Two-loop hadronic corrections to Bhabha scattering

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$$e^+ + e^- \quad \longrightarrow \quad e^+ + e^-$$

Important process to determine the luminosity:

- at high-energy colliders (LEP,SLC) in the small-angle region
- at flavor factories (BABAR, BELLE, DAΦNE,...) in the large-angle region
- at ILC in the large-angle region (luminosity spectrum)

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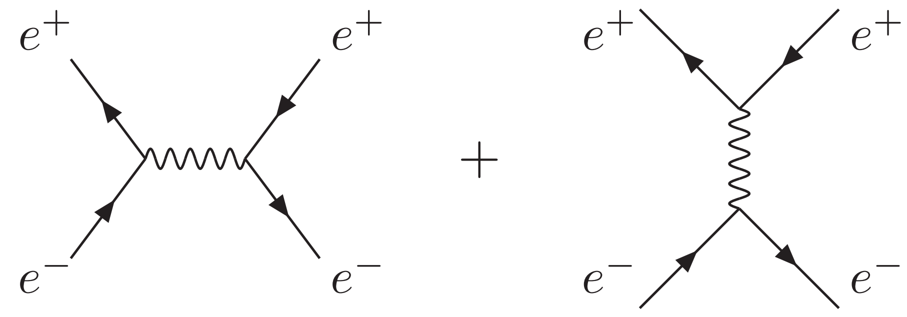
Theoretical compitations at two-loop in QED:

- Corrections with massless electrons (Bern-Dixon-Ghinculov '00)
- Photonic corrections with small electron mass (Penin '06, Becher-Melnikov '07)
- Corrections from light leptons (Actis-Czakon-Gluza-Riemann '07, Becher-Melnikov '07)
- Leptonic corrections (Bonciani-Ferroglia-Penin '07)
- Hadronicnic corrections (Actis-Czakon-Gluza-Riemann '07, Kühn-U. '08)

