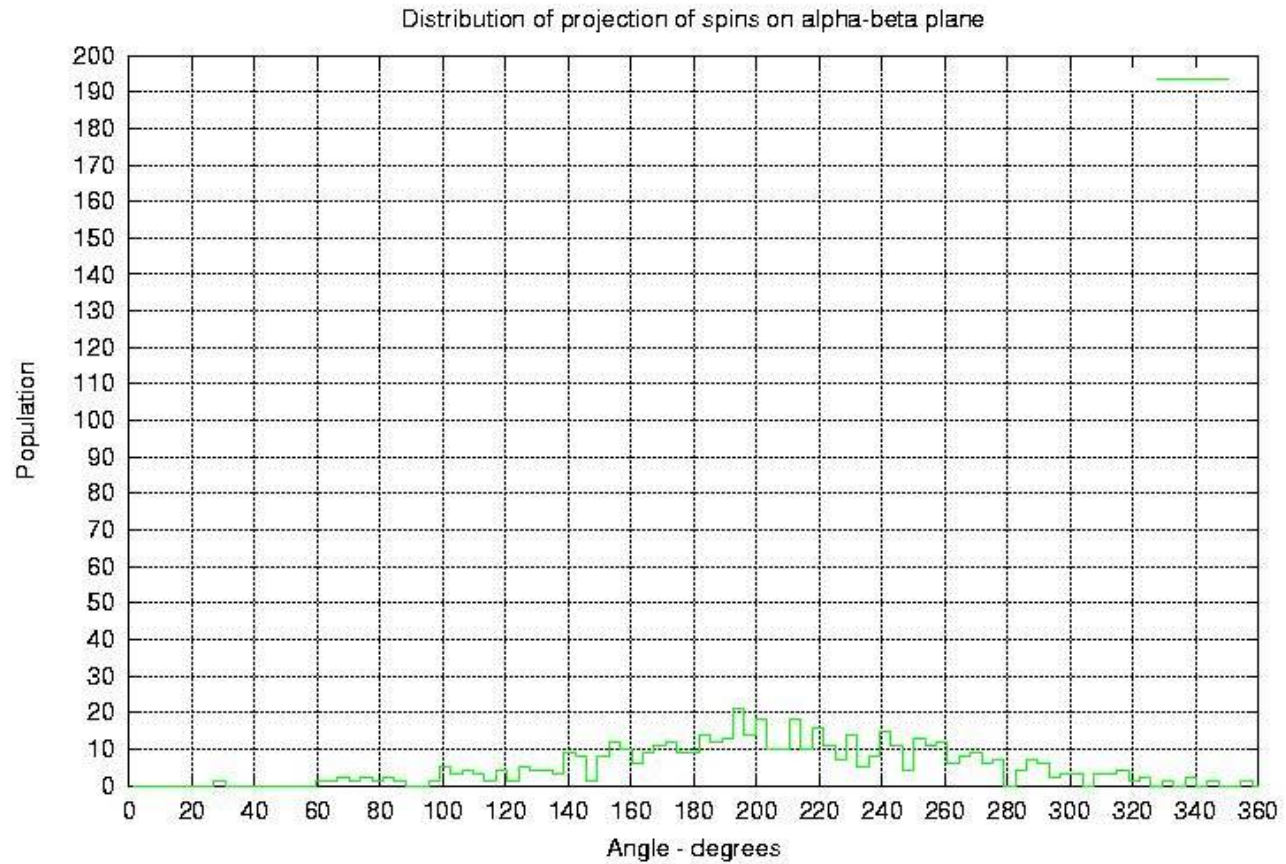
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