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Damping Rings I
Part 1: Introduction and DR Basics
Part 2: Low Emittance Ring Design

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Lecture Overview

Damping Rings Lecture I

- Part 1: Introduction and DR Basics
  - Overview
  - Damping Rings Introduction
  - General Linear Beam Dynamics
- Part 2: Low Emittance Ring Design
  - Radiation Damping and Equilibrium Emittance
  - ILC Damping Ring Lattice

Damping Rings Lecture II

- Part 1: Technical Systems
  - Systems Overview and Review of Selected Systems
  - R&D Challenges
- Part 2: Beam Dynamics Issues
  - Overview of Impedance and Instability Issues
  - Review of Selected Collective Effects
  - R&D Challenges

Damping Rings Lecture I

Our objectives for today's lecture are to:

Examine the role of the damping rings in the ILC accelerator complex;

Review the parameters of the ILC damping rings and identify *key challenges* in the design and construction of these machines;

Review the physics of storage rings including the linear beam dynamics and radiation damping;

Apply the above principles to the case of the ILC damping rings to begin to understand the major *design choices* that have been made

Outline of DR Lecture I, Part 1

Damping Rings Introduction

- Role of Damping Rings
- ILC Damping Ring Parameters
- Damping Rings Overview

General Linear Beam Dynamics

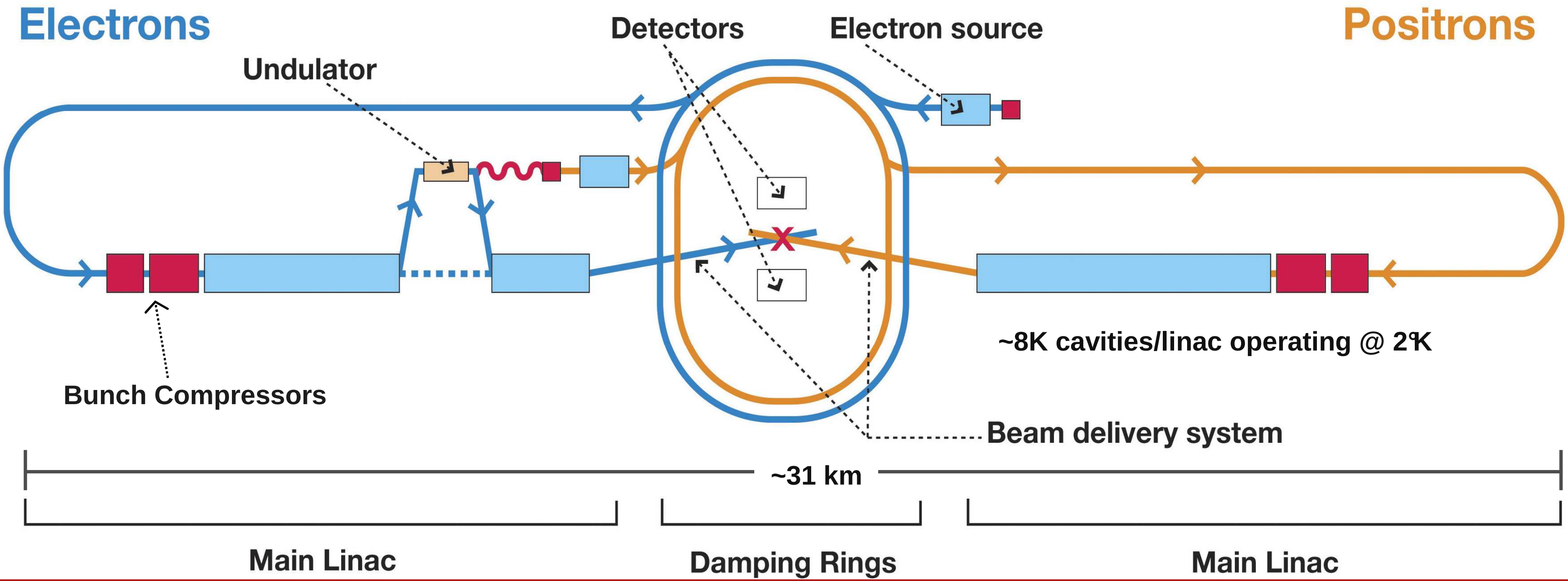
- Storage Ring Equations of Motion
- Betatron Motion
- Twiss Parameters
- Emittance
- Coupling
- Dispersion
- Chromaticity

The ILC Reference Design

Machine Configuration

- Helical Undulator polarized e+ source
- Two ~6.5 km damping rings in a central complex
- RTML running length of linac
- 2 ×11.2 km Main Linac
- Single Beam Delivery System
- 2 Detectors in Push-Pull Configuration

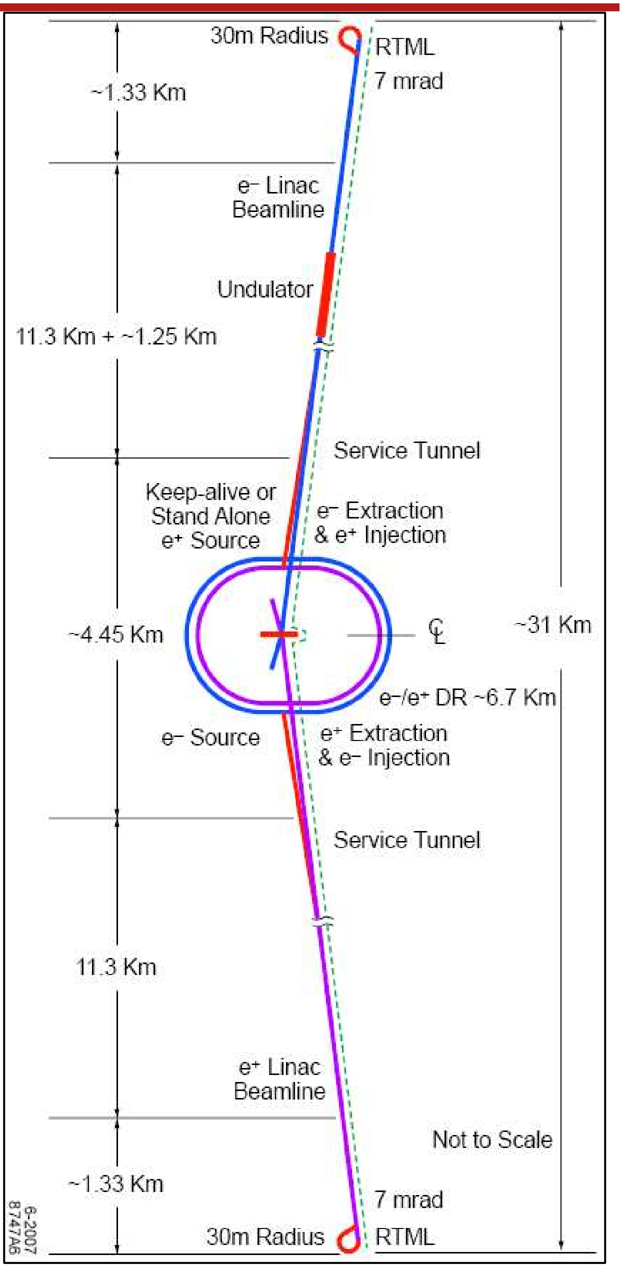
Parameter	Unit	
Center-of-mass energy range	GeV	200 - 500
Peak luminosity[a]	$cm^{-2}s^{-1}$	2×10^{34}
Average beam current in pulse	mA	9.0
Pulse rate	Hz	5.0
Pulse length (beam)	ms	~ 1
Number of bunches per pulse		1000 - 5400
Charge per bunch	nC	1.6 - 3.2
Accelerating gradient[a]	MV/m	31.5
RF pulse length	ms	1.6
Beam power (per beam)[a]	MW	10.8
Typical beam size at IP[a] $(h \times v)$	nm	640×5.7
Total AC Power consumption[a]	MW	230



Role of the Damping Rings

The damping rings

- Accept e^+ and e^- beams with large transverse and longitudinal emittance and produce the ultra-low emittance beams necessary for high luminosity collisions at the IP
- Damp longitudinal and transverse jitter in the incoming beams to provide very stable beams for delivery to the IP
- Delay bunches from the source to allow feed-forward systems to compensate for pulse-to-pulse variations



DR Reference Design Parameters

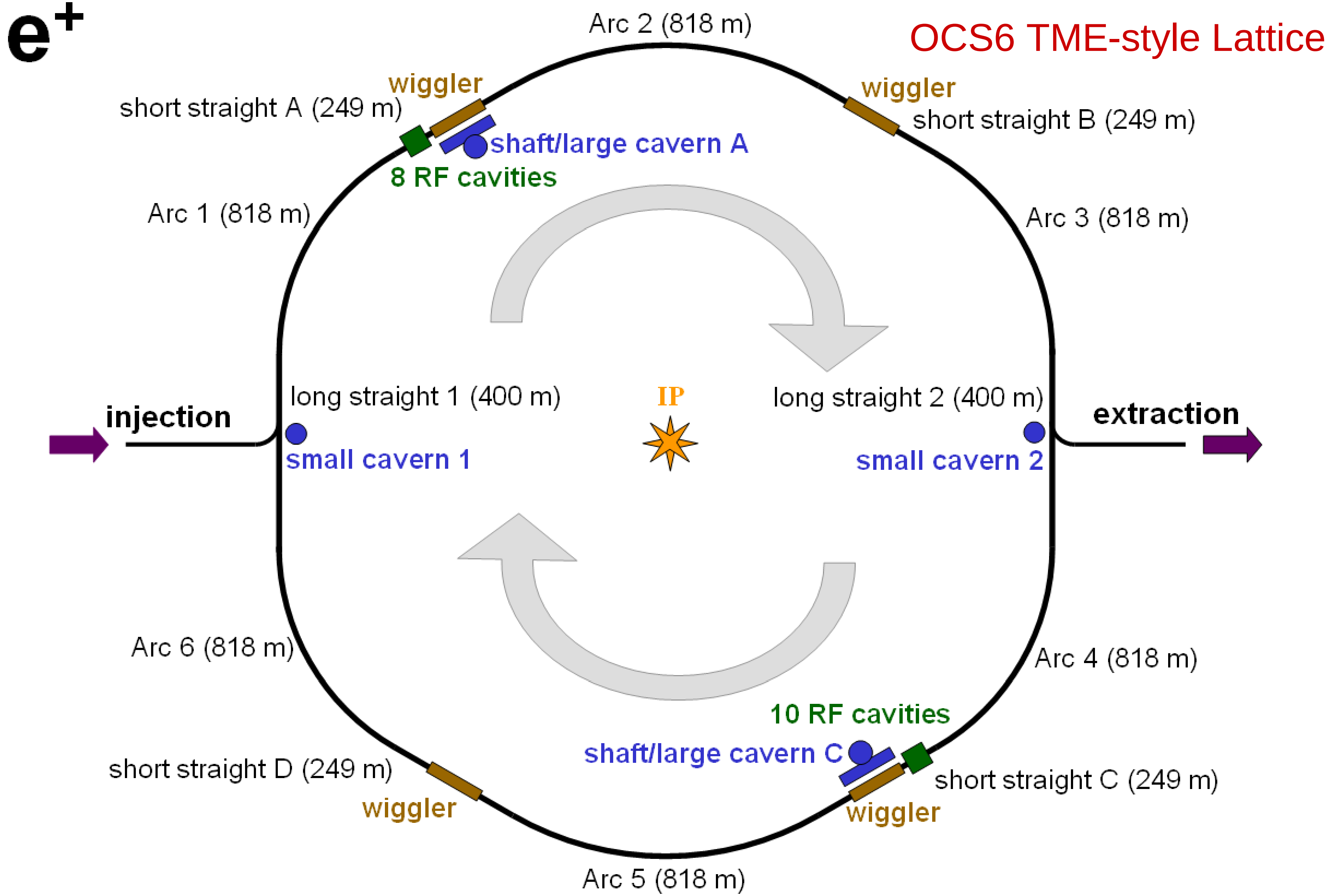
By the end of this lecture, the goal is for each of you to be able to explain the reasons that the parameters in this table have the values that are specified.

By the end of the second lecture tomorrow, you should be able to identify and explain why several of these parameters are candidates for further optimization.

So, let's begin our tour of ring dynamics and what these parameters mean…

Parameter	Units	Value
Energy	GeV	5.0
Circumference	km	6.695
Nominal # of bunches & particles/bunch		$2625@2.0\times10^{10}$
Maximum # of bunches & particles/bunch		$5534@1.0\times10^{10}$
Average current	A	0.4
Energy loss per turn	MeV	8.7
Beam power	MW	3.5
Nominal bunch current	mA	0.14
RF Frequency	MHz	650
Total RF voltage	MV	24
RF bucket height	%	1.5
Injected betatron amplitude, $A_x + A_y$	m·rad	0.09
Equilibrium normalized emittance, $\gamma\varepsilon_x$	μm·rad	5.0
Chromaticity, χ_x/χ_y		-63/-62
Partition numbers, J_x J_y J_z		0.9998 1.0000 2.0002
Harmonic number, h		14,516
Synchrotron tune, ν_s		0.067
Synchrotron frequency, f_s	kHz	3.0
Momentum compaction, α_c		4.2×10^{-4}
Horizontal/vertical betatron tunes, ν_x/ν_y		52.40/49.31
Bunch length, σ_z	mm	9.0
Momentum spread, σ_p/p		1.28×10^{-3}
Horizontal damping time, τ_x	ms	25.7
Longitudinal damping time, τ_z	ms	12.9

The RDR Damping Ring Layout



Damping Ring Design Inputs

A number of parameters in the previous table are (*essentially)* design *inputs* for the damping rings (or can be directly inferred from such inputs). The table below summarizes these critical interface issues.

We will examine these requirements from the perspective of the collision point first and then look at requirements coming from other sub-systems downstream and upstream of the DRs.

Particles per bunch	1×10^{10} - 2×10^{10}	Upper limit set by disruption at IP.
Max. Avg. current in main linac	~9 mA	Upper limit set by RF technology.
Machine repetition rate	5 Hz	Set by cryogenic cooling capacity. Partially determines required damping time.
Max. Linac RF pulse length	~1 ms	Upper limit set by RF technology.
Min. Particles per machine pulse	~5.6×10^{13}	Lower limit set by luminosity goal.
Injected normalized emittance	0.01 m-rad	Set by positron source. Partially determines required damping time.
Injected energy spread	±0.5%	Set by positron source.
Injected betatron amplitude (A_x+A_y)	0.09 m-rad	Set by positron source.
Extracted normalized emittances	8 μm horizontally 20 nm vertically	Set by luminosity goal.
Max. Extracted bunch length	9 mm (⇨6 mm)	Upper limit set by bunch compressors.
Max. Extracted energy spread	0.15%	Upper limit set by bunch compressors.