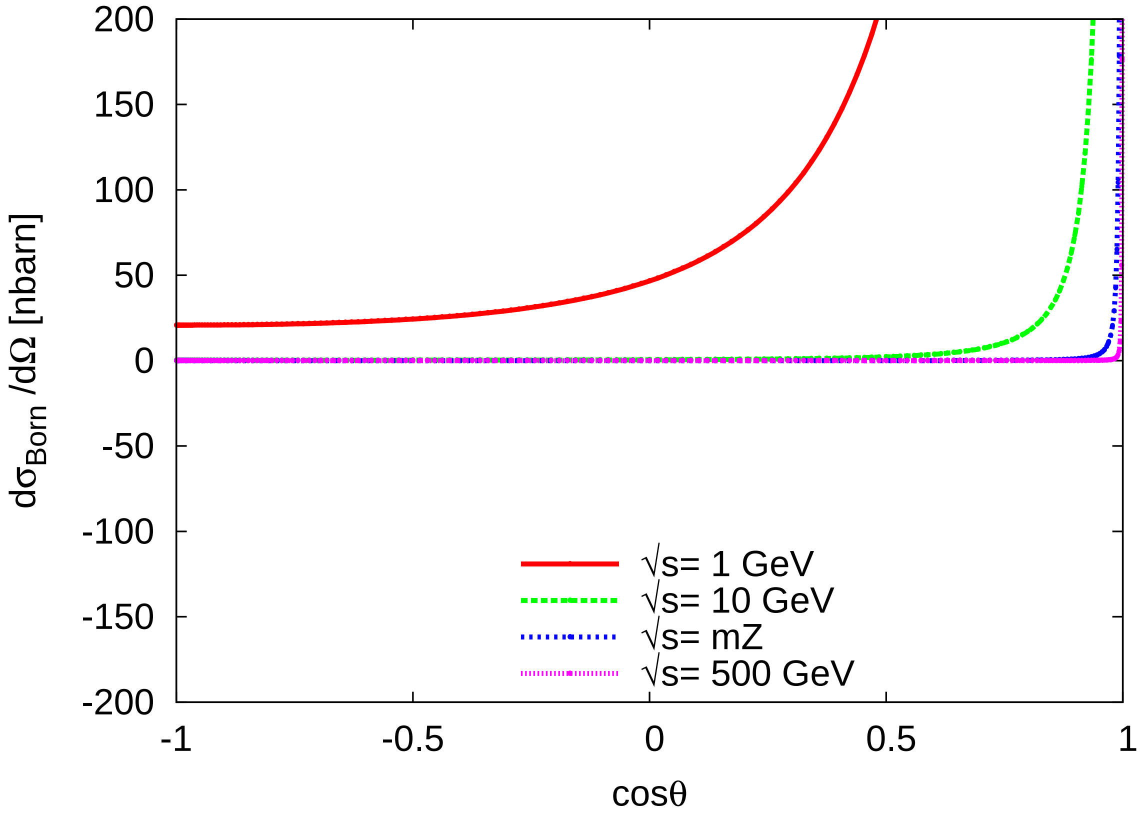
The Born cross section

$\mathcal{A}_{\rm Born} =$

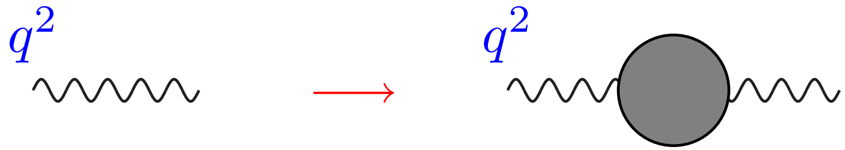


$$\frac{d\sigma_{\rm Born}}{d\Omega} = \frac{\alpha^2}{s}\left(\frac{1-x+x^2}{x}\right)^2$$

$$x = -\frac{t}{s} = \frac{1-\cos\theta}{2}$$



Hadronic corrections

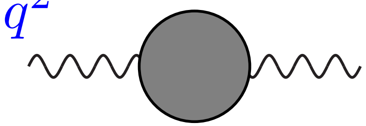


Hadronic corrections

q^2

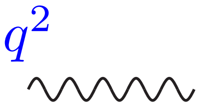
$\longrightarrow$

q^2

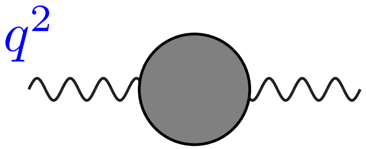


$$\frac{-\,i g_{\alpha\beta}}{q^2+i\epsilon} \qquad \longrightarrow \qquad \frac{-\,i g_{\alpha\delta}}{q^2+i\epsilon}\; i\,(q^2 g^{\delta\epsilon} - q^{\delta} q^{\epsilon})\,\Pi(q^2)\;\frac{-\,i g_{\epsilon\beta}}{q^2+i\epsilon}$$

Hadronic corrections

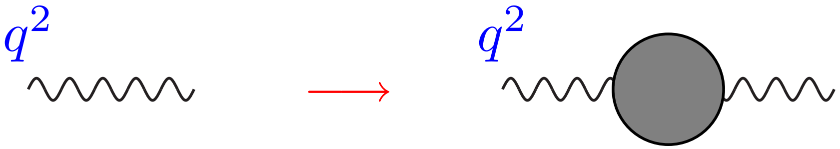


$\longrightarrow$



$$\frac{-\,i g_{\alpha\beta}}{q^2 + i\epsilon} \quad \longrightarrow \quad \frac{-\,i g_{\alpha\delta}}{q^2 + i\epsilon}\, i\,(q^2 g^{\delta\epsilon} - q^\delta q^\epsilon)\,\Pi(q^2)\,\frac{-\,i g_{\epsilon\beta}}{q^2 + i\epsilon} \quad \longrightarrow \quad \frac{-\,i g_{\alpha\beta}}{q^2 + i\epsilon}\,\Pi(q^2)$$

Hadronic corrections

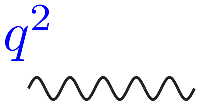


$$\frac{-\,ig_{\alpha\beta}}{q^2+i\epsilon} \longrightarrow \frac{-\,ig_{\alpha\delta}}{q^2+i\epsilon}\, i\,(q^2 g^{\delta\epsilon} - q^{\delta} q^{\epsilon})\,\Pi(q^2)\,\frac{-\,ig_{\epsilon\beta}}{q^2+i\epsilon} \longrightarrow \frac{-\,ig_{\alpha\beta}}{q^2+i\epsilon}\,\Pi(q^2)$$

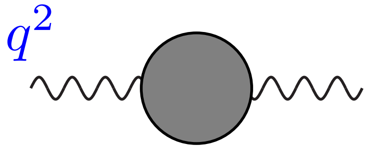
Dispersion relation

$$\Pi(q^2) = -\,\frac{q^2}{\pi}\,\int_{4m^2}^{\infty}\frac{dz}{z}\,\frac{\mathrm{Im}\Pi(z)}{q^2-z+i\epsilon}, \qquad \mathrm{Im}\Pi(z) = -\,\frac{\alpha}{3}\,R(z)$$