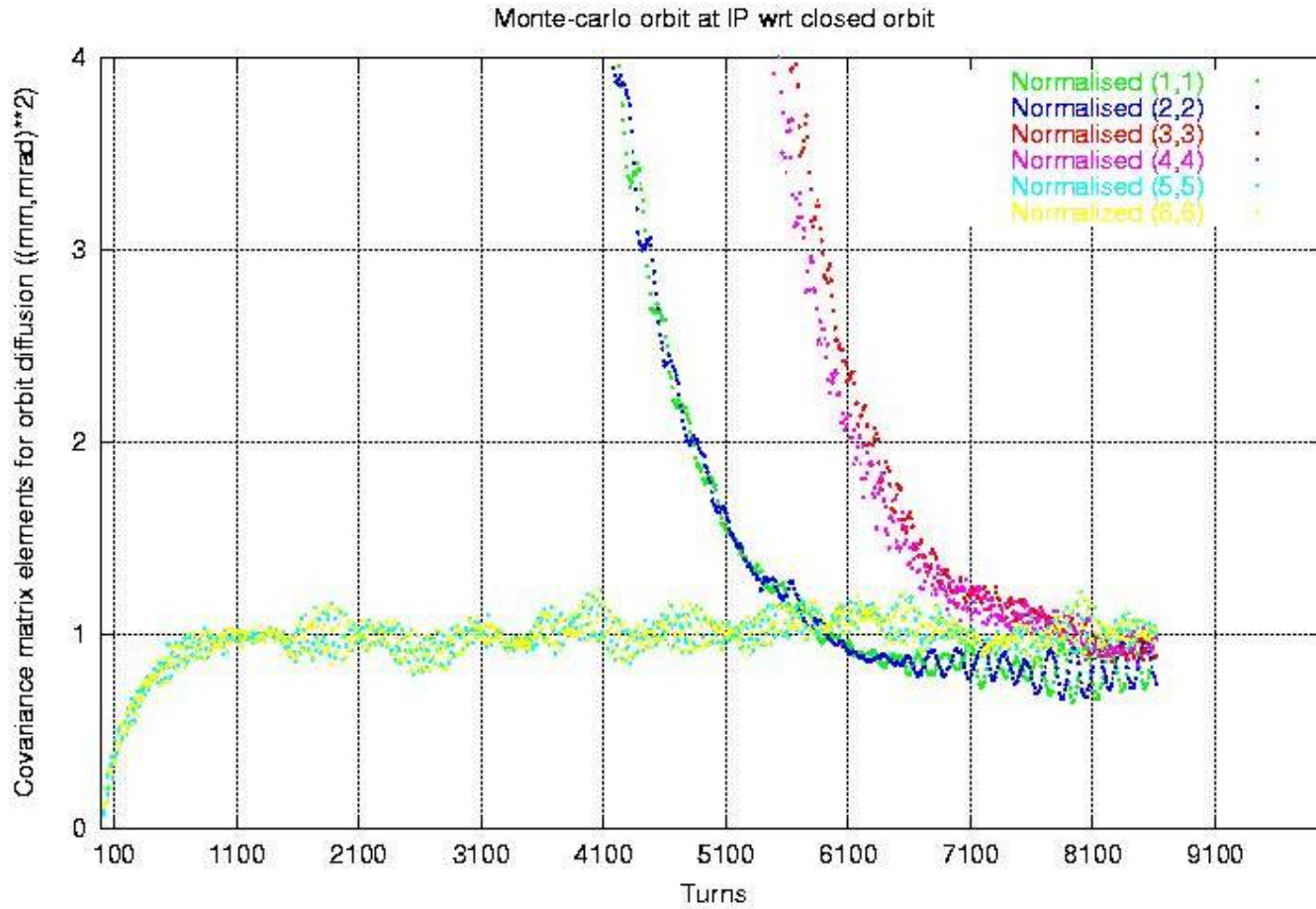
OCS6