Don't forget, however, that these parameters are the result of a great deal of back-and-forth negotiation between sub-systems and between accelerator and HEP physicists. Thus they represent a mix of technological limits and physics desires…

Downstream Requirements

The principle parameter driver is the production of luminosity at the collision point

$$\mathcal{L} = \frac{N^2 f_{coll}}{4\pi\sigma_x\sigma_y}\mathcal{H}_D$$

where

N is the number of particles per bunch (*assumed equal for all bunches*)

f_{coll} is the overall collision rate at the interaction point (IP)

σ_x and σ_y are the horizontal and vertical beam sizes (*assumed equal for all bunches*)

$\mathcal{H}_D$ is the luminosity enhancement factor

Ideally we want:

- High intensity bunches
- High repetition rate
- Small transverse beam sizes

Parameters at the Interaction Point

The parameters at the interaction point have been chosen to provide a nominal luminosity of 2×10^{34} $cm^{-2}s^{-1}$. With

$N = 2\times10^{10}$ particles/bunch

$\sigma_x \sim 640$ nm $\Leftrightarrow \beta_x^* = 20$ mm, $\varepsilon_x = 20$ pm-rad

$\sigma_y \sim 5.7$ nm $\Leftrightarrow \beta_y^* = 0.4$ mm, $\varepsilon_y = 0.08$ pm-rad

$\mathcal{H}_D \sim 1.7$

$$\mathcal{L} = \frac{N^2 f_{coll}}{4\pi\sigma_x\sigma_y}\mathcal{H}_D = \left(1.4\times10^{30}\,cm^{-2}\right)\times f_{coll}$$

In order to achieve the desired luminosity, an average collision rate of ~14kHz is required (we will return to this parameter shortly). The beam sizes at the IP are determined by the strength of the final focus magnets and the emittance, phase space volume, of the incoming bunches.

A number of issues impact the choice of the final focus parameters. For example, the beam-beam interaction as two bunches pass through each other can enhance the luminosity, however, it also disrupts the bunches. If the beams are too badly disrupted, safely transporting them out of the detector to the beam dumps becomes quite difficult. Another effect is that of beamstrahlung which leads to significant energy losses by the particles in the bunches and can lead to unacceptable detector backgrounds. Thus the above parameter choices represent a complicated optimization.

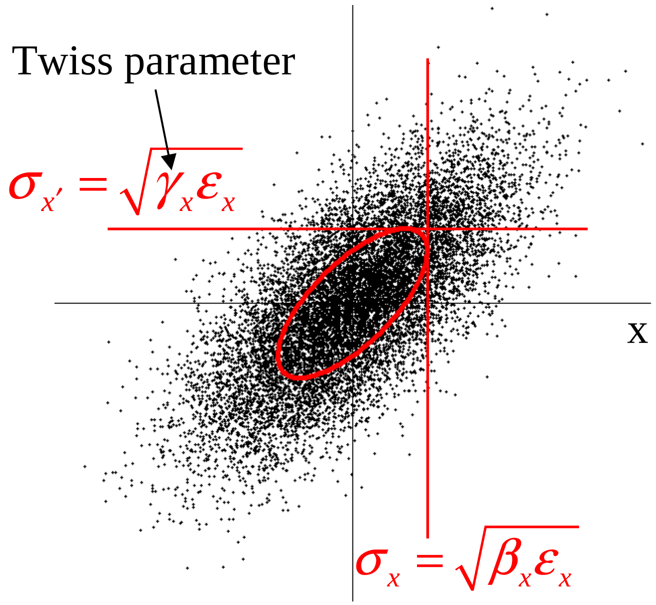
Emittance Transport from the DR to the IP

The geometric emittances required at the IP are:

$\varepsilon_x = 20$ pm-rad

$\varepsilon_y = 0.08$ pm-rad

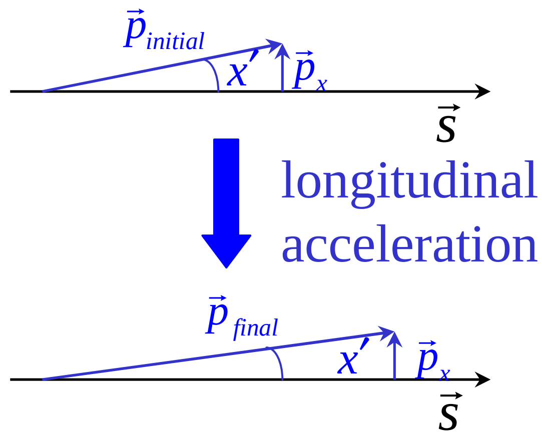
We need to use the relativistic invariant quantity, the *normalized emittance*, in order to project this to the requirements for the damping ring.



Note: We will take a more detailed look at emittance in the DR later in this lecture

Normalized Emittance:

Use of the conjugate phase-space coordinates (x,p_x) from the Hamiltonian instead of (x,x') gives:



$$p_x = px' = mc\beta\gamma x'$$

Thus we define the normalized emittance as

$\varepsilon_n = \beta\gamma\varepsilon_{geo} \approx \gamma\varepsilon_{geo}$ for a relativistic electron

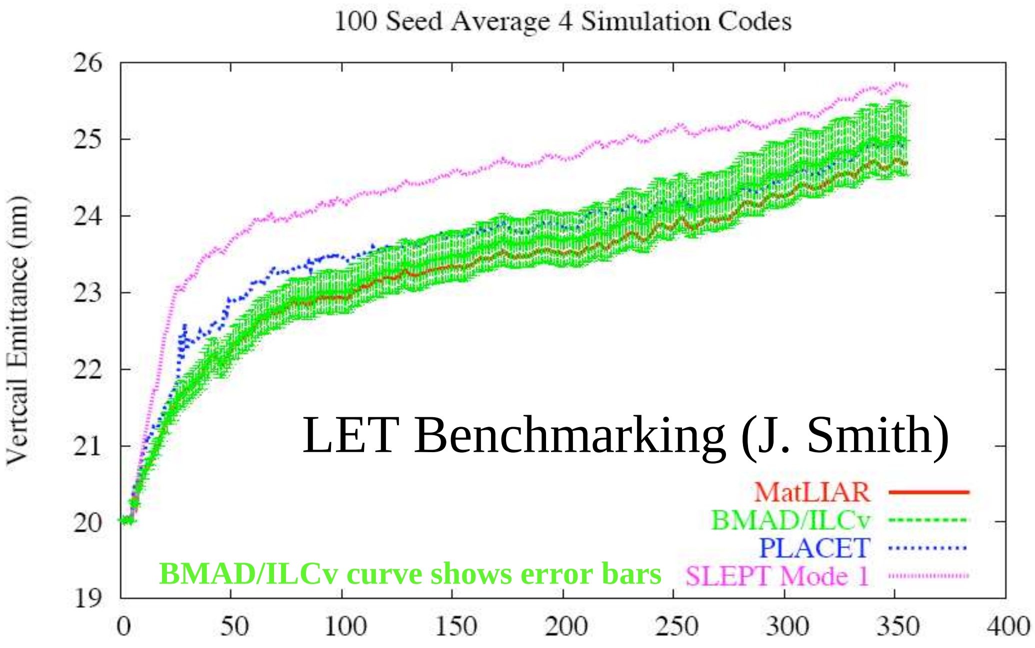
Emittance Transport from the DR to the IP

We can now infer the requirements for the equilibrium emittance requirements for the ILC DRs

	ε_{geo} @ IP (250 GeV)	ε_n @ IP	Equilibrium ε_n @ DR	Equilibrium ε_{geo} @ DR (5 GeV)
x	20 pm-rad	10 μm-rad	½ × (10 μm-rad)	0.5 nm
y	0.08 pm-rad	40 nm-rad	½ × (40 nm-rad)	2 pm

Allow for 100% vertical emittance growth downstream of DRs

DR extracted emittances must allow for downstream emittance growth during transport as well as for the finite damping time during the machine pulse cycle



Main Linac (ML) Parameters

The bunch-train structure is largely determined by the design of the superconducting RF system of the main linac (ML)

	Primary Limitation
– 1 ms RF pulse – 9 mA average current in each pulse	RF power system
– 5 Hz repetition rate	Cryogenic load

This leads to the nominal bunch train parameters:

n_b = 2625 bunches per pulse

Δt_b ~ 380 ns for uniform loading through pulse

The resulting collision rate at the IP is then

f_{coll} = 13.1 kHz

consistent with the target luminosity. The 5 Hz repetition rate places the primary constraint on the DR damping times. In order for the bunches in each pulse to experience 8 full damping cycles, a transverse damping time of ≤25 ms is required.

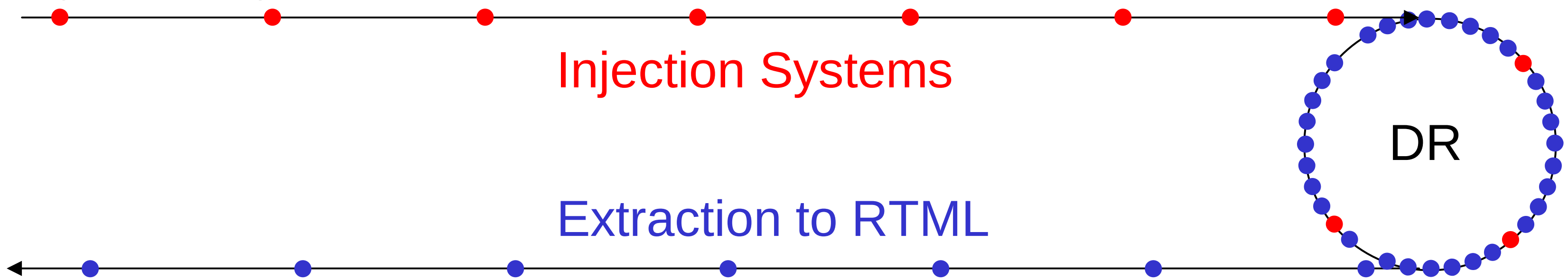
Baseline Bunch Train

From the discussion on the preceding page, we can now see the basic bunch train structure

- 1 msec pulse
- ~3000 uniformly spaced bunches
- ~350 ns between bunches

⇨ Train Length of $\sim$300km $\gg$ ML length > DR Circumference

Thus, the damping rings must act as a *reservoir* to store the full train. Because we cannot afford to build a 300+ km ring, we must *fold* the long bunch train into a much shorter ring ⇨ key trade-offs between bunch spacing and ring circumference.



Note that there will be significant overlap between the injection and extraction cycles:

– Structure of machine
– Maintain relatively constant beam loading

Bunch Compressors

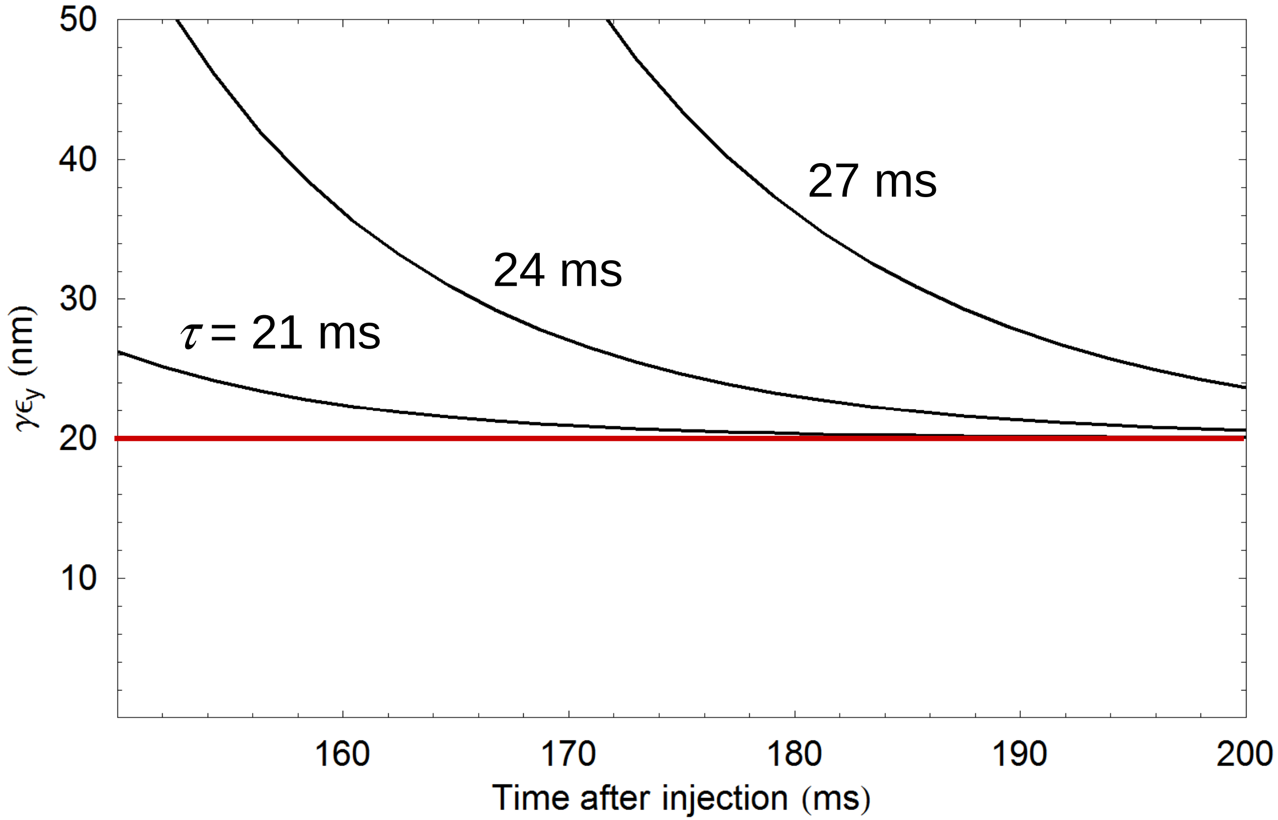
Shortly after extraction from the damping ring, the bunches will traverse the bunch compressors. These devices take the relatively long bunches of the damping rings ($\sigma_z \sim$ fraction of a centimeter) and manipulate the longitudinal phase space to provide bunches that are compatible with the very small focal point at the IP ($\sigma_z \sim$ 200-500 microns). Technical and cost limitations place serious constraints on how long the bunch from the DR can be and the maximum energy spread.

RDR DR Bunch length: 9 mm ⇨ 2-stage bunch compressor
Extracted energy spread within the bunch compressor acceptance

From the downstream point of view, lowering the bunch length to 6mm would allow the cheaper and simpler solution of using a single stage bunch compressor. From the DR point of view, shorter bunches require smaller values of the ring momentum compaction (impacts sensitivity to collective effects) or higher RF voltage (more RF units, hence greater cost).