Hadronic corrections



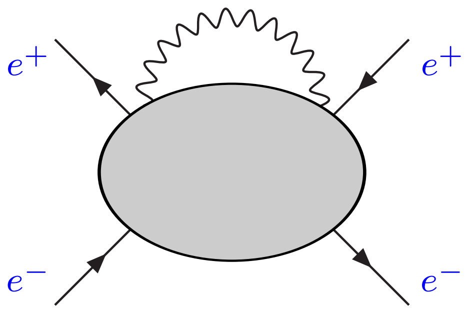
$\longrightarrow$



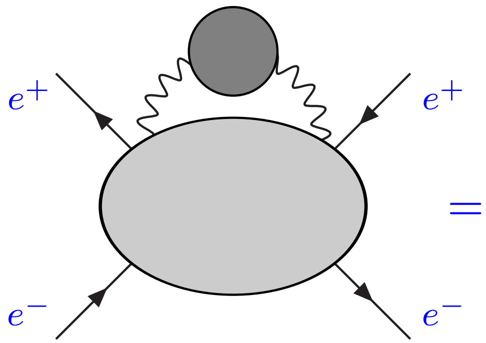
$$\frac{-ig_{\alpha\beta}}{q^2+i\epsilon} \longrightarrow \frac{-ig_{\alpha\delta}}{q^2+i\epsilon}\, i\,(q^2 g^{\delta\epsilon} - q^\delta q^\epsilon)\,\Pi(q^2)\,\frac{-ig_{\epsilon\beta}}{q^2+i\epsilon} \longrightarrow \frac{-ig_{\alpha\beta}}{q^2+i\epsilon}\,\Pi(q^2)$$

Dispersion relation

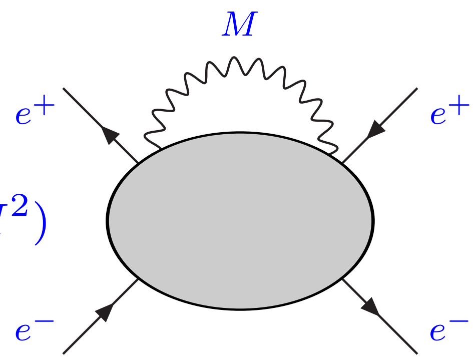
$$\Pi(q^2) = \frac{\alpha}{3\pi}\int_{4m^2}^{\infty} dz\, \frac{q^2}{z}\, \frac{R(z)}{q^2 - z + i\epsilon},$$



