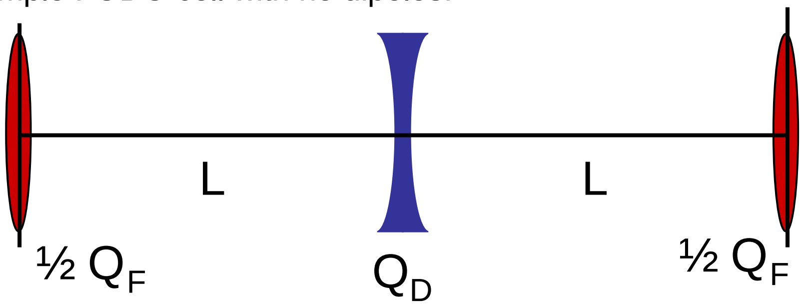
Damping Rings Lecture I Homework Problems with Solutions

1. Consider a simple FODO cell with no dipoles:



(a) Using the expressions on slide 28, write out the thin lens approximation ($\ell \rightarrow 0$) for the transfer matrix of a focusing and a de-focusing quadrupole.

<u>Solution:</u>

The transfer matrix for a focusing quadrupole is given by:

$$\mathbf{M}_{QF} = \left.\begin{pmatrix} \cos\left(\sqrt{k}\ell\right) & \frac{1}{\sqrt{k}}\sin\left(\sqrt{k}\ell\right) \\ -\sqrt{k}\sin\left(\sqrt{k}\ell\right) & \cos\left(\sqrt{k}\ell\right) \end{pmatrix}\right|_{\ell\rightarrow 0} = \begin{pmatrix} 1 & 0 \\ -k\ell & 1 \end{pmatrix}$$

Although the length of the quadrupole is shrinking to zero, it's strength does not. Thus we identify:

$$-k\ell = -\frac{1}{f}$$

and obtain:

$$\mathbf{M}_{QF} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

Applying the same arguments to the defocusing case yields:

$$\mathbf{M}_{QD} = \left.\begin{pmatrix} \cosh\left(\sqrt{|k|}\ell\right) & \frac{1}{\sqrt{|k|}}\sinh\left(\sqrt{|k|}\ell\right) \\ \sqrt{|k|}\sinh\left(\sqrt{|k|}\ell\right) & \cosh\left(\sqrt{|k|}\ell\right) \end{pmatrix}\right|_{\ell\rightarrow 0} \Rightarrow \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

(b) For the case where f_{QF}=-f_{QD}=f, write out the transfer matrix for the above FODO cell.

Solution:

First a special note – there was some confusion about what was meant by f_{QF}=-f_{QD}. In the convention that we used for the transfer matrices in the lectures, f simply indicates the *magnitude* of the focal length. Sticking to this convention, I should have written f_{QF}=f_{QD}.

The FODO cell as drawn begins halfway through the first focusing quadrupole and ends halfway through the second focusing quadrupole. Thus we can write the overall transfer matrix as:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} 1-\frac{L^2}{2f^2} & L\left(2+\frac{L}{f}\right) \\ -\frac{L}{2f^2}\left(1-\frac{L}{2f}\right) & 1-\frac{L^2}{2f^2} \end{pmatrix}$$

(c) Recalling the form of the transfer matrix given on slide 31, identify the elements of your transfer matrix. What is the condition for stability in terms of f and L?

Solution:

Looking at the general solution for the transfer matrix, we want to match the parameters such that we can determine a stability condition. We identify

$$\cos\Phi = 1 - \frac{L^2}{2f^2}$$

where the symmetry of the diagonal elements immediately shows that $\alpha(0) = \alpha(2L) = 0$ which applies at points in the middle of the focusing quadrupoles. We see that stable solutions can be obtained only for:

$$-1 \le \cos\Phi \le 1 \quad \Rightarrow \quad \frac{L}{2f} \le 1$$

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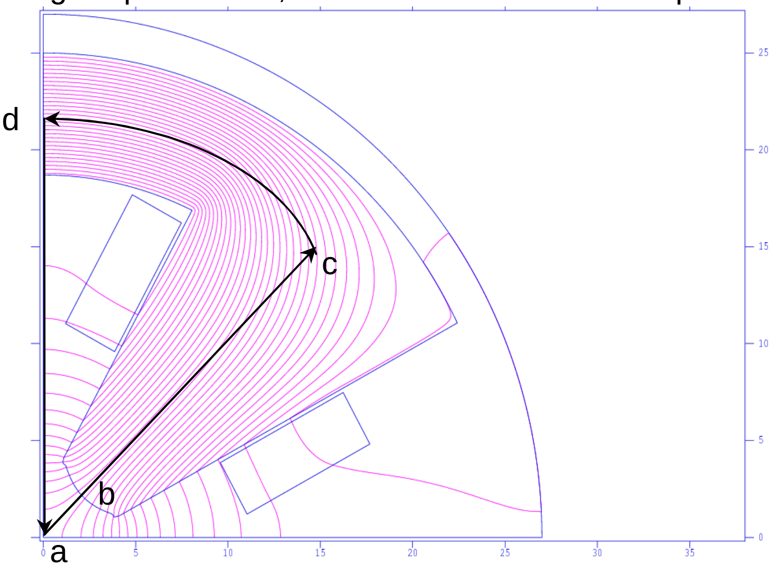
2. Several quadrupole types were required for the ILC DR Reference Design Report. The high field quadrupole required kL=0.31 m^{-1} for p_0=5GeV/c. The length of the quad is 0.3 m and the pole tip radius is 30 mm. Using Ampere's Law, estimate the number of amp-turns required to power this magnet.