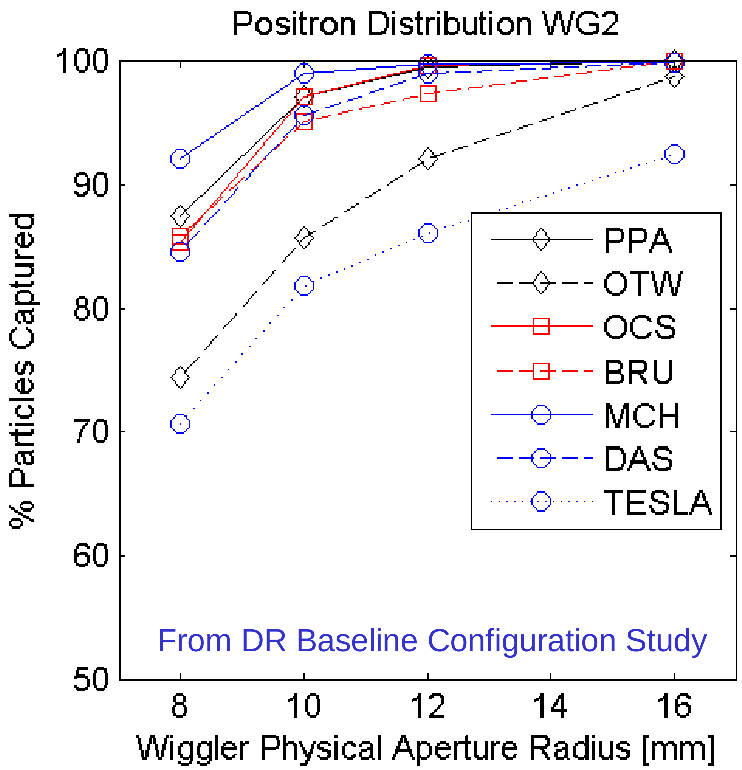
Upstream Requirements

The key upstream requirement is the emittance of the beams produced by the injectors. Positron production via a heavy metal target results in much larger emittances due to scattering in the target for positrons than for electrons whose emittance can be controlled by the design of the injector gun and its cathode. The approach to the target extraction emittance is shown for various DR damping times assuming the target e^+ injected emittance ($\varepsilon_n = 0.01$ m-rad).



Upstream Requirements

In addition to the need to damp the large emittance beams that are injected from the positron source, the injected beams are expected to have potentially large betatron amplitudes and energy errors. This requires that the acceptance of the damping ring to be sufficiently large to accommodate these oscillations immediately after injection. It places important constraints on the minimum aperture of the vacuum system and the minimum good field regions of all of the magnets (including the damping wigglers).



Particle capture rates assuming that the limiting physical aperture in the damping rings is due to the vacuum chambers in the wiggler regions. The choice of a superferric wiggler design, with large physical aperture, allows for a DR design with full acceptance.

Storage Ring Basics

Now we will begin our review of storage ring basics. In particular, we will cover:

- Ring Equations of Motion
- Betatron Motion
- Emittance
- Transverse Coupling
- Dispersion and Chromaticity
- Momentum Compaction Factor
- Radiation Damping and Equilibrium Beam Properties

Equations of Motion

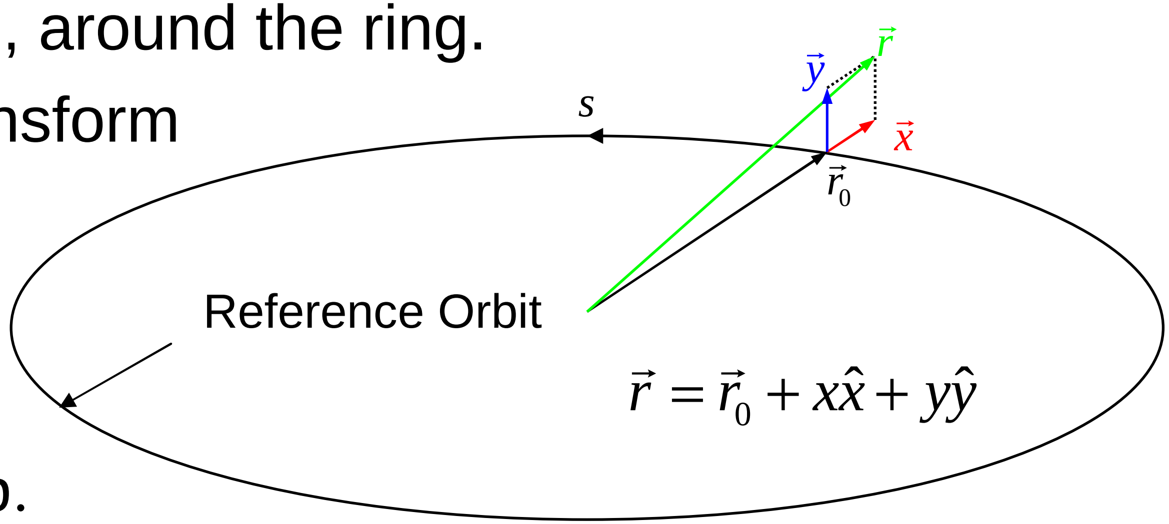
Particle motion in electromagnetic fields is governed by the Lorentz force: $$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

with the corresponding Hamiltonian: $$\mathcal{H} = c\left[m^2c^2 + \left(\vec{P} - e\vec{A}\right)^2\right]^{1/2} + e\Phi$$

$$\dot{x} = \frac{\partial\mathcal{H}}{\partial P_x},\ \dot{P}_x = -\frac{\partial\mathcal{H}}{\partial x},\ldots$$

For circular machines, it is convenient to convert to a curvilinear coordinate system and change the independent variable from time to the location, s-position, around the ring.

In order to do this we transform to the *Frenet-Serret* coordinate system. The local radius of curvature is denoted by ρ.



Equations of Motion

With a suitable canonical transformation, we can re-write the Hamiltonian as:

$$\tilde{\mathcal{H}} = -\left(1+\frac{x}{\rho}\right)\left[\frac{\left(\mathcal{H} - e\Phi\right)^2}{c^2} - m^2c^2 - \left(p_x - eA_x\right)^2 - \left(p_y - eA_y\right)^2\right]^{1/2} - eA_s$$

Using the relations $E = \mathcal{H} - e\Phi$, $p = \sqrt{\frac{E^2}{c^2} - m^2c^2}$

and expanding to 2nd order in p_x and p_y yields:

$$\tilde{\mathcal{H}} \approx -p\left(1+\frac{x}{\rho}\right) + \frac{1+x/\rho}{2p}\left[\left(p_x - eA_x\right)^2 - \left(p_y - eA_y\right)^2\right] - eA_s$$

which is now periodic in s.

Equations of Motion

Thus, in the absence of synchrotron motion, we can generate the equations of motion with:

$$x' = \frac{\partial \tilde{\mathcal{H}}}{\partial p_x}, \quad p_x' = -\frac{\partial \tilde{\mathcal{H}}}{\partial x}, \quad y' = \frac{\partial \tilde{\mathcal{H}}}{\partial p_y}, \quad p_y' = -\frac{\partial \tilde{\mathcal{H}}}{\partial y}$$

which yields:

$$x'' - \frac{\rho + x}{\rho^2} = \pm \frac{B_y}{B\rho} \frac{p_0}{p} \left(1 + \frac{x}{\rho}\right)^2, \quad \text{top / bottom sign for + / - charges}$$

and

$$y'' = \mp \frac{B_x}{B\rho} \frac{p_0}{p} \left(1 + \frac{x}{\rho}\right)^2$$

Note: $1/B\rho$ is the *beam rigidity* and is taken to be positive

Specific field configurations are applied in an accelerator to achieve the desired manipulation of the particle beams. Thus, before going further, it is useful to look at the types of fields of interest via the multipole expansion of the transverse field components.

Magnetic Field Multipole Expansion

Magnetic elements with 2-dimensional fields of the form

$$\vec{B} = B_x(x,y)\,\hat{x} + B_y(x,y)\,\hat{y}$$

can be expanded in a complex multipole expansion:

$$B_y(x,y) + iB_x(x,y) = B_0 \sum_{n=0}^{\infty} (b_n + ia_n)(x+iy)^n$$

with $b_n = \frac{1}{n!B_0} \frac{\partial^n B_y}{\partial x^n}\bigg|_{(x,y)=(0,0)}$ and $a_n = \frac{1}{n!B_0} \frac{\partial^n B_x}{\partial x^n}\bigg|_{(x,y)=(0,0)}$

In this form, we can normalize to the main guide field strength, $-B\hat{y}$, by setting $b_0=1$ to yield:

$$\frac{1}{B\rho}(B_y + iB_x) = \frac{e}{p_0}(B_y + iB_x) = \mp\frac{1}{\rho}\sum_{n=0}^{\infty}(b_n + ia_n)(x+iy)^n \quad \text{for } \pm q$$

Multipole Moments

Upright Fields

Dipole:

$$\frac{e}{p_0}B_x = 0 \qquad \frac{e}{p_0}B_y = \kappa_x$$

Quadrupole:

$$\frac{e}{p_0}B_x = ky \qquad \frac{e}{p_0}B_y = kx$$

Sextupole:

$$\frac{e}{p_0}B_x = mxy \qquad \frac{e}{p_0}B_y = \frac{1}{2}m\left(x^2 - y^2\right)$$

Octupole:

$$\frac{e}{p_0}B_x = \frac{1}{6}r\left(3x^2y - y^3\right)$$

$$\frac{e}{p_0}B_y = \frac{1}{6}r\left(x^3 - 3xy^2\right)$$

Skew Fields

Dipole ($\theta = 90°$):

$$\frac{e}{p_0}B_x = -\kappa_y \qquad \frac{e}{p_0}B_y = 0$$

Quadrupole ($\theta = 45°$):

$$\frac{e}{p_0}B_x = -k_{skew}x \qquad \frac{e}{p_0}B_y = k_{skew}y$$

Sextupole ($\theta = 30°$):

$$\frac{e}{p_0}B_x = -\frac{1}{2}m_{skew}\left(x^2 - y^2\right)$$

$$\frac{e}{p_0}B_y = m_{skew}xy$$

Octupole ($\theta = 22.5°$):

$$\frac{e}{p_0}B_x = -\frac{1}{6}r_{skew}\left(x^3 - 3xy^2\right)$$

$$\frac{e}{p_0}B_y = \frac{1}{6}r_{skew}\left(3x^2y - y^3\right)$$

Equations of Motion (Hill's Equation)

We next want to consider the equations of motion for a ring with only guide (dipole) and focusing (quadrupole) elements:

$$B_y = \mp B_0 + \frac{p_0}{e} kx = B_0 \left(\rho kx \mp 1 \right) \quad \text{and} \quad B_x = \frac{p_0}{e} ky = B_0 \rho kx$$

Taking $p=p_0$ and expanding the equations of motion to first order in x/ρ and y/ρ gives:

$$x'' + K_x(s)\, x = 0, \qquad K_x(s) = \frac{1}{\rho^2(s)} \mp k(s)$$

$$y'' + K_y(s)\, y = 0, \qquad K_y(s) = \pm k(s)$$

also commonly denoted as k_1

where the upper/low signs are for a positively/negatively charged particle.

The focusing functions are periodic in s: $K_{x,y}(s+L) = K_{x,y}(s)$

Solutions to Hill's Equation

Some introductory comments about the solutions to Hill's equations:

- The solutions to Hill's equation describe the particle motion around a reference orbit, the *closed orbit*. This motion is known as betatron motion. We are generally interested in small amplitude motions around the closed orbit (as has already been assumed in the derivation of the preceding pages).
- Accelerators are generally designed with discrete components which have locally uniform magnetic fields. In other words, the focusing functions, $K(s)$, can typically be represented in a piecewise constant manner. This allows us to locally solve for the characteristics of the motion and implement the solution in terms of a *transfer matrix*. For each segment for which we have a solution, we can then take a particle's initial conditions at the entrance to the segment and transform it to the final conditions at the exit.

Solutions to Hill's Equation

Let's begin by considering constant $K=k$:

$$x'' + kx = 0$$

where x now represents either x or y. The 3 solutions are:

$$x(s) = a\sin\left(\sqrt{k}\,s\right) + b\cos\left(\sqrt{k}\,s\right), \qquad k > 0 \qquad \text{Focusing Quadrupole}$$

$$x(s) = as + b, \qquad k = 0 \qquad \text{Drift Region}$$

$$x(s) = a\sinh\left(\sqrt{|k|}\,s\right) + b\cosh\left(\sqrt{|k|}\,s\right), \qquad k < 0 \qquad \text{Defocusing Quadrupole}$$

For each of these cases, we can solve for initial conditions and recast in 2×2 matrix form:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\vec{x} = \mathbf{M}\left(s \middle| s_0\right)\vec{x}_0$$

Transfer Matrices

We can now re-write the solutions of the preceding page in transfer matrix form:

$$\mathbf{M}\left(s|s_0\right)=\begin{cases}\begin{pmatrix}\cos\left(\sqrt{k}\,\ell\right) & \dfrac{1}{\sqrt{k}}\sin\left(\sqrt{k}\,\ell\right)\\ -\sqrt{k}\sin\left(\sqrt{k}\,\ell\right) & \cos\left(\sqrt{k}\,\ell\right)\end{pmatrix} & \text{Focusing Quadrupole}\\ \begin{pmatrix}1 & \ell\\ 0 & 1\end{pmatrix} & \text{Drift Region}\\ \begin{pmatrix}\cosh\left(\sqrt{|k|}\,\ell\right) & \dfrac{1}{\sqrt{|k|}}\sinh\left(\sqrt{|k|}\,\ell\right)\\ \sqrt{|k|}\sinh\left(\sqrt{|k|}\,\ell\right) & \cosh\left(\sqrt{|k|}\,\ell\right)\end{pmatrix} & \text{Defocusing Quadrupole}\end{cases}$$

where $\ell = s - s_0$.