$\longrightarrow$



$$= \frac{\alpha}{3\pi}\int_{4m^2}^{\infty} \frac{dM^2}{M^2}\, R(M^2)$$

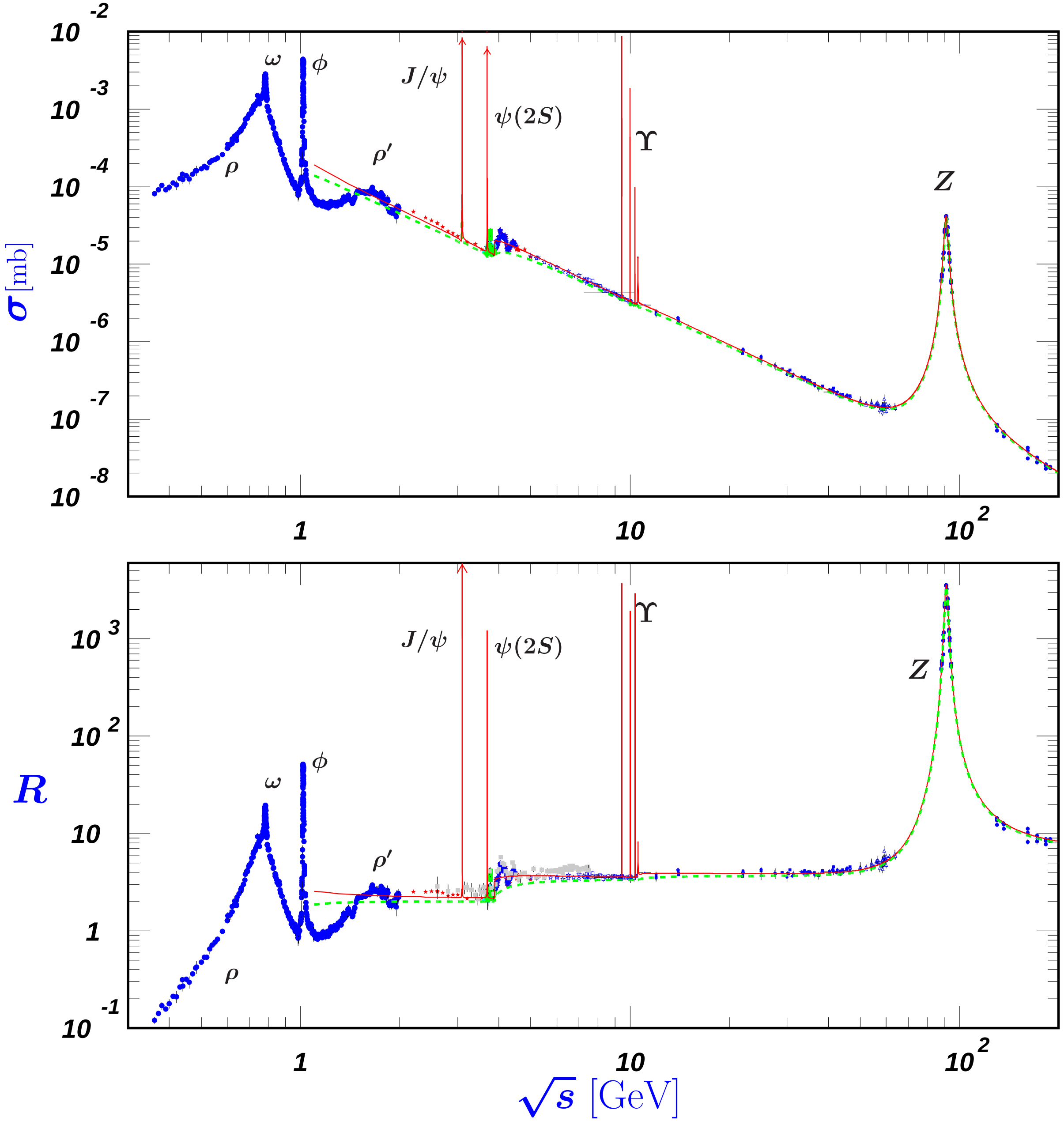


$$R(s) = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-, \text{QED}sp)}$$

$\downarrow$

leptonic case

$$R_l(s) = \left(1 + \frac{4m_l^2}{2\,s}\right)\sqrt{1 - \frac{4m_l^2}{s}}.$$



Three classes of contributions:

1. Vacuum polarization insertion
2. Reducible vertex and box corrections
3. Irreducible vertex and box corrections

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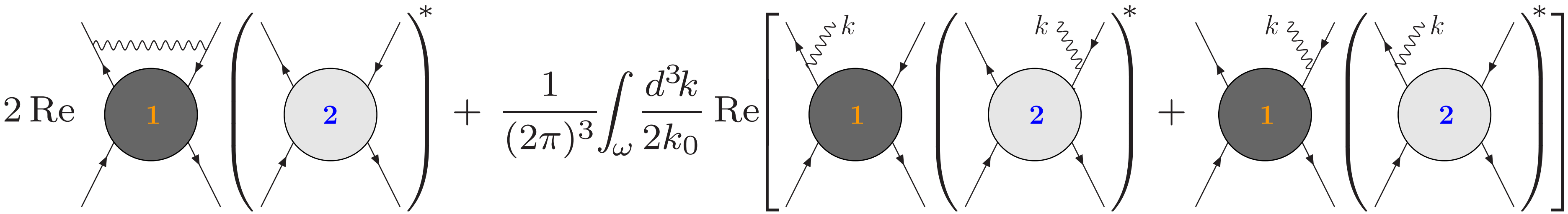
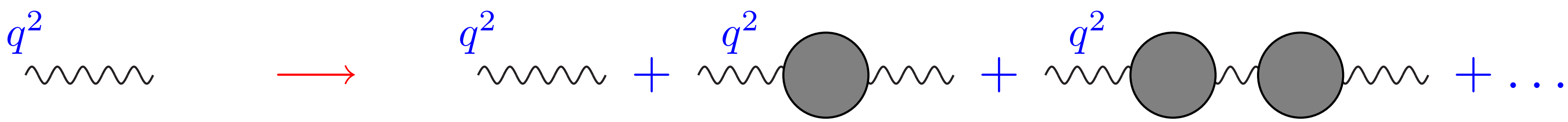
Inclusion of real soft photons



$$k = (k_0, \vec{k}) \qquad k_0^2 = \vec{k}^2 + \lambda^2 \qquad |\vec{k}| < \omega$$