OCS6

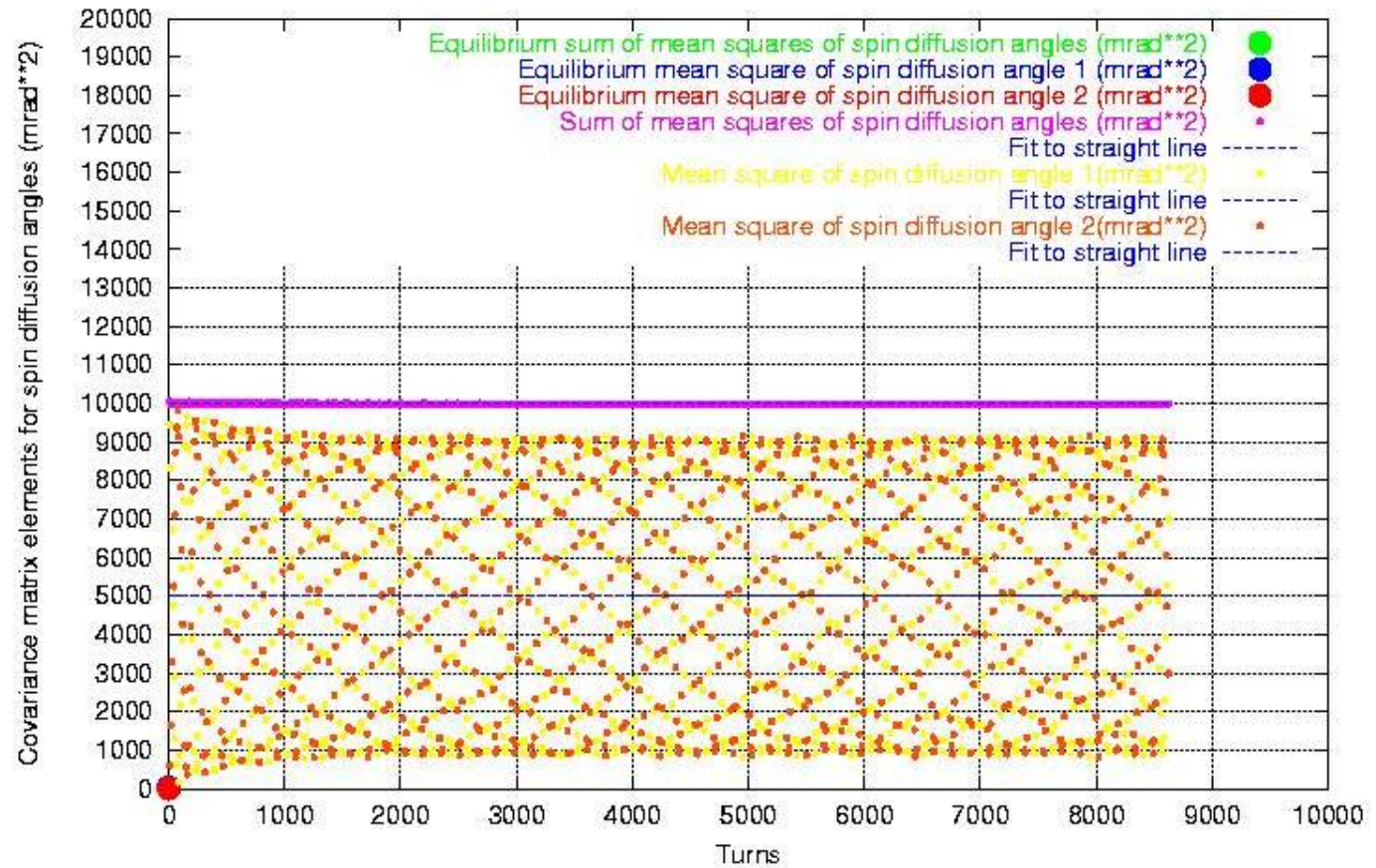


OCS6

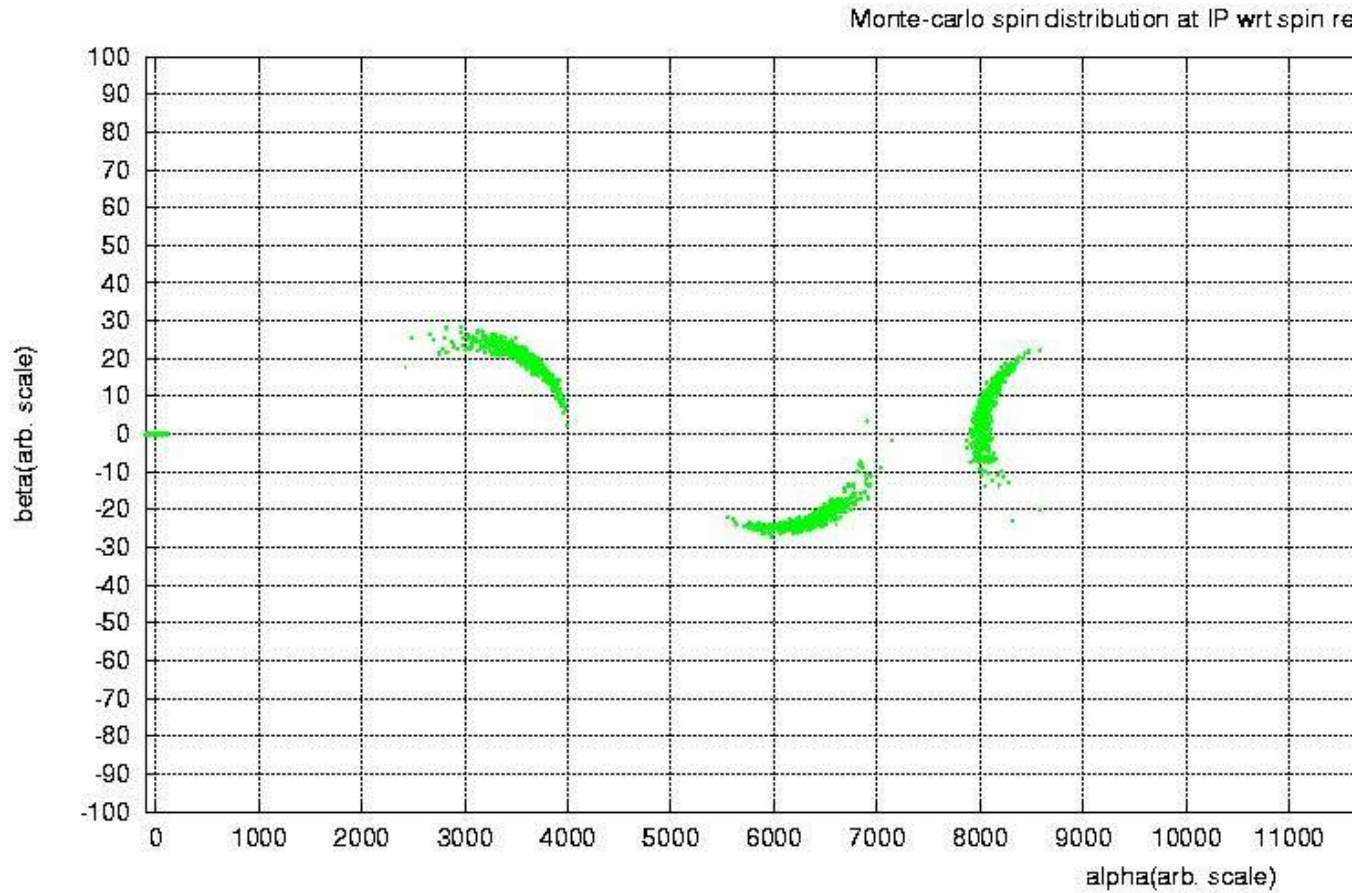


OCS6

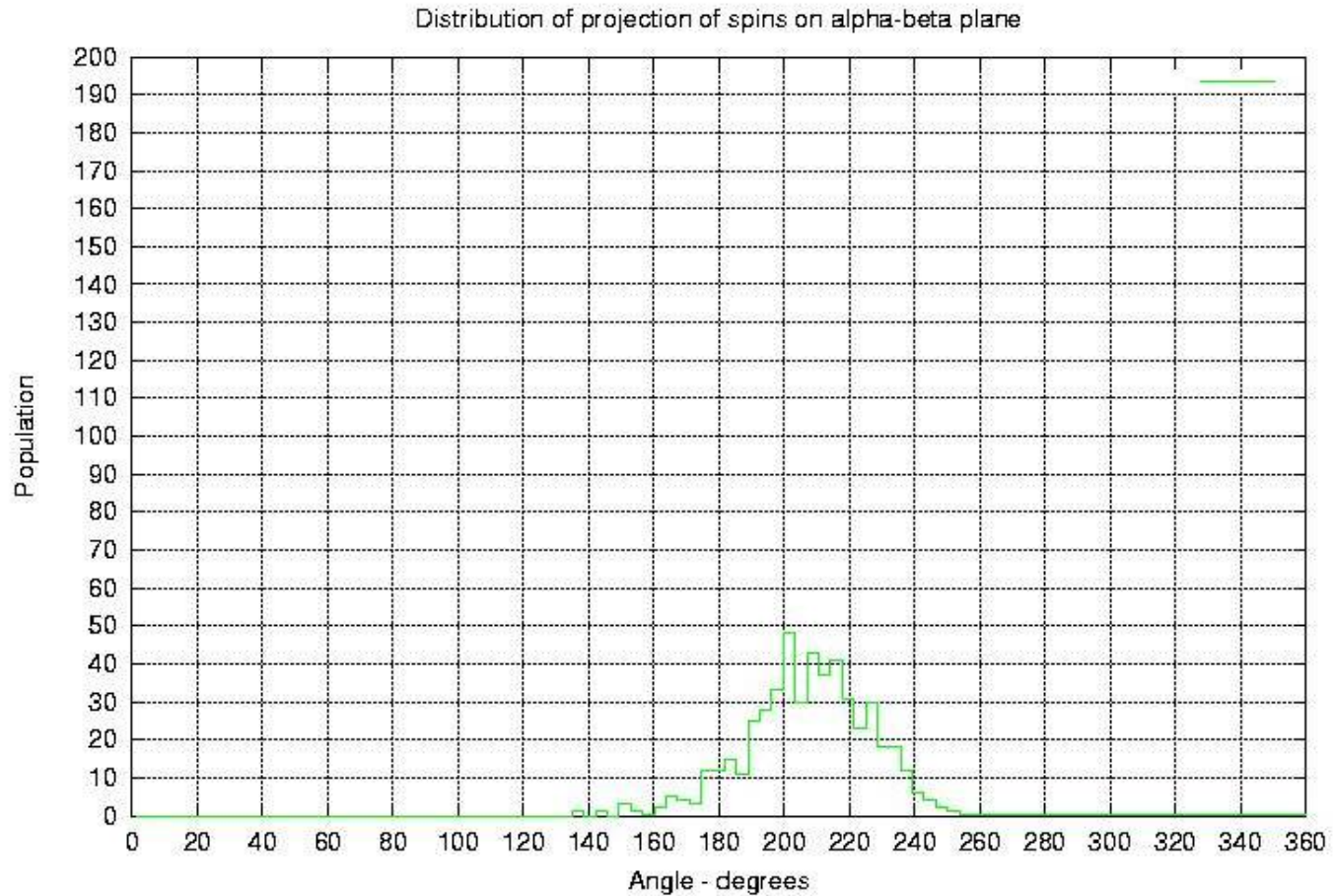
"Monte-carlo spin diffusion at IP wrt spin reference frame (n_0, m, l)"



OCS6



OCS6



This lack of decoherence was predicted already at SPIN94:

http://www.desy.de/~mpybar/psdump/bloom19_update.pdf

http://arxiv.org/PS_cache/physics/pdf/9709/9709025v1.pdf

For vanishing initial bunch length and energy spread, the asymptotic angular spread is:

$$\sigma_{\psi} = \frac{a\gamma\sigma_{\delta}}{Q_s}$$

If the initial longitudinal phase space distribution is the equilibrium distribution

$$\sigma_{\psi} = \frac{\sqrt{2} a\gamma\sigma_{\delta}}{Q_s}$$

The SLICKTRACK simulations agree.

The final angular spread increases strongly with the initial bunch length.

Linac and beam delivery system: Larisa Malysheva

Just apply SLICKTRACK M-C methods to the Linac and BDS.

Put in correct initial covariance matrices.

Ensure that the starting $\hat{n}_0(s)$ leads to longitudinal $\hat{n}_0(s)$ at the IP.

At very high energy, might expect loss of polarisation: $\Delta \mathcal{G}_{\text{spin}} = (1 + a \gamma) \Delta \mathcal{G}_{\text{orb}}$

Only around 0.06% loss of polarisation:

Mainly due to beam divergence – not stochastics.

Final comments I

Simulation of orbit and spin motion is standard and well developed.