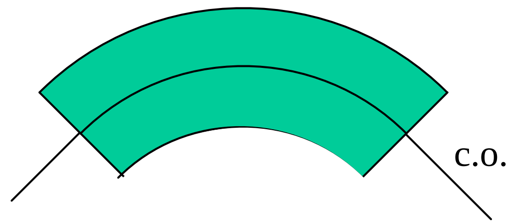
Transfer Matrices

Examples:

- Thin lens approximation: $\ell \to 0, \qquad f = \lim_{\ell \to 0} \frac{1}{|K|\ell}$

$$\mathbf{M}_{\text{focusing}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \qquad \mathbf{M}_{\text{defocusing}} = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

- Sector dipole (entrance and exit faces ⊥ to closed orbit):



$$\mathbf{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix} \approx \begin{pmatrix} 1 & \ell \\ -\frac{\ell}{\rho^2} & 1 \end{pmatrix}$$

where $\theta = \frac{\ell}{\rho}$

Transfer Matrices

Transport through an interval $s_0 \rightarrow s_2$ can be written as the product of 2 transport matrices for the intervals $s_0 \rightarrow s_1$ and $s_1 \rightarrow s_2$:

$$\mathbf{M}\left(s_2 \middle| s_0\right) = \mathbf{M}\left(s_2 \middle| s_1\right)\mathbf{M}\left(s_1 \middle| s_0\right)$$

and the determinant of each transfer matrix is: $\left|\mathbf{M}_i\right| = 1$

Many rings are composed of repeated sets of identical magnetic elements. In this case it is particularly straightforward to write the *one-turn matrix* for P superperiods, each of length L, as:

$$\mathbf{M}_{ring} = \left[\mathbf{M}\left(s + L \middle| s\right)\right]^P$$

with the boundary condition that: $\mathbf{M}\left(s + L \middle| s\right) = \mathbf{M}\left(s\right)$

The multi-turn matrix for m revolutions is then: $\left[\mathbf{M}\left(s\right)\right]^{mP}$

Twiss Parameters

The generalized one turn matrix can be written as:

$$\mathbf{M}=\begin{pmatrix} \cos\Phi+\alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi-\alpha\sin\Phi \end{pmatrix}=\mathbf{I}\cos\Phi+\mathbf{J}\sin\Phi$$

Identity matrix

This is the most general form of the matrix. α, β, and γ are known as either the Courant-Snyder or Twiss parameters (note: they have nothing to do with the familiar relativistic parameters) and Φ is the betatron phase advance. The matrix **J** has the properties:

$$\mathbf{J}=\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \qquad \mathbf{J}^2=-\mathbf{I} \Leftrightarrow \boxed{\beta\gamma=1+\alpha^2}$$

The n-turn matrix can be expressed as: $\mathbf{M}^n=\mathbf{I}\cos(n\Phi)+\mathbf{J}\sin(n\Phi)$
which leads to the stability requirement for betatron motion:

$$\boxed{\left|\mathrm{Trace}(\mathbf{M})\right|=2\cos\Phi\leq 2}$$

The Envelope Equations

We will look for 2 independent solutions to Hill's Equation of the form:
$$x(s) = aw(s)e^{i\psi(s)} \quad \text{and} \quad x^*(s) = aw(s)e^{-i\psi(s)}$$

Then w and ψ satisfy:

$$w'' + Kw - \frac{1}{w^3} = 0$$
$$\psi' = \frac{1}{w^2}$$

Betatron envelope

and

phase equations

Since any solution can be written as a superposition of the above solutions, we can write [with $w_i = w(s_i)$]:

$$\mathbf{M}(s_2|s_1) = \begin{pmatrix} \frac{w_2}{w_1}\cos\psi - w_2 w_1' \sin\psi & w_1 w_2 \sin\psi \\ -\frac{(1 + w_1 w_1' w_2 w_2')}{w_1 w_2}\sin\psi - \left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1}\right)\cos\psi & \frac{w_1}{w_2}\cos\psi + w_1 w_2' \sin\psi \end{pmatrix}$$

The Envelope Equations

Application of the previous transfer matrix to a full turn and direct comparison with the Courant-Snyder form yields:

$$w^2 = \beta$$

$$\alpha = -ww' = -\frac{\beta'}{2}$$

the betatron envelope equation becomes

$$\frac{1}{2}\beta'' + K\beta - \frac{1}{\beta}\left[1 + \frac{\beta'^2}{4}\right] = 0$$

and the transfer matrix in terms of the Twiss parameters can immediately be written as:

$$\mathbf{M}\left(s_2 \middle| s_1\right) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\Delta\psi + \alpha_1 \sin\Delta\psi\right) & \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\Delta\psi + \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1\beta_2}}\cos\Delta\psi & \sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\Delta\psi - \alpha_2 \sin\Delta\psi\right) \end{pmatrix}$$

General Solution to Hill's Equation

The general solution to Hill's equation can now be written as:

$$x(s) = A\sqrt{\beta_x(s)}\cos\left[\psi_x(s)+\phi_0\right] \quad \text{where} \quad \psi_x(s) = \int_0^s \frac{ds}{\beta_x(s)}$$

We can now define the *betatron tune* for a ring as:

$$Q_x = \nu_x = \frac{\Phi_{turn}}{2\pi} = \frac{1}{2\pi}\int_s^{s+C} \frac{ds}{\beta_x(s)} \quad \text{where} \quad C = \text{ring circumference}$$

If we make the coordinate transformation:

$$z = \frac{x}{\sqrt{\beta_x}} \quad \text{and} \quad \xi(s) = \frac{1}{\nu_x}\int_0^s \frac{ds}{\beta_x(s)}$$

we see that particles in the beam satisfy the equation for simple harmonic motion:

$$\frac{d^2 z}{d\xi^2} + \nu_x^2 z = 0$$

The Courant-Snyder Invariant

With K real, Hill's equation is conservative. We can now take

$$x(s) = A\sqrt{\beta_x(s)}\cos\left[\psi_x(s)+\phi_0\right] \text{ and}$$

$$x'(s) = -\frac{A}{\sqrt{\beta_x(s)}}\left\{\alpha(s)\cos\left[\psi_x(s)+\phi_0\right]+\sin\left[\psi_x(s)+\phi_0\right]\right\}$$

After some manipulation, we can combine these two equations to give:

Conserved quantity

$$A^2 = \varepsilon = \frac{x^2}{\beta_x(s)} + \left[\frac{\alpha_x(s)}{\sqrt{\beta_x(s)}}x + \sqrt{\beta_x(s)}x'\right]^2$$

Recalling that $\beta\gamma = 1+\alpha^2$ yields:

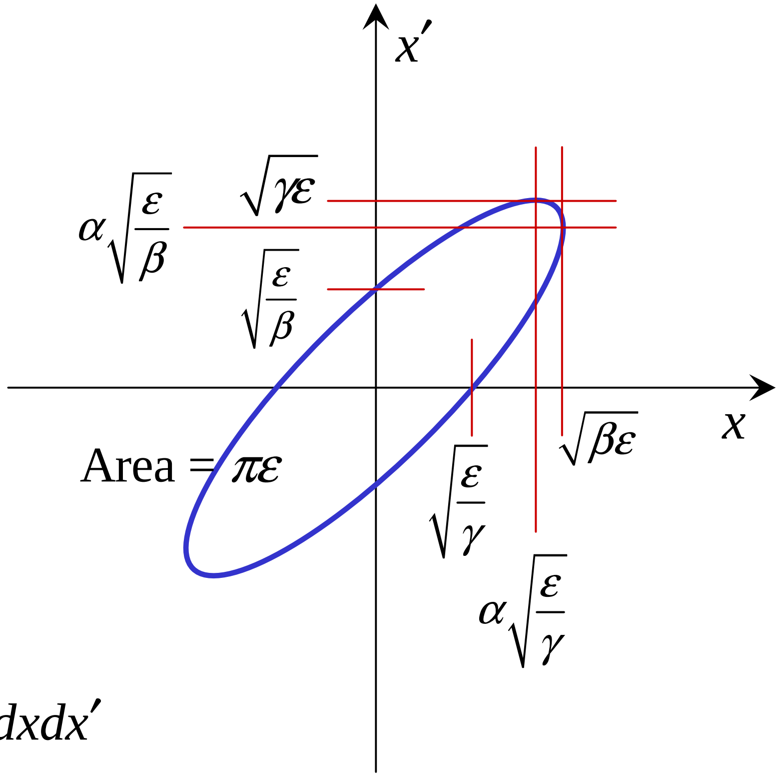
$$A^2 = \varepsilon = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s)$$

Emittance

The equation

$$\gamma(s)x^2(s)+2\alpha(s)x(s)x'(s)+\beta(s)x'^2(s)=\varepsilon$$

describes an ellipse with area $\pi\varepsilon$.



For an ensemble of particles, each following its own ellipse, we can define the moments of the beam as:

$$\langle x\rangle=\int x\rho(x,x')dxdx' \qquad \langle x'\rangle=\int x'\rho(x,x')dxdx'$$

$$\sigma_x^2=\int\left(x-\langle x\rangle\right)^2\rho(x,x')\,dxdx' \qquad \sigma_{x'}^2=\int\left(x'-\langle x'\rangle\right)^2\rho(x,x')\,dxdx'$$

$$\sigma_{xx'}^2=\int\left(x-\langle x\rangle\right)\left(x'-\langle x'\rangle\right)\rho(x,x')\,dxdx'=r\sigma_x\sigma_{x'}$$

The rms emittance of the beam is then $\varepsilon_{rms}=\sqrt{\sigma_x^2\sigma_{x'}^2-\sigma_{xx'}^2}=\frac{\langle A^2\rangle}{2}$

which is the area enclosed by the ellipse of an *rms particle*.

Coupling

Up to this point, the equations of motion that we have considered have been independent in x and y. An important issue for all accelerators, and particularly for damping rings which attempt to achieve a very small vertical emittance, is coupling between the two planes. For the damping ring, we are primarily interested in the coupling that arises due to small rotations of the quadrupoles. This introduces a *skew quadrupole* component to the equations of motion.

$$x'' + K_x(s)x = 0 \quad \Rightarrow \quad x'' + K_x(s)x + k_{skew}y = 0$$

$$y'' + K_y(s)y = 0 \quad \Rightarrow \quad y'' + K_y(s)y + k_{skew}x = 0$$

Another skew quadrupole term arises from "feed-down" when the closed orbit is displaced vertically in a sextupole magnet. In this case the effective skew quadrupole moment is given by the product of the sextupole strength and the closed orbit offset

$$k_{skew} = my_{co}$$

Coupling

For uncoupled motion, we can convert the 2D (x,x') and (y,y') transfer matrices to 4D form for the vector (x,x',y,y'):

$$\mathbf{M}_{\mathrm{4D}}\left(s|s_0\right)=\begin{pmatrix}\mathbf{M}_{\mathrm{focusing}} & 0\\ 0 & \mathbf{M}_{\mathrm{defocusing}}\end{pmatrix}=\begin{pmatrix}\mathbf{M}_{\mathrm{F}} & 0\\ 0 & \mathbf{M}_{\mathrm{D}}\end{pmatrix}$$

where we have arbitrarily chosen this case to be focusing in x. The matrix is block diagonal and there is no coupling between the two planes. If the quadrupole is rotated by angle θ, the transfer matrix becomes:

$$\mathbf{M}_{\mathrm{skew}}=\begin{pmatrix}\mathbf{M}_{\mathrm{F}}\cos^2\theta+\mathbf{M}_{\mathrm{D}}\sin^2\theta & \sin\theta\cos\theta\left(\mathbf{M}_{\mathrm{D}}-\mathbf{M}_{\mathrm{F}}\right)\\ \sin\theta\cos\theta\left(\mathbf{M}_{\mathrm{D}}-\mathbf{M}_{\mathrm{F}}\right) & \mathbf{M}_{\mathrm{D}}\cos^2\theta+\mathbf{M}_{\mathrm{F}}\sin^2\theta\end{pmatrix}$$

and motion in the two planes is coupled.

Coupling and Emittance

Later in this lecture we will look in greater detail at the sources of vertical emittance for the ILC damping rings.

In the absence of coupling and ring errors, the vertical emittance of a ring is determined by the the radiation of photons and the fact that emitted photons are randomly radiated into a characteristic cone with half-angle $\theta_{1/2}$~$1/\gamma$. This quantum limit to the vertical emittance is generally quite small and can be ignored for presently operating storage rings.

Thus the presence of betatron coupling becomes one of the primary sources of vertical emittance in a storage ring.

Dispersion

In our initial derivation of Hill's equation, we assumed that the particles being guided had the design momentum, p_0, thus ignoring longitudinal contributions to the motion. We now want to address off-energy particles. Thus we take the equation of motion:

$$x'' - \frac{\rho + x}{\rho^2} = \pm \frac{B_y}{B\rho} \frac{p_0}{p} \left(1 + \frac{x}{\rho}\right)^2$$

and expand to lowest order in $\delta = \frac{\Delta p}{p_0}$ and $\frac{x}{\rho}$ which yields:

$$x'' + K(s)x = \frac{\delta}{\rho}$$

We have already obtained a homogenous solution, $x_\beta(s)$. If we denote the particular solution as $D(s)\delta$, the general solution is:

$$x = x_\beta(s) + D(s)\delta$$

Dispersion Function and Momentum Compaction

The dispersion function satisfies: $D'' + K(s)D = 1/\rho$

with the boundary conditions: $D(s+L) = D(s);\ D'(s+L) = D'(s)$

The solution can be written as the sum of the solution to the homogenous equation and a particular solution:

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = \mathbf{M}(s_2|s_1) \begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix}$$

which can be expressed in a 3×3 matrix form as:

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{M}(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix},$$ **where** $\bar{d} = \begin{pmatrix} d \\ d' \end{pmatrix}$

Momentum Compaction

We can now consider the difference in path length experienced by such an off-momentum particle as it traverses the ring. The path length of an on-momentum particle is given by: $C = \oint \frac{x_{c.o.}}{\rho} ds$

For the off-momentum case, we then have: $\Delta C = \delta \times \oint \frac{D(s)}{\rho} ds = I_1 \delta$

I_1 is the first *radiation integral*.

The momentum compaction factor, α_c, is defined as:

$$\alpha_c = \frac{\Delta C / C}{\delta} = \frac{I_1}{C}$$

The Synchrotron Radiation Integrals

I_1 is the first of 5 “radiation integrals” that we will study in this lecture. These 5 integrals describe the key properties of a storage ring lattice including:

- Momentum compaction
- Average power radiated by a particle on each revolution
- The radiation excitation and average energy spread of the beam
- The *damping partition numbers* describing how radiation damping is distributed among longitudinal and transverse modes of oscillation
- The natural emittance of the lattice

In later sections of this lecture we will work through the key aspects of radiation damping in a storage ring

Chromaticity

An off-momentum particle passing through a quadrupole will be under/over-focused for positive/negative momentum deviation. This is chromatic aberration. Hill's equation becomes:

$$x'' + \left[K_0(s)(1-\delta) \right] x = 0$$

We will evaluate the chromaticity by first looking at the impact of local gradient errors on the particle beam dynamics.

Effect of a Gradient Error

We consider a local perturbation of the focusing strength $K = K_0 + \Delta K$. The effect of ΔK can be represented by including a thin lens transfer matrix in the one-turn matrix. Thus we have

$$\mathbf{M}_{\Delta K} = \begin{pmatrix} 1 & 0 \\ -\Delta K \ell & 1 \end{pmatrix}$$

and

$$\mathbf{M}_{1-turn} = \begin{pmatrix} \cos\Phi + \alpha \sin\Phi & \beta \sin\Phi \\ -\gamma \sin\Phi & \cos\Phi - \alpha \sin\Phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\Phi_0 + \alpha \sin\Phi_0 & \beta \sin\Phi_0 \\ -\gamma \sin\Phi_0 & \cos\Phi_0 - \alpha \sin\Phi_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Delta K \ell & 1 \end{pmatrix}$$

With $\Phi = \Phi_0 + \Delta\Phi$, we can take the trace of the one-turn matrix to give:

$$\cos(\Phi_0 + \Delta\Phi) = \cos\Phi_0 - \frac{1}{2}\beta \Delta K \ell \sin\Phi_0$$

Effect of a Gradient Error

Using the relation: $\cos(\Phi_0 + \Delta\Phi) = \cos\Delta\Phi\cos\Phi_0 - \sin\Delta\Phi\sin\Phi_0$

we can identify: $\Delta\Phi \approx \frac{1}{2}\beta\Delta K\ell$

Thus we can write: $\Delta Q = \frac{1}{4\pi}\beta\Delta K\ell$

and we see that the result of gradient errors is a shift in the betatron tune. For a distributed set of errors, we then have:

$$\Delta Q = \frac{1}{4\pi}\oint \beta\Delta K ds$$

which is the result we need for evaluating chromatic aberrations. *Note that the tune shift will be positive/negative for a focusing/defocusing quadrupole.*

Chromaticity

We can now write the betatron tune shift due to chromatic aberration as:

$$\Delta Q = \frac{1}{4\pi}\oint \beta \Delta K ds \approx -\frac{\delta}{4\pi}\oint \beta K ds$$

The chromaticity is defined as the change in tune with respect to the momentum deviation:

$$C = \frac{\partial Q}{\partial \delta}$$

Because the focusing is weaker for a higher momentum particle, the natural chromaticity due to quadrupoles is always negative. This can be a source of instabilities in an accelerator. However, the fact that a momentum deviation results in a change in trajectory (the dispersion) as well as the change in focusing strength, provides a route to mitigate this difficulty.