Ampere's law can be written as:

$$NI_{enclosed} = \oint \mathbf{H \bullet dl} = \frac{1}{\mu_0}\int_{air} \mathbf{B \bullet dl} + \frac{1}{\mu}\int_{iron} \mathbf{B \bullet dl}$$

The last integral in iron can be ignored because the permeability of iron is quite large. Similarly, the portion of the integral between points d and a will be zero due to symmetry (ie, the magnetic field lines are perpendicular to the path of integration). Thus we only need to evaluate the integral between points a and b.

In order to evaluate the integral, we need to solve for the radial magnetic field from a to b. The x and y components of the field are: $\frac{e}{p_0}B_x = ky$ $\qquad \frac{e}{p_0}B_y = kx$

which gives: $$B_r = \frac{p_0}{e}k\sqrt{x^2+y^2} = \frac{p_0}{e}kr$$

and $$NI_{enclosed} = \frac{p_0}{2e}ka^2$$

where a is the pole tip radius. Using the conversion Bρ[Tm]=E[GeV]/0.3, the fact that k~1m^{-2}, a=0.03m and E=5GeV, lets us conclude that:

$$NI_{enclosed} = \left(\frac{5}{0.3}Tm\right)\left(\frac{1m^{-2}}{4\pi\times 10^{-7}\ Tm/Amp-Turn}\right)\left(\frac{(0.03m)^2}{2}\right) = 6000\,Amp-Turns$$

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3. A key contribution of damping wigglers in the damping ring is to lower the damping time of the beam. For a ring having total wiggler length of L_w and wigglers with peak field B_w, show the dependence of the damping time on the wiggler parameters. You may assume the wiggler field has a simple sinusoidal form in s. In other words: $B=B_w\sin(k_w s)$ where B_w is the peak wiggler field and the period of the wiggler is given by $2\pi/k_s$. Hint: It is often easiest to scale quantities with respect to the beam rigidity…

Solution:

Recall that $\alpha_i = \dfrac{U_0}{2E_0T_0}$ and $U_0 = \dfrac{C_\gamma E^4}{2\pi} I_2 = \dfrac{C_\gamma E^4}{2\pi}\oint \dfrac{1}{\rho^2}ds$

For a wiggler-dominated ring, we can write

$$I_2 \approx I_{2,wig} = \frac{1}{(B\rho)^2}\int_0^{L_w} B_w^2 \sin^2(k_w s)\,ds = \frac{B_w^2 L_w}{(B\rho)^2}\left\langle \sin^2\theta \right\rangle = \frac{B_w^2 L_w}{2(B\rho)^2} \propto E_0^2 B_w^2 L_w$$

Thus we can immediately write: $\tau_i \propto \dfrac{T_0}{E_0 B_w^2 L_w}$

Note that there is no dependence on the wiggler period.

4. In addition to shortening the damping time, the wigglers also act to increase the energy spread of the beam.

(a) In the same level of approximation as in the preceding problem, now calculate the contribution of the wiggler to the energy spread of the beam.

Solution:

The energy spread is given by: $\sigma_\delta^2 = C_q \gamma^2 \dfrac{I_3}{I_2}$ where $I_3 = \oint \dfrac{1}{\left|\rho^3\right|}ds$

We have evaluated I_2 in the previous problem so now we need to evaluate I_3.

$$I_{3,wig} = \frac{1}{(B\rho)^3}\int_0^{L_w} B_w^3 \left|\sin^3(k_w s)\right| ds = \frac{B_w^3 L_w}{(B\rho)^3}\left\langle \left|\sin^3\theta\right| \right\rangle = \frac{4B_w^3 L_w}{3\pi(B\rho)^3}$$

Thus $$\sigma_\delta^2 = \frac{4C_q\gamma^2 B_w}{3\pi B\rho} \quad \Rightarrow \quad \sigma_\delta \propto \sqrt{E_0 B_w}$$

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4. (b) Discuss the trade-off between damping time and energy spread in determining the energy of the damping ring. Recall that the ILC DR design employs approximately 200 m of wigglers with a peak field of 1.6 T and that the energy spread that can be tolerated by the downstream systems is <0.15%.

<u>Discussion:</u> The relations from problems 3 and 4a show that lowering the damping time by increasing the peak wiggler field has the necessary consequence of increasing the energy spread of the ring. If we consider only varying the wiggler parameters, we can lower the peak wiggler field while increasing the total length of wiggler in order to maintain a given damping time while simultaneously lowering the energy spread of the ring. If we look at the scaling of the energy spread to the damping time, we see that

$$\frac{\sigma_\delta}{\tau} \propto E^{3/2}$$

which generally places an upper limit on the energy of the damping ring. At the same time, a lower limit on the energy is present because, assuming that T_0 is constrained due to the need to fill the ring with a minimum number of bunches, $B_w^2 L_w$ will quickly become too large to be practical (either due to the peak magnetic field required or the number of individual wigglers required). This is one factor in pushing the damping rings that have been proposed towards operating energies in the few to several GeV range.