It looks as though scattering (noise) can be included by modifying/extending existing software for depolarisation.

$$\Delta \mathcal{G}_{\text{spin}} = (1 + a \gamma) \Delta \mathcal{G}_{\text{orb}}$$

At the low energy of damping rings, it is probable that the depolarisation is negligible even with misalignments and with skew fields.

But without very careful correction, misalignments and skew fields lead to a vertical emittance which is orders of magnitude larger than required.

Final comments II

Dangerous to claim that the horizontal components of spins decohere in the damping rings

Recent work suggests that there is eventual decoherence in storage mode – due to higher order dependence of spin maps on the orbital variable (work with E. Forest).

Need to include filamentation due to spread in tunes.

Spin is now in PTC-FPP: algorithms being developed with E. Forest.

Plans to upgrade SLICKTRACK with PTC-FPP technology with Kai Meng Hock and thereby check the effect of non-linear fields and non-linear motion for the large injected positron beams with their large energy spread.

Other codes: BMAD, Merlin: *Beware of black boxes*: we exposed a bug in BMAD!

Without the S–T terms, the corresponding form for the
Polarisation Density $\vec{\mathcal{P}}$:

$$\frac{\partial \vec{\mathcal{P}}}{\partial s} = \mathcal{L}_{\text{FP,orb}} \vec{\mathcal{P}} + \vec{\Omega}(\vec{u}; s) \times \vec{\mathcal{P}}$$

Barber + Heinemann 1990's

$$\vec{P}(s) = \int d^6u \vec{\mathcal{P}}(\vec{u}; s).$$

This equation:

- can be derived in a classical picture,
- is homogeneous in $\vec{\mathcal{P}}$ i.e. it's “universal”,
- is valid far from spin–orbit equilibrium,
- contains the whole of depolarisation!

Back to the T-BMT !

$$\frac{d \vec{S}}{d t} = \frac{e}{m \gamma} \vec{S} \times \left((1 + a \gamma) \vec{B} - a(\gamma - 1) (\hat{t} \cdot \vec{B}) \hat{t} - \left(a \gamma + \frac{\gamma}{1 + \gamma} \right) \frac{\beta \hat{t} \times \vec{E}}{c} \right)$$

$$\vec{\omega}_O = -\dot{\hat{t}} \times \hat{t}; \quad \dot{\hat{t}} = \frac{e}{m \gamma} \left(\frac{\vec{E} - (\vec{E} \cdot \hat{t}) \hat{t}}{c \beta} + \hat{t} \times \vec{B} \right) \Rightarrow \vec{\omega}_O = \frac{-e}{m \gamma} \left(\vec{B}_\perp + \frac{\vec{E} \times \hat{t}}{c \beta} \right)$$

$$\frac{d \vec{S}}{d t} = \vec{\tilde{\Omega}}_s \times \vec{S} \quad \text{with}$$

$$\vec{\tilde{\Omega}}_s = -\vec{\Omega} = -\frac{e}{m \gamma} \left((1 + a \gamma) \vec{B} - a(\gamma - 1) (\hat{t} \cdot \vec{B}) \hat{t} - \left(a \gamma + \frac{\gamma}{1 + \gamma} \right) \frac{\beta \hat{t} \times \vec{E}}{c} \right)$$

Then w.r.t. the **particle's** orbit we have $\frac{d \vec{S}}{d t} = (\vec{\tilde{\Omega}}_s - \vec{\omega}_O) \times \vec{S}$

$$\text{with } \vec{\tilde{\Omega}}_s - \vec{\omega}_O = -\frac{e}{m \gamma} \left(a \gamma \vec{B} - \left(a \gamma - \frac{g}{2} \right) (\hat{t} \cdot \vec{B}) \hat{t} - \left(a \gamma - \frac{\gamma}{\gamma^2 - 1} \right) \frac{\beta \hat{t} \times \vec{E}}{c} \right)$$

$$\text{or } \vec{\tilde{\Omega}}_s - \vec{\omega}_O = -\frac{e}{m \gamma} \left(a \gamma \vec{B}_\perp + \frac{g}{2} \vec{B}_\parallel - \left(a \gamma - \frac{\gamma}{\gamma^2 - 1} \right) \frac{\beta \hat{t} \times \vec{E}}{c} \right)$$