Sextupoles

Recall that the magnetic field in a sextupole can be written as:

$$\frac{e}{p_0}B_x = mxy \qquad \frac{e}{p_0}B_y = \frac{1}{2}m\left(x^2 - y^2\right)$$

Using the orbit of an off-momentum particle $x = x_\beta(s) + D(s)\delta$

we obtain $$\frac{e}{p_0}B_x = mD(s)\delta y_\beta(s) + mx_\beta(s)y$$

and $$\frac{e}{p_0}B_y = mD(s)\delta x_\beta(s) + \frac{1}{2}mD^2(s)\delta^2 + \frac{1}{2}m\left[x_\beta^2(s) - y_\beta^2(s)\right]$$

where the first terms in each expression are a quadrupole feed-down term for the off-momentum particle. Thus the sextupoles can be used to compensate the chromatic error. The change in tune due to the sextupole is

$$\Delta Q = \frac{\delta}{4\pi}\oint mD(s)\beta(s)\,ds$$

Outline of DR Lecture I, Part 2

Radiation Damping and Equilibrium Emittance

- Radiation Damping
- Synchrotron Equations of Motion
- Synchrotron Radiation Integrals
- Quantum Excitation and Equilibrium Emittance
- Summary of Beam Parameters and Radiation Integrals

ILC Damping Ring Lattice

- Damping Ring Design Optimization
- The OCS Lattice
- The DCO Lattice
- Summary of Parameters and Design Choices

Synchrotron Radiation and Radiation Damping

Up to this point, we have treated the transport of a relativistic electron (or positron) around a storage ring as a conservative process. In fact, the bending field results in the particles radiation synchrotron radiation.

The energy lost by an electron beam on each revolution is replaced by radiofrequency (RF) accelerating cavities. Because the synchrotron radiation photons are emitted in a narrow cone (of half-angle $1/\gamma$) around the direction of motion of a relativistic electron while the RF cavities are designed to restore the energy by providing momentum kicks in the $\hat{s}$ direction, this results in a gradual loss of energy in the transverse directions. This effect is known as *radiation damping*.

Synchrotron Radiation

We will only concern ourselves with electron/positron rings. The instantaneous power radiated by a relativistic electron with energy E in a magnetic field resulting in bending radius ρ is:

$$P_\gamma = \frac{cC_\gamma E^4}{2\pi\rho^2} = \frac{e^2c^3}{2\pi} C_\gamma E^2 B^2$$ where $C_\gamma = 8.85\times10^{-5}\, m/(GeV)^3$

We can integrate this expression over one revolution to obtain the energy loss per turn:

$$U_0 = \frac{C_\gamma E^4}{2\pi}\oint\frac{ds}{\rho^2} = \frac{C_\gamma E^4}{2\pi} I_2$$ where I_2 is the 2nd radiation integral

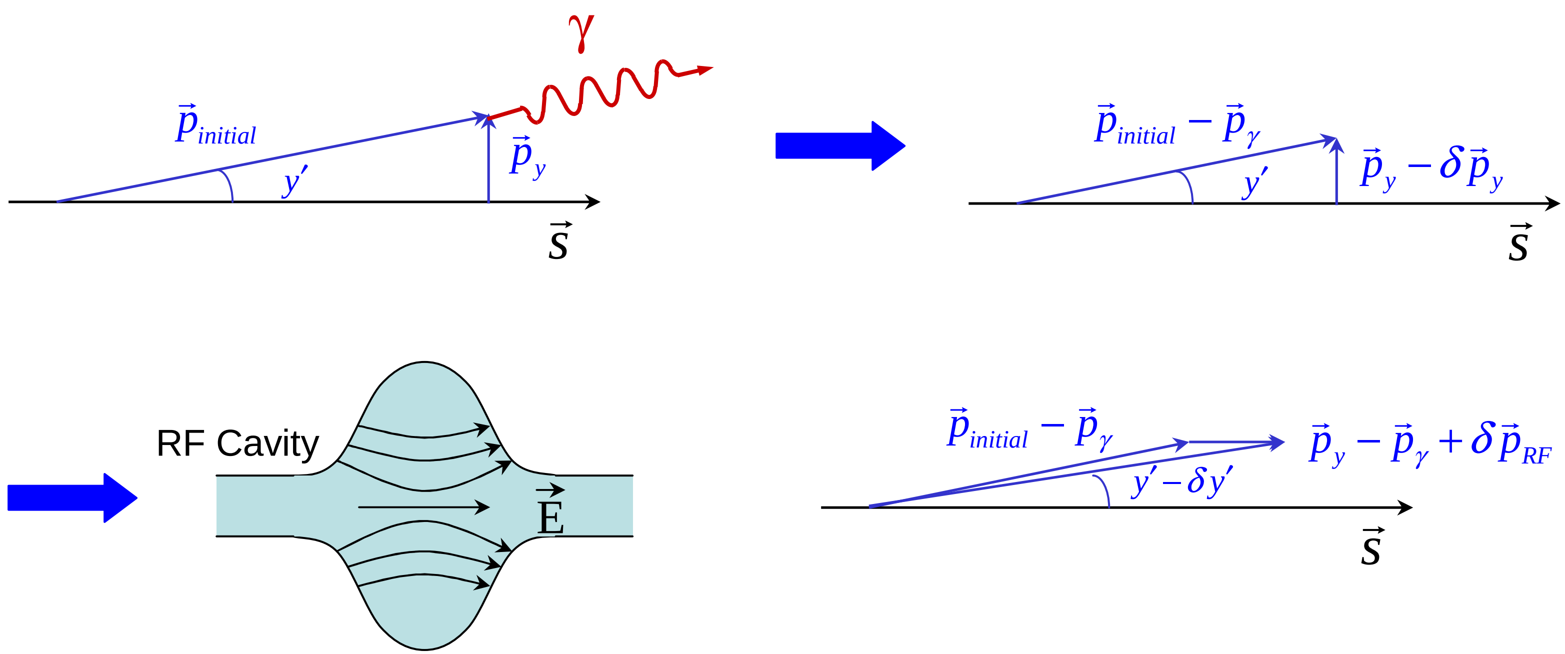
For a lattice with uniform bending radius (iso-magnetic) this yields:

$$U_0[eV] = 8.85\times10^4 \frac{E^4[GeV]}{\rho[m]}$$

If this energy were not replaced, the particles would lose energy and gradually spiral inward until they would be lost by striking the vacuum chamber wall. The RF cavities replace this lost energy by providing momentum kicks to the beam in the longitudinal direction.

Radiation Damping of Vertical Betatron Motion

We look first at the vertical dimension where, for an ideal machine, we do not need to consider effects of vertical dispersion.



The change in y' after the RF cavity can be written as:

$$\delta y' = -y' \frac{\delta p_{RF}}{p} = -y' \frac{\delta E}{E}$$

Radiation Damping (Vertical)

Recall that an oscillation with amplitude A is described by:

$$A^2 = \gamma y^2 + 2\alpha y y' + \beta y'^2$$

If we assume that the β-function is slowly varying, so that

$\alpha = -\beta'/2 \sim 0$, we can write:

$$\delta\left(A^2\right) \approx \underbrace{\delta\left(\gamma y^2\right)}_{=0} + \delta\left(\beta y'^2\right)$$

$$\Rightarrow A\delta A = \beta y'^2 \frac{\delta y'}{y'} = -\beta y'^2 \frac{\delta E}{E}$$

and (using the solution to Hill's equation we obtained previously):

$$y'(s) \approx -\frac{A}{\sqrt{\beta_y(s)}} \sin\left[\psi_y(s) + \phi_0\right]$$

Substituting and averaging then gives:

$$\frac{\delta A}{A} = -\frac{1}{2}\frac{\delta E}{E_0}$$

Radiation Damping (Vertical)

Thus the damping decrement, ie, the fractional decrease in amplitude in one revolution, is:

$$\alpha_y = \frac{\langle \delta A \rangle}{A T_0} = \frac{U_0}{2 E_0 T_0}$$

We can re-write this in exponential decay form as:

$$A(t) = A(0) \exp(-\alpha_y t)$$

or equivalently, the damping of the vertical emittance is given by:

$$\boxed{\varepsilon(t) = \varepsilon(0) \exp(-2\alpha_y t)}$$

Radiation Damping (Transverse)

The situation for horizontal radiation damping is somewhat more complicated than the vertical case because of the presence of dispersion generated by the bending magnets. A similar procedure to that followed for the vertical case yields the result:

$$\alpha_x = \frac{U_0}{2E_0T_0}\left(1-\mathcal{D}\right)$$

$$\mathcal{D} = \frac{I_4}{I_2} \quad \text{with} \quad I_2 = \oint \frac{ds}{\rho^2} \quad \text{and} \quad I_4 = \oint \frac{D}{\rho}\left(\frac{1}{\rho^2}+2k\right)ds$$

It is usual to write the transverse damping decrements as:

$$\alpha_i = \frac{U_0}{2E_0T_0}J_i \quad \text{with} \quad J_x = 1-\mathcal{D} \quad \text{and} \quad J_y = 1$$

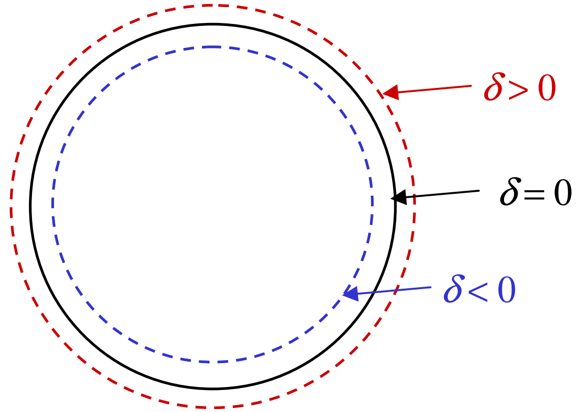
The transverse emittances will damp as:

$$\frac{d\varepsilon_i}{dt} = -2\alpha_i\varepsilon_i$$

Synchrotron Motion

As particles circulate in a ring, the phase of their passage through the RF accelerating cavities must stay synchronized with respect to the RF frequency in order for their orbits to be stable. This stability is provided by the principle of phase focusing. In the relativistic limit we take:

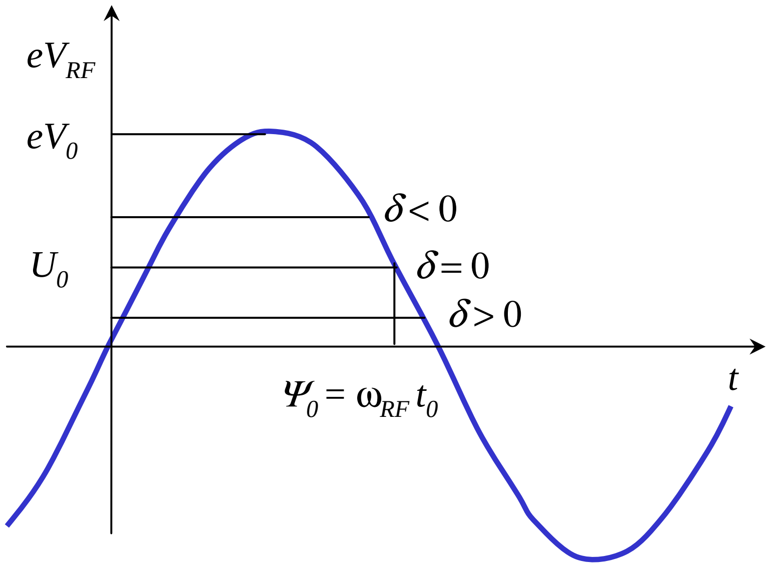
$$\delta = \frac{\Delta p}{p} = \frac{\Delta E}{E}$$



The arrival time for each particle is given by:

$$\frac{\Delta t}{T_0} = \frac{\Delta C}{C} = \alpha_c \delta$$

where α_c is the momentum compaction factor. Thus particles with $\delta>0$ will be delayed and will receive a smaller kick from the RF while particles with $\delta<0$ will arrive early and receive a larger kick as long as the default arrival time in the RF cavity is as shown on the right. This leads to synchrotron oscillations around a stable point.



Synchrotron Equations of Motion

For our description of the longitudinal motion, we will use the variables:

$$\delta = \frac{\Delta E}{E_0} \quad \text{and} \quad \tau = t - t_0$$

where the *0* subscripts are for the synchronous particle.

Thus we can write:

$$\frac{d\tau}{dt} = -\alpha_c \delta$$

and

$$\frac{d\delta}{dt} = \frac{eV_{RF}(\tau) - U(E)}{E_0 T_0}$$

Note that we write the energy loss term as a function of E

where we have assumed that any synchrotron oscillations are far slower than the revolution time (a good assumption in practice) so that using the average energy loss per turn is valid. For small values of τ the RF voltage can be linearized as:

$$V_{RF}(\tau) = \frac{U_0}{e} + \tau \left.\frac{dV}{dt}\right|_{t=t_0} = \frac{U_0}{e} + \tau\omega_{RF} V_0 \cos\Psi_s$$

where $\sin\Psi_s = \frac{U_0}{eV_0}$

Synchrotron Equation of Motion

We can now write:

$$\frac{d^2\delta}{dt^2} + 2\alpha_E \frac{d\delta}{dt} + \omega_s^2\delta = 0$$

Synchrotron EOM

where:

$$\alpha_E = \frac{1}{2T_0}\left.\frac{dU(E)}{dE}\right|_{E=E_0}$$

$$\omega_s^2 = \frac{e\alpha_c\omega_{RF}V_0\cos\Psi_s}{E_0T_0}$$

The solutions to the synchrotron EOM can be written as:

$$\delta(t) = A_E e^{-\alpha_E t}\cos(\omega_s t - \Psi_s)$$

with

$$\tau(t) = \frac{-\alpha_c A_E}{E_0\omega_s} e^{-\alpha_E t}\sin(\omega_s t - \Psi_s)$$

which describes the oscillation in energy and time of a particle with respect to the ideal synchronous particle.

Energy Oscillation Damping

There are a couple points to note about the synchrotron EOM.

- First, we note that the synchrotron motion is intrinsically damped towards the motion of the synchronous particle. In the δ-τ plane, an off-energy particle will exponentially spiral towards the origin – the synchronous particle's parameters
- Second, the damping coefficient, α_E, is dependent on the energy of the particle. This happens in two ways. First the power radiated depends on energy. Secondly, the time it takes an electron to complete a revolution around the ring depends on the circumference of the orbit which also depends on the energy. Thus we still have some work to do to understand the rate of damping.