1. Vacuum polarization insertion

- **Resummation:**



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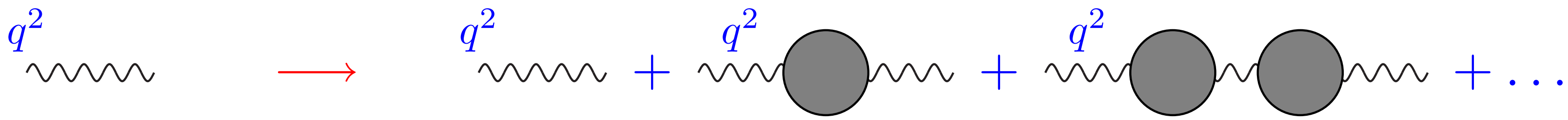

$$\frac{-\,ig_{\alpha\beta}}{q^2+i\epsilon} \longrightarrow \frac{-\,ig_{\alpha\beta}}{q^2+i\epsilon}\left[1+\Pi(q^2)+\Pi(q^2)^2+\ldots\right]$$

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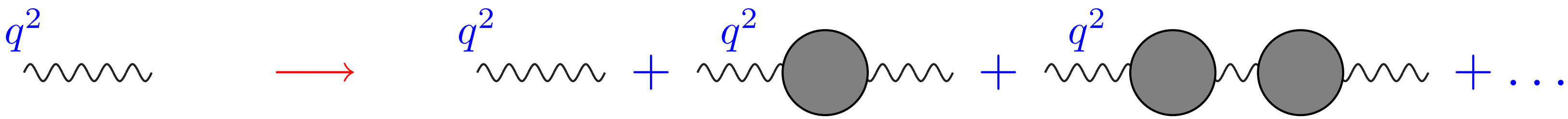

$$\frac{-ig_{\alpha\beta}}{q^2+i\epsilon} \longrightarrow \frac{-ig_{\alpha\beta}}{q^2+i\epsilon}\left[1+\Pi(q^2)+\Pi(q^2)^2+\dots\right] = \frac{-ig_{\alpha\delta}}{q^2+i\epsilon}\,\frac{1}{1-\Pi(q^2)}$$

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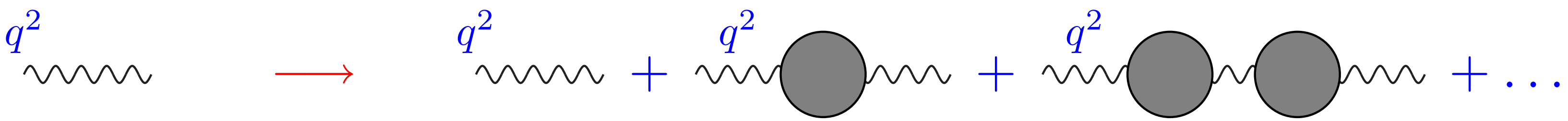

$$\frac{-ig_{\alpha\beta}}{q^2+i\epsilon} \longrightarrow \frac{-ig_{\alpha\beta}}{q^2+i\epsilon}\left[1+\Pi(q^2)+\Pi(q^2)^2+\dots\right] = \frac{-ig_{\alpha\delta}}{q^2+i\epsilon}\,\frac{1}{1-\Pi(q^2)}$$

$$\frac{d\sigma_\Pi}{d\Omega} = \frac{\alpha^2}{s}\left\{ \frac{1-2x+2x^2}{2}\left|\frac{1}{1-\Pi(s)}\right|^2 + \frac{2-2x+x^2}{2\,x^2}\left|\frac{1}{1-\Pi(t)}\right|^2 - \frac{(1-x)^2}{x}\mathrm{Re}\,\frac{1}{[1-\Pi(s)]\,[1-\Pi(t)]} \right\}$$

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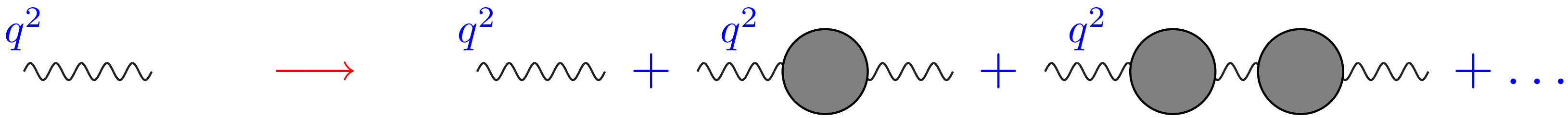

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