We start by writing the energy lost in one turn as:

$$U = \int_0^T P_\gamma dt$$

Radiation Damping of Synchrotron Motion

We want to convert the integral over time to an integral over s. For a particle that is not on the closed orbit, the path length that it traverses can be written as:

$$d\ell = \left(1 + \frac{x}{\rho}\right) ds \qquad \Rightarrow \qquad dt = \frac{d\ell}{c} = \frac{1}{c}\left(1 + \frac{x}{\rho}\right) ds$$

where x represents the orbit displacement due to the energy deviation. We can thus write the time differential as:

$$dt = \left(1 + \frac{D\delta}{\rho}\right) ds$$

and the energy loss per turn becomes:

$$U = \frac{1}{c} \oint P_\gamma \left(1 + \frac{D\delta}{\rho}\right) ds$$

Radiation Damping

Evaluating $\left.\frac{dU}{dE}\right|_{E=E_0}$ yields (after a bit of work):

$$\alpha_E = \frac{1}{2T_0}\frac{dU}{dE} = \frac{U_0}{2T_0 E_0} J_E$$

where $$J_E = 2 + \mathcal{D} = 2 + \frac{I_4}{I_2}$$

and $$I_2 = \oint \frac{1}{\rho^2} ds \qquad I_4 = \oint \frac{D}{\rho}\left(\frac{1}{\rho^2} + 2k\right) ds \qquad k = \frac{1}{B\rho}\frac{dB_y}{dx}$$

Thus an energy deviation will damp with a time constant

$$\tau_E = \frac{2T_0 E_0}{J_E U_0}$$

Summary of Radiation Damping

We can now summarize the radiation damping rates for each of the beam degrees of freedom:

$$\alpha_E = \frac{U_0}{2T_0E_0} J_E \qquad J_E = 2 + \mathcal{D} \qquad \mathcal{D} = 1 + \frac{I_4}{I_2}$$

$$\alpha_x = \frac{U_0}{2T_0E_0} J_x \qquad J_x = 1 - \mathcal{D}$$

$$\alpha_y = \frac{U_0}{2T_0E_0} J_y \qquad J_y = 1$$

and we can immediately write:

$$J_E + J_x + J_y = 4$$

Robinson's Theorem

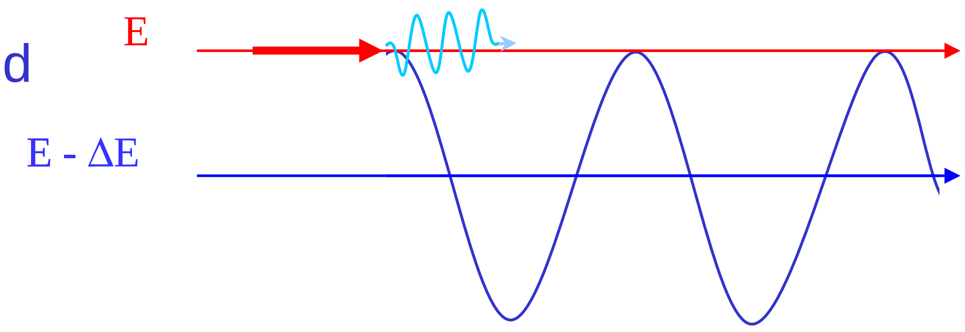
For separated function lattices, $\mathcal{D} \ll 1$ and the longitudinal damping occurs at roughly twice the rate of the damping in the two transverse dimensions.

Radiation damping plays a very special role in electron/positron rings because it provides a direct mechanism to take *hot* injected beams and reduce the equilibrium parameters to a regime useful for high luminosity colliders and high brightness light sources. At the same time, the radiated power plays a dominant role in the design of the technical systems – we will discuss some aspects of this further in tomorrow's lecture.

Equilibrium Beam Properties

Now that we have determined the radiation damping rates, we can explore the equilibrium properties of the beam

- The emission of photons by the beam is a random process around the ring
- Photons are emitted within a cone around the direction of the beam particle with a characteristic angle $1/\gamma$



- This quantized process excites oscillations in each dimension
- In the absence of resonance or collective effects, which also serve to *heat* the beam, the balance between quantum excitation and radiation damping results in the equilibrium beam properties that are characteristic of a given lattice

Quantum Excitation - Longitudinal

We will first look at the impact of quantum excitation in the longitudinal dimension.

For the very short timescales corresponding to photon emission, we can take the equations of motion we previously obtained for synchrotron motion and write:

$$\delta_E^2(t) + \frac{E_0^2 \omega_s^2}{\alpha_c^2} \tau^2(t) = A_E^2$$

where A_E is a constant of the motion.

We want to consider the change in A_E due to the emission of individual photons. The emission of an individual photon will not affect the time variable, however, it will cause an instantaneous change in the value of δ_E.

Quantum Excitation - Longitudinal

Thus we can write:

$$\Delta\delta = A_0 \cos\omega_s\left(t - t_0\right) - \frac{u}{E_0}\cos\omega_s\left(t - t_1\right) = A_1\cos\omega_s\left(t - t_1\right)$$

where u is the energy radiated at time t_1. Thus

$$A_1^2 = A_0^2 + \left(\frac{u}{E_0}\right)^2 - \frac{2A_0 u}{E_0}\cos\omega_s\left(t_1 - t_0\right)$$

and

$$\Delta A^2 = \left\langle A^2 - A_0^2 \right\rangle = \frac{u^2}{E_0^2}$$

We can thus write the average change in synchrotron amplitude due to photon emission as:

$$\frac{d\left\langle A^2 \right\rangle}{dt} = \mathcal{N}\left(\frac{u}{E_0}\right)^2$$

where $\mathcal{N}$ is the rate of photon emission and u is the photon energy.

Quantum Excitation - Longitudinal

If we now include the radiation damping term, the net change in the synchrotron amplitude can be written as:

$$\frac{d\left\langle A^2\right\rangle}{dt} = -2\alpha_E\left\langle A^2\right\rangle + \mathcal{N}\frac{u^2}{E_0^2}$$

The equilibrium properties of a bunch are obtained when the rate of growth from quantum excitation and the rate of damping from radiation damping are equal. For an ensemble of particles where we identify the RMS energy amplitude with the energy spread, we can then write the equilibrium condition as:

$$\sigma_\delta^2 = \left(\frac{\sigma_E}{E_0}\right)^2 = \frac{\left\langle A^2\right\rangle}{2} = \frac{\left\langle \mathcal{N}\left\langle u^2\right\rangle\right\rangle_s}{4\alpha_E E_0^2}$$

Photon Emission

$\left\langle \mathcal{N} \left\langle u^2 \right\rangle \right\rangle_s$ is the ring-wide average of the photon emission rate, $\mathcal{N}$, times the mean square energy loss associated with each emission. In other words:

$$\mathcal{N} = \int_0^\infty n(u)du \quad \text{and} \quad \mathcal{N} \left\langle u^2 \right\rangle = \int_0^\infty u^2 n(u)\, du$$

where $n(u)$ is the photon emission rate at energy u, and

$$\left\langle \mathcal{N} \left\langle u^2 \right\rangle \right\rangle_s = \frac{1}{C} \oint \mathcal{N} \left\langle u^2 \right\rangle ds$$

where C is the ring circumference. Derivations of the photon spectrum emitted in a magnetic field are available in many texts and we will simply quote the result:

$$\mathcal{N} \left\langle u^2 \right\rangle = 2C_q \gamma^2 \frac{E_0 P_\gamma}{\rho} \quad \text{where} \quad C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 3.84 \times 10^{-13}\, m$$

Energy Spread and Bunch Length

Integrating around the ring then yields the beam energy spread:

$$\sigma_\delta^2 = \left(\frac{\sigma_E}{E_0}\right)^2 = C_q \gamma^2 \frac{I_3}{J_E I_2} \quad \text{where} \quad I_3 = \oint \frac{ds}{|\rho|^3}$$

Using our solution to the synchrotron equations of motion, the bunch length is related to the energy spread by:

$$\sigma_\ell = \frac{c\alpha_c}{\omega_s E_0} \quad \text{where} \quad \omega_s^2 = \frac{e\alpha_c \omega_{RF} V_0 \cos\Psi_s}{E_0 T_0}$$

We note that the bunch length scales inversely with the square root of the RF voltage.

Quantum Excitation - Horizontal

In order to evaluate the impact of the radiated photon on the motion of the emitting electron, we recall

$$A^2 = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s)$$

The change in closed orbit due to losing a unit of energy, u, is given by:

$$\delta x = -D(s)\frac{u}{E_0}$$

$$\delta x' = -D'(s)\frac{u}{E_0}$$

and we can then write:

$$\delta A^2 = \left(\gamma D^2 + 2\alpha DD' + \beta D'^2\right)\frac{u^2}{E_0^2} = \mathcal{H}(s)\frac{u^2}{E_0^2}$$

where $\mathcal{H}(s)$ is the *curly-H* function.

Horizontal Emittance

We can then write an excitation term for the rms emittance as:

$$\left.\frac{d\varepsilon_x}{dt}\right|_{QE} = \frac{1}{2}\frac{d\left\langle A^2\right\rangle}{dt} = \frac{\left\langle \mathcal{N}\mathcal{H}\left\langle u^2\right\rangle\right\rangle_s}{2E_0^2}$$

Equating this expression to the damping rate yields (after some calculation) the equilibrium horizontal emittance:

$$\varepsilon_x = C_q \frac{\gamma^2 \left\langle \dfrac{\mathcal{H}}{\rho^3}\right\rangle}{J_x \left\langle \dfrac{1}{\rho^2}\right\rangle} = C_q \frac{\gamma^2 I_5}{J_x I_2}$$

where we have defined the next synchrotron radiation integral:

$$I_5 = \oint \frac{\mathcal{H}}{\rho^3} ds$$

Quantum Excitation - Vertical

In the vertical dimension, where we assume the ideal case of no vertical dispersion, the quantum excitation of the emittance is determined by the opening angle of the emitted photons. The resulting perturbation to the vertical motion can be described as:

$$\delta y = 0 \qquad\qquad \delta y' = \frac{u}{E_0}\theta_\gamma$$

and we can write:

$$\delta\left\langle A^2 \right\rangle = \left(\frac{u\theta_\gamma}{E_0}\right)^2 \beta_y$$

Thus, proceeding as we have on the preceding pages, we can write the expression for the equilibrium emittance as:

$$\varepsilon_y = \frac{\left\langle \mathcal{N}\left\langle u^2\right\rangle \beta_y \right\rangle_s \left\langle \theta_\gamma^2 \right\rangle}{4E_0^2} = \frac{\left\langle \mathcal{N}\left\langle u^2\right\rangle \beta_y \right\rangle_s}{4\gamma^2 E_0^2}$$

$$\varepsilon_y \approx \frac{C_q}{2J_y I_2}\oint \frac{\beta_y}{\rho^3}ds$$

Vertical Emittance & Emittance Coupling

For typical storage ring parameters, the vertical emittance due to quantum excitation is negligible. Assuming a typical β_y values of a few 10's of meters and bending radius of ~100m, we can estimate $\varepsilon_y \leq 0.1\ \mathrm{pm}$. The observed sources of vertical emittance are:

- ***emittance coupling*** whose source is ring errors which couple the vertical and horizontal betatron motion
- ***vertical dispersion*** due to vertical misalignment of the quadrupoles and sextupoles and angular errors in the dipoles

The vertical and horizontal emittances in the presence of a collection of such errors around a storage ring is commonly described as:

$$\varepsilon_y = \frac{\kappa}{1+\kappa}\varepsilon_0; \quad \varepsilon_x = \frac{1}{1+\kappa}\varepsilon_0 \quad \text{for} \quad 0<\kappa<1$$

ε_0 is the natural emittance.

Radiation Integrals and Equilibrium Quantities

Summary of Radiation Integrals:

$$I_1 = \oint \frac{D(s)}{\rho} ds$$

$$I_2 = \oint \frac{1}{\rho^2} ds$$

$$I_3 = \oint \frac{1}{|\rho|^3} ds$$

$$I_4 = \oint \frac{D(s)}{\rho} \left(\frac{1}{\rho^2} + 2k \right) ds$$

$$I_5 = \oint \frac{\mathcal{H}}{|\rho|^3} ds$$

$$\mathcal{H} = \gamma D^2 + 2\alpha D D' + \gamma D'^2$$

Summary of Equilibrium Beam Properties:

$$\alpha_c = \frac{I_1}{C}$$

$$U_0 = \frac{C_\gamma E^4}{2\pi} I_2 \quad \text{where} \quad C_\gamma = 8.85 \times 10^{-5}\, m / (GeV)^3$$

$$\alpha_i = \frac{U_0}{2 E_0 T_0} J_i, \quad i = x, y, E$$

$$J_x = 1 - \mathcal{D}; \quad J_y = 1; \quad J_E = 2 + \mathcal{D}; \quad \mathcal{D} = \frac{I_4}{I_2}$$

$$\left(\frac{\sigma_E}{E} \right)^2 = \frac{C_q \gamma^2 I_3}{J_E I_2} \quad \text{where} \quad C_q = 3.84 \times 10^{-13}\, m$$

$$\sigma_\ell = \frac{c \alpha_c}{\omega_s E_0} \quad \text{where} \quad \omega_s^2 = \frac{e \alpha_c \omega_{RF} V_0 \cos \Psi_s}{E_0 T_0}; \quad \sin \Psi_s = \frac{U_0}{e V_0}$$

$$\varepsilon_x = \frac{C_q \gamma^2 I_5}{J_x I_2}; \quad \varepsilon_y \approx \frac{C_q}{2 J_y I_2} \oint \frac{\beta_y}{\rho^3} ds \quad \text{(quantum excitation)}$$

Emittance Scaling in Lattices

The natural emittance of a lattice is given by:

$$\varepsilon_0 = \frac{C_q \gamma^2}{J_x} \frac{I_5}{I_2}$$

The ratio $\frac{I_5}{I_2}$ can be tailored to provide very low emittance. It can be shown that the natural emittance scales approximately as:

$$\varepsilon_0 \approx F \frac{C_q \gamma^2}{J_x} \theta^3$$

where F is a function of the lattice design and θ is the bending angle from the dipoles in each lattice cell. The natural emittance can be made small by having small bending angles in the dipoles of each lattice cell and by optimizing F. The theoretical minimum emittance (TME) lattice has

$$F \approx \frac{1}{12\sqrt{15}}$$

Unfortunately, designing a very low emittance lattice in this way may have serious impact on the cost and/or performance of a low emittance ring.

Achieving Ultra-Low Emittance

The path to low emittance that is pursued in a damping ring, is to provide insertion devices, wigglers, which dominate the radiation damping of the machine. For a sinusoidal wiggler, we can write the energy loss around the ring as:

$$U_0 = \frac{C_\gamma E^4}{2\pi}\left(\oint_{dipoles} \frac{1}{\rho^2} ds + \int_0^{L_{wiggler}} \frac{1}{\rho_{wig}^2} ds \right) = U_{dip} + U_{wig}$$

The overall length of the wiggler section, along with the wiggler period and peak field, can be adjust to make the second term dominate the radiation losses in the ring and hence the damping rate. The expressions

$$\varepsilon_{dip} = C_q \gamma^2 \frac{I_{5dip}}{I_{2dip}} \quad \text{and} \quad \varepsilon_{wig} = C_q \gamma^2 \frac{I_{5wig}}{I_{2wig}}$$

give the emittance contributions of the dipole and wiggler regions, respectively. We can then write the natural emittance of the ring as:

$$\varepsilon_0 = \frac{\varepsilon_{dip}}{1+F} + \frac{\varepsilon_{wig} F}{1+F} \quad \text{where} \quad F = \frac{U_{wig}}{U_{dip}}$$

Thus, if the wiggler radiation dominates, the emittance contribution due to the dipoles is reduced by a factor of F and the ring emittance is dominated by the intrinsic wiggler emittance. In fact, the wiggler emittance can be quite small by placing the wigglers in zero dispersion regions with small β_x.

The Damping Rings Lattice

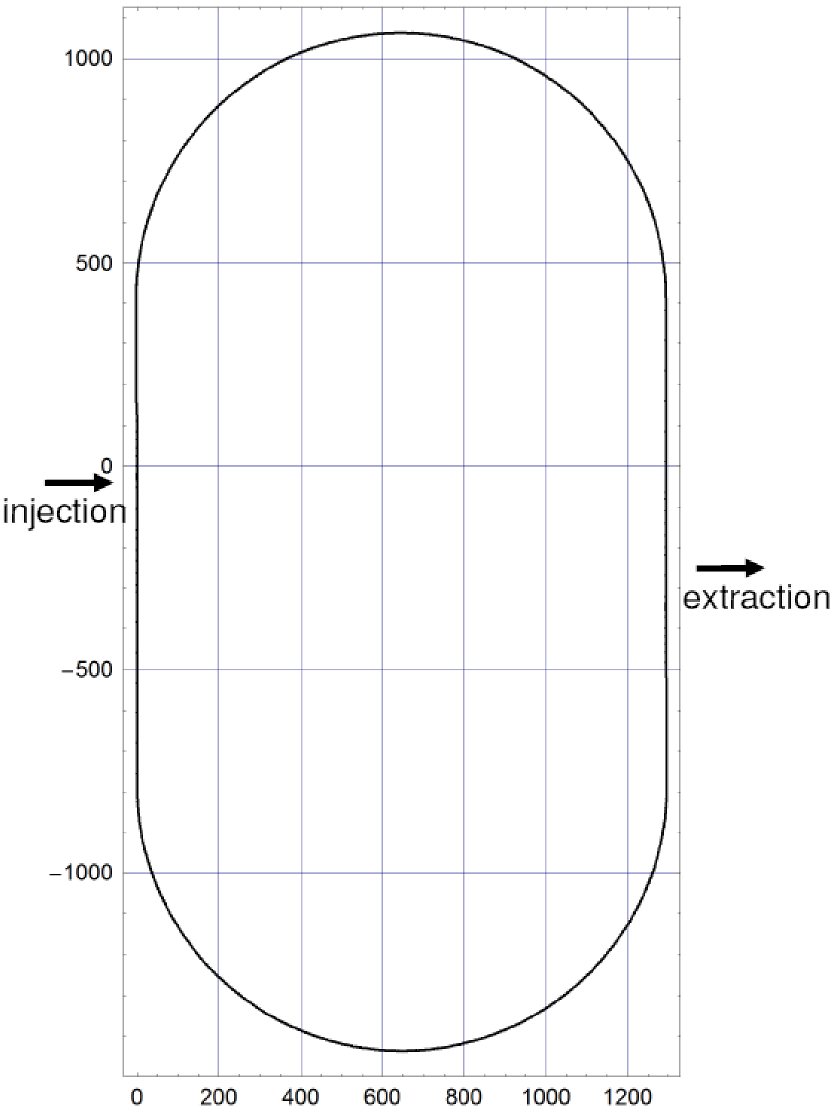
At the time of the ILC Reference Design Report, the ILC damping rings lattice was based on a variant of the TME (theoretical minimum emittance) lattice. As noted earlier, however, there is flexibility in the choice of lattice style in a wiggler dominated ring.

Thus, the present damping ring design employs a FODO lattice. The FODO-based design offers greater flexibility in setting the momentum compaction of the damping rings and was chosen to be the basis for further ILC DR design work.

It should be noted that much of the design work for each of these lattices is associated with the injection/extraction straights, RF and wiggler regions, and other specialty segments of the accelerator.

The DCO Lattice

Wolski, Korostelev



- Arcs consist of a total of 192 FODO cells
- Flexibility in tuning momentum compaction factor, given by phase advance per arc cell:
  - **72° phase advance: α_p=2.8×10^{-4}**
  - **90° phase advance: α_p=1.7×10^{-4}**
  - **100° phase advance: α_p=1.3×10^{-4}**
- No changes in dipole strengths needed for different working points.
- Racetrack structure has two similar straights containing:
  - **injection and extraction in opposite straights**
  - **phase trombones**
  - **circumference chicanes**
  - **rf cavities**
  - **"doglegs" to separate wiggler from rf and other systems**
  - **wiggler**

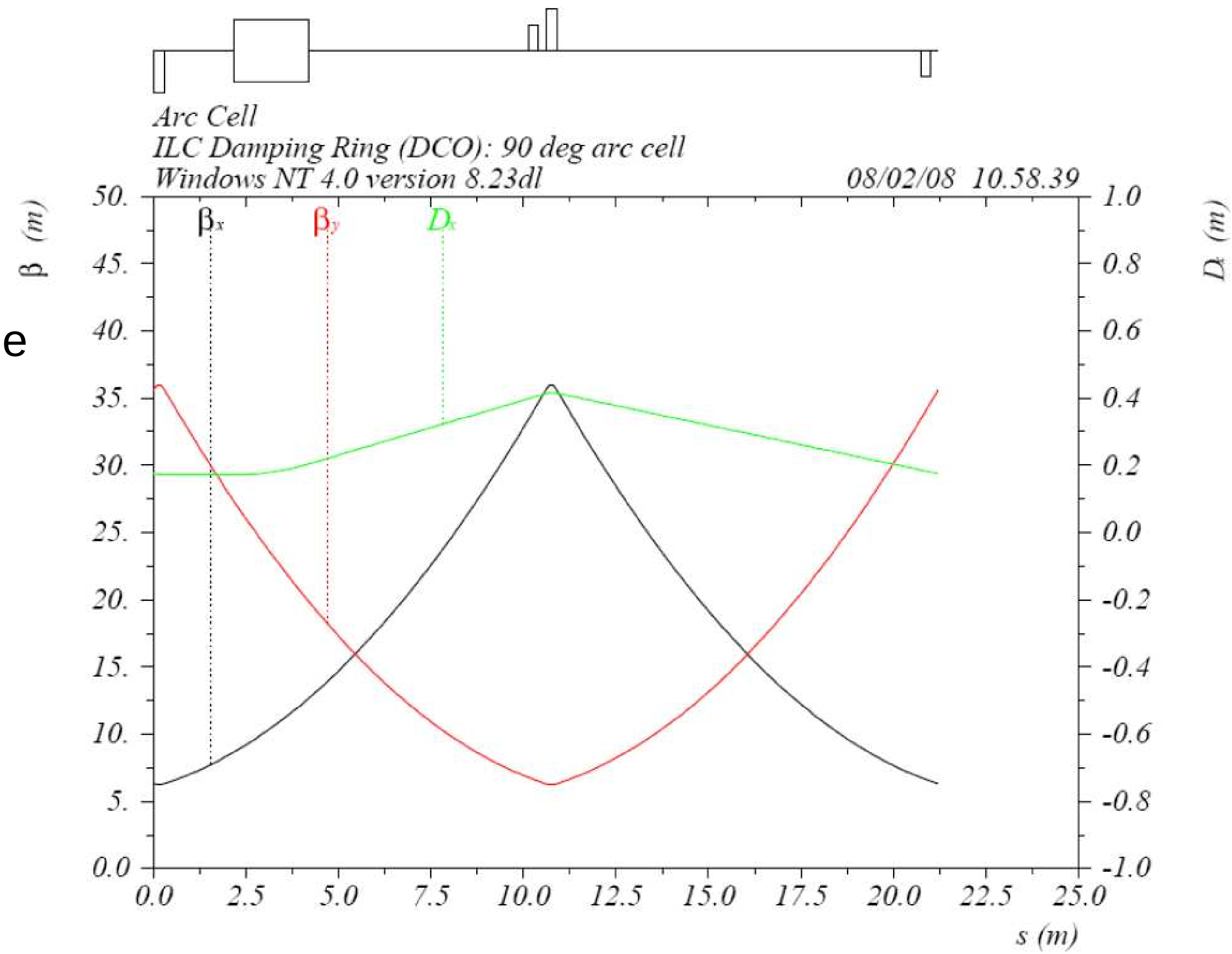
DCO Design Parameters

Beam energy	5 GeV
Circumference	6476.440 m
RF frequency	650 MHz
Harmonic number	14042
Transverse damping time	21.0 ms
Natural rms bunch length	6.00 mm
Natural rms energy spread	1.27×10^{-3}

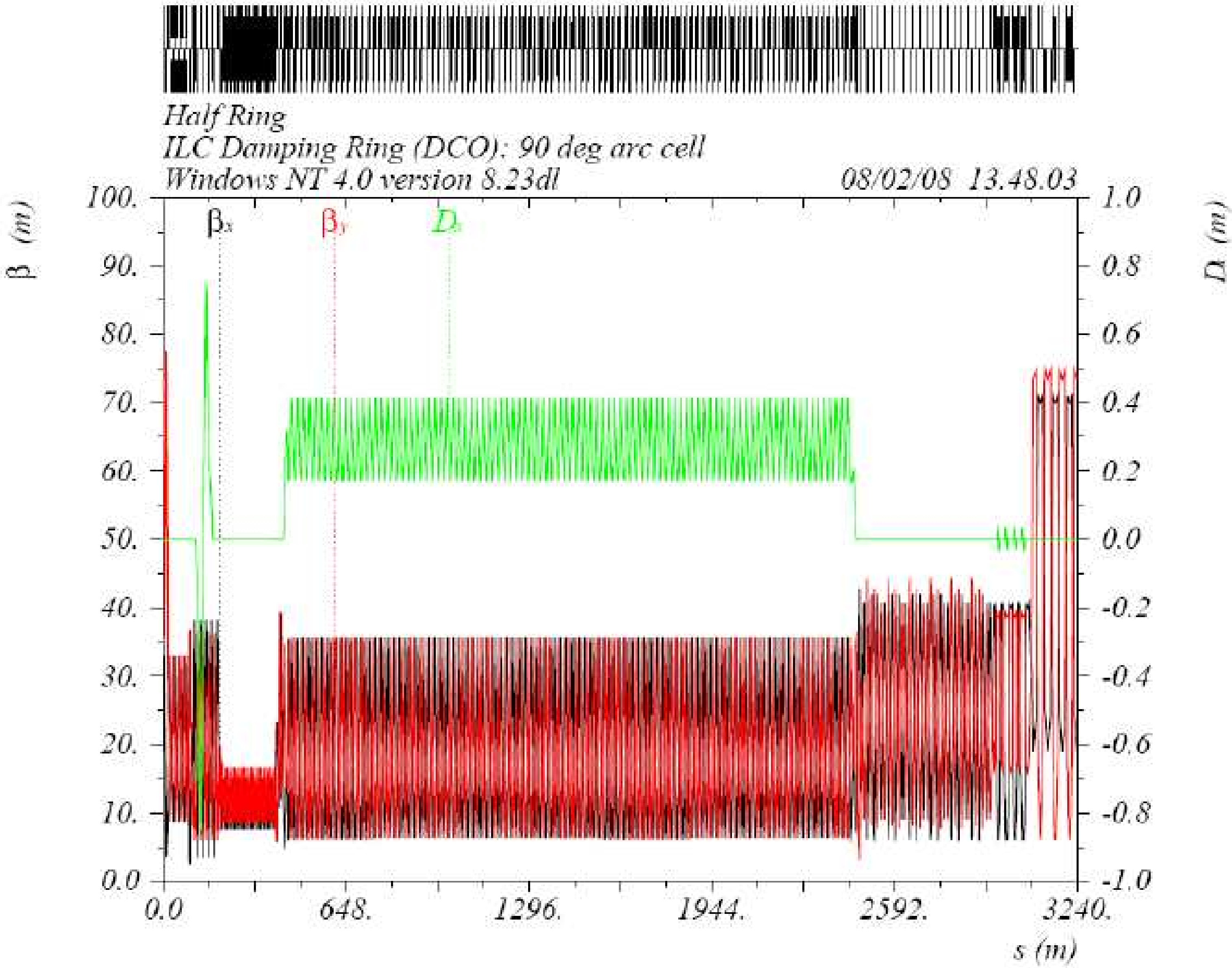
Phase advance per arc cell (approximate)	72°	90°	100°
Momentum compaction factor	2.80×10^{-4}	1.73×10^{-4}	1.29×10^{-4}
Normalised natural emittance	6.53 μm	4.70 μm	4.27 μm
RF voltage	31.6 MV	21.1 MV	17.2 MV
RF acceptance	2.35%	1.99%	1.72%
Synchrotron tune	0.061	0.038	0.028
Horizontal tune	64.750	75.200	80.450
Natural horizontal chromaticity	-76.5	-95.1	-106.9
Vertical tune	61.400	71.400	75.900
Natural vertical chromaticity	-75.6	-93.4	-103.5

Arc Cell

The arc cell design is a slightly non-standard FODO cell which utilizes relatively little dipole in each cell to help control the dispersion in the design.

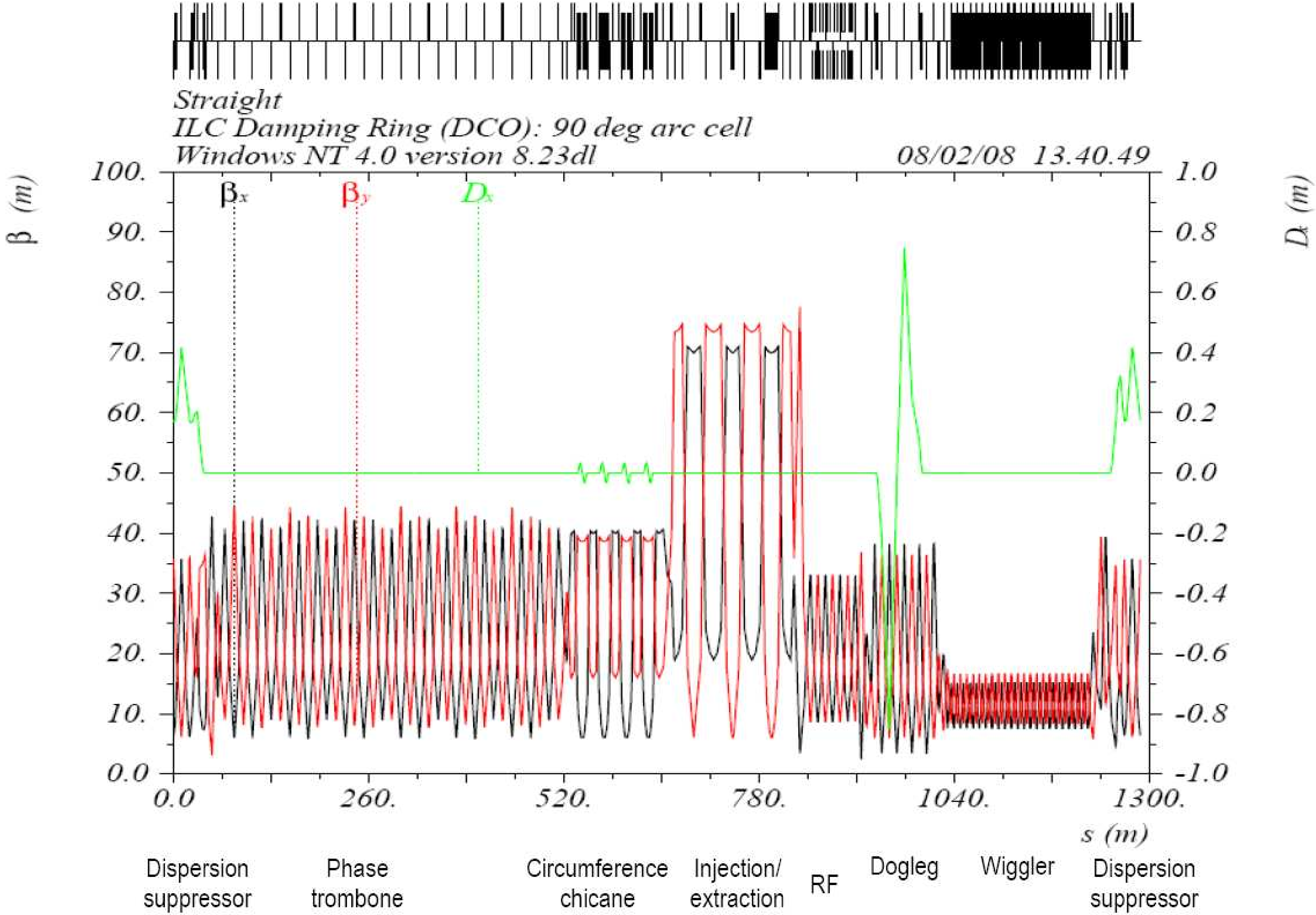


Half Ring



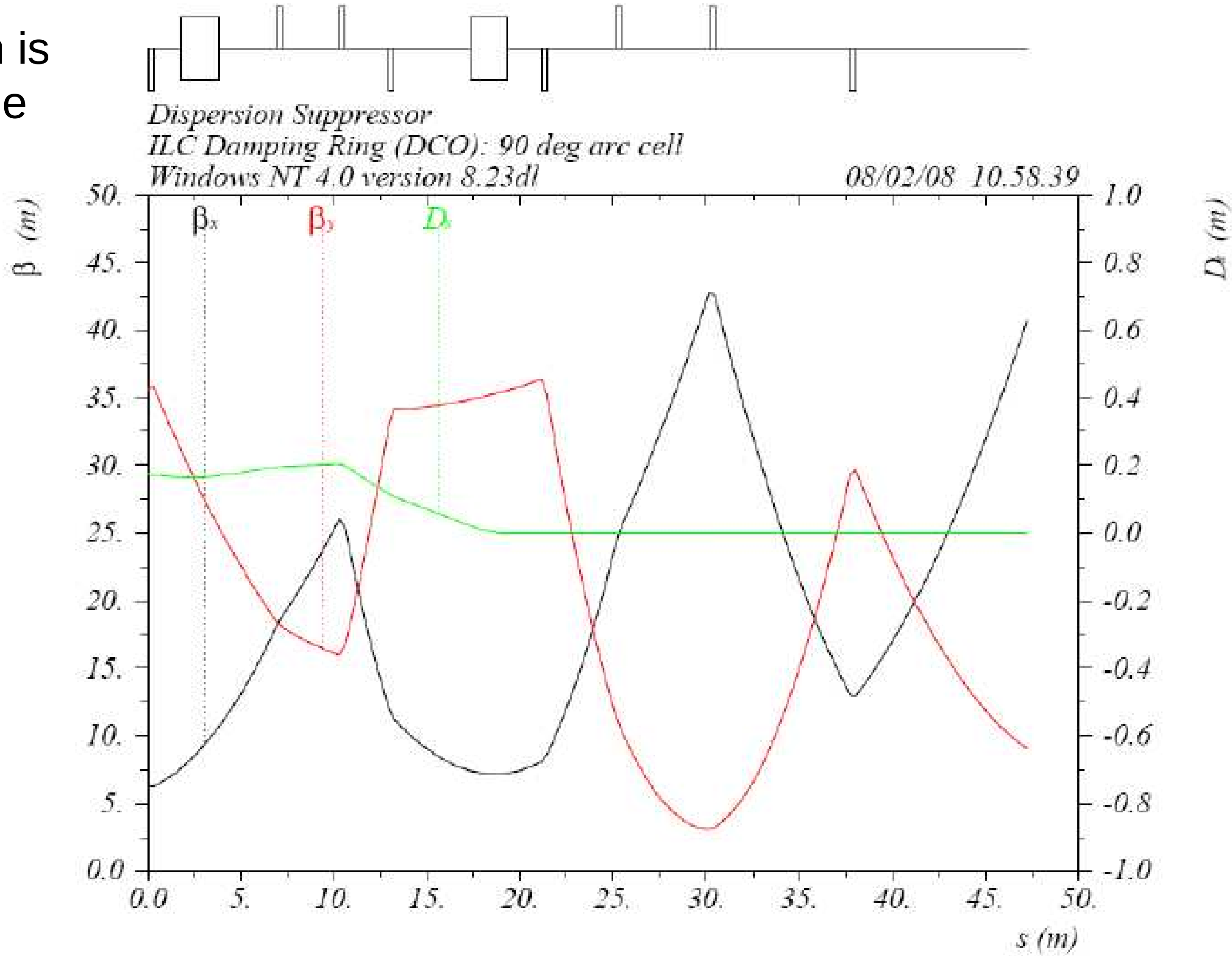
Half Ring
ILC Damping Ring (DCO): 90 deg arc cell
Windows NT 4.0 version 8.23dl
08/02/08 13.48.03
β (m)
D (m)
β_x
β_y
D_x
100.
90.
80.
70.
60.
50.
40.
30.
20.
10.
0.0
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
0.0
648.
1296.
1944.
2592.
3240.
s (m)

DCO Straight Section



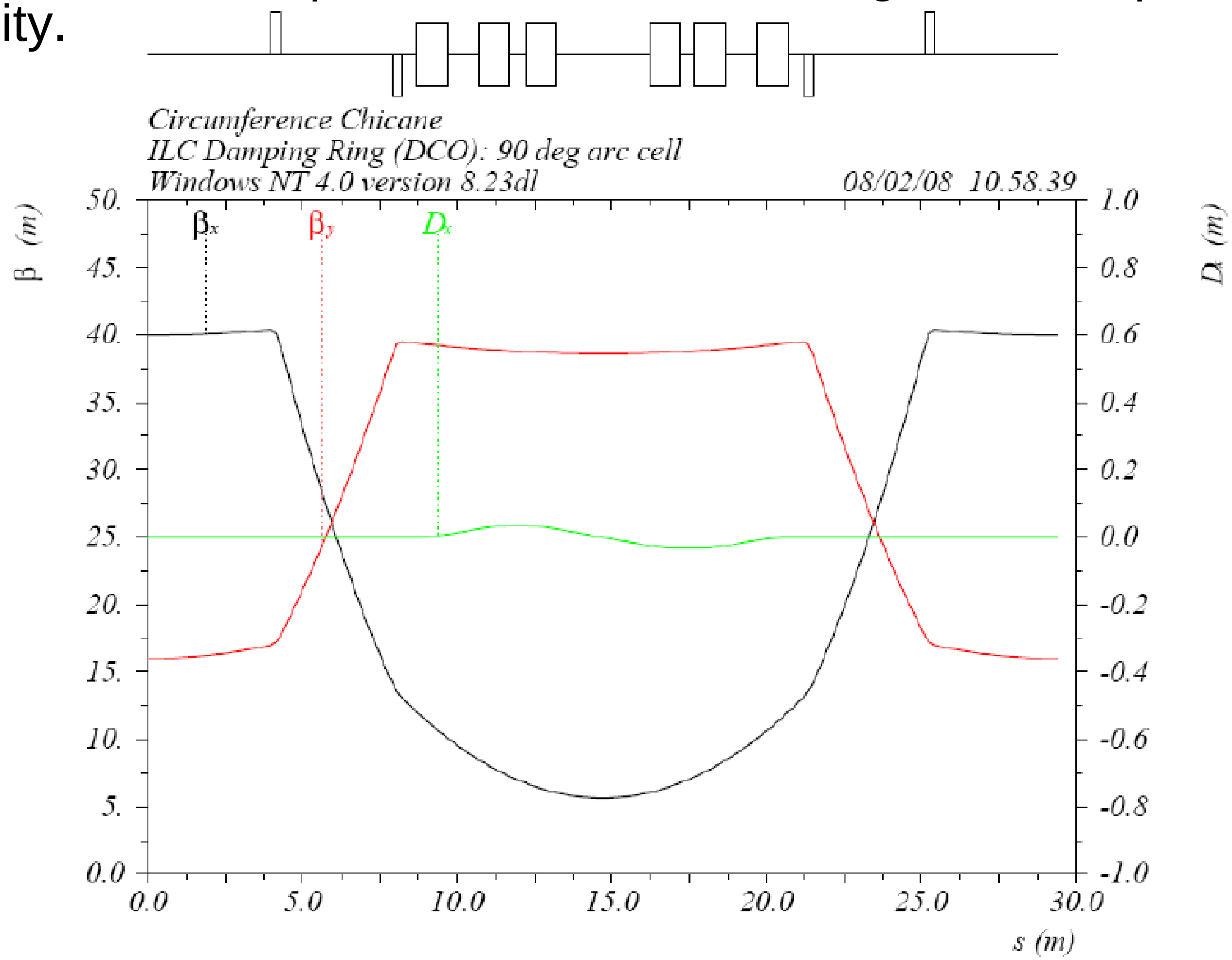
Dispersion Suppressor Section

A dispersion suppressor section is utilized to match the arcs with the zero dispersion straight sections



Chicanes

Because the ring RF frequency must be locked to the main linac RF, an important feature of the DR lattice is the need to adjust the circumference of the ring while maintaining a fixed RF frequency. Estimates of our ability to maintain the circumference suggest that adjustments on the order of ±1 cm are required. A set of 4 chicanes, with 6 dipoles each, in each straight section provide this range of flexibility.



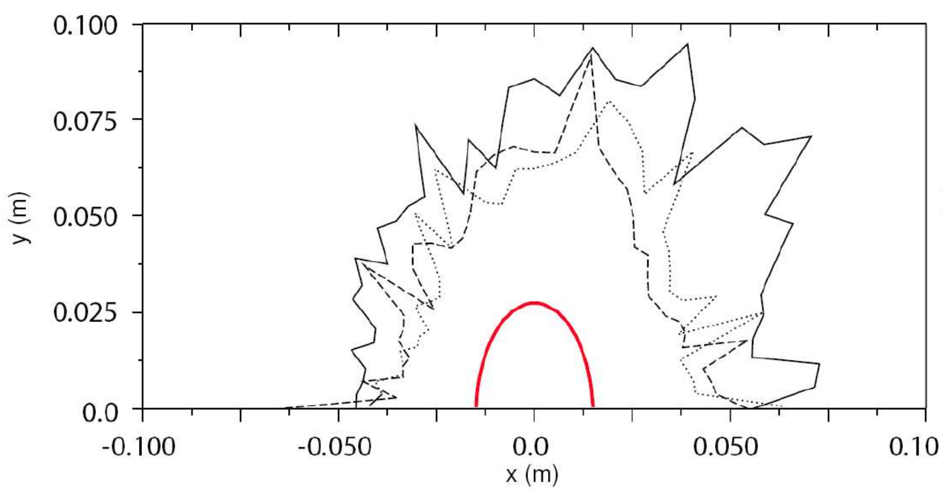
Other Features of the DCO Lattice

Other key features of the DCO lattice include:

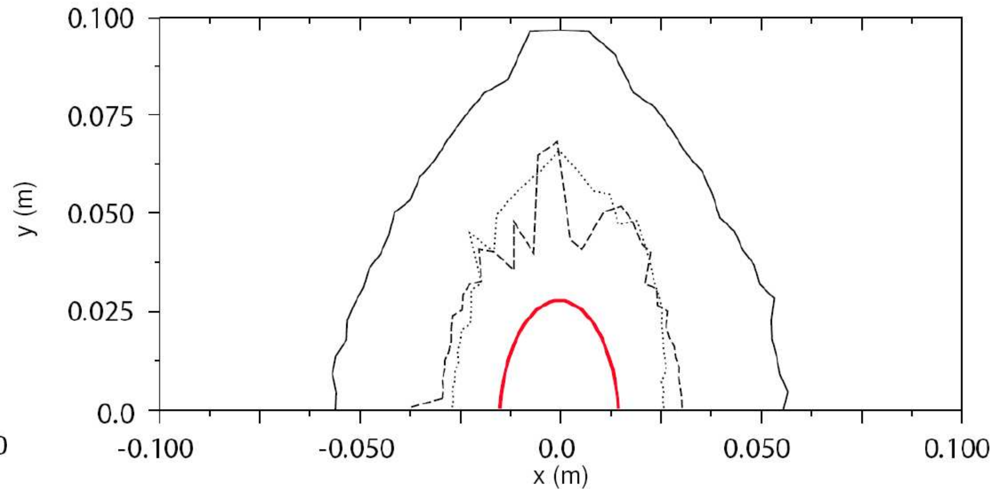
- Space in the injection and extraction optics to accommodate up to 33 kicker modules
  - Each module includes a stripline kicker of 30 cm length and 20 mm gap
  - 30 modules with the plates operating at ±7 kV are required for operation
- Space in the straights for up to 24 RF cavities.
  - Assuming 1.7 MV per module, 19 cavities are required to provide a 6 mm bunch length in the high momentum compaction ($\alpha_c = 2.8\times10^{-4}$) configuration
- The dogleg sections provide 2 m transverse shift of the beamline after each wiggler straight
  - The dogleg will allow installation of a photon dump to handle the forward radiation from each wiggler section
  - It will also serve to protect sensitive downstream hardware from the wiggler radiation fan.
  - This arrangement allows the RF and wiggler sections to be quite close and hence minimizes the amount of cryogenic transfer line required.

Dynamic Aperture

72° arc cell with $\alpha_c = 2.8 \times 10^{-4}$



90° arc cell with $\alpha_c = 1.7 \times 10^{-4}$



Dynamic aperture plots show the maximum initial amplitudes of stable trajectories. It is customary to overlay either the injected or equilibrium beam size on the plot. Significant margin is usually desirable in a design because machine errors will degrade it.

- Dotted lines indicate particles with ±0.5% energy deviations
- Solid black line indicates on energy particles
- Red ellipse shows the maximum injected coordinates for the positron beam

An ongoing area of optimization is the relatively poor DA for the 100° arc cell

Summary

During today's lecture, we have reviewed the basics of storage ring physics with particular attention on the effect know as radiation damping which is central to the operation of storage and damping rings. We have also had an overview of the key design elements presently incorporated into the damping ring lattice. The homework problems will provide an opportunity to become more familiar with some of these issues.

Tomorrow we will look in greater detail at specific systems and specific physics effects which play significant roles in the successful operation of a damping ring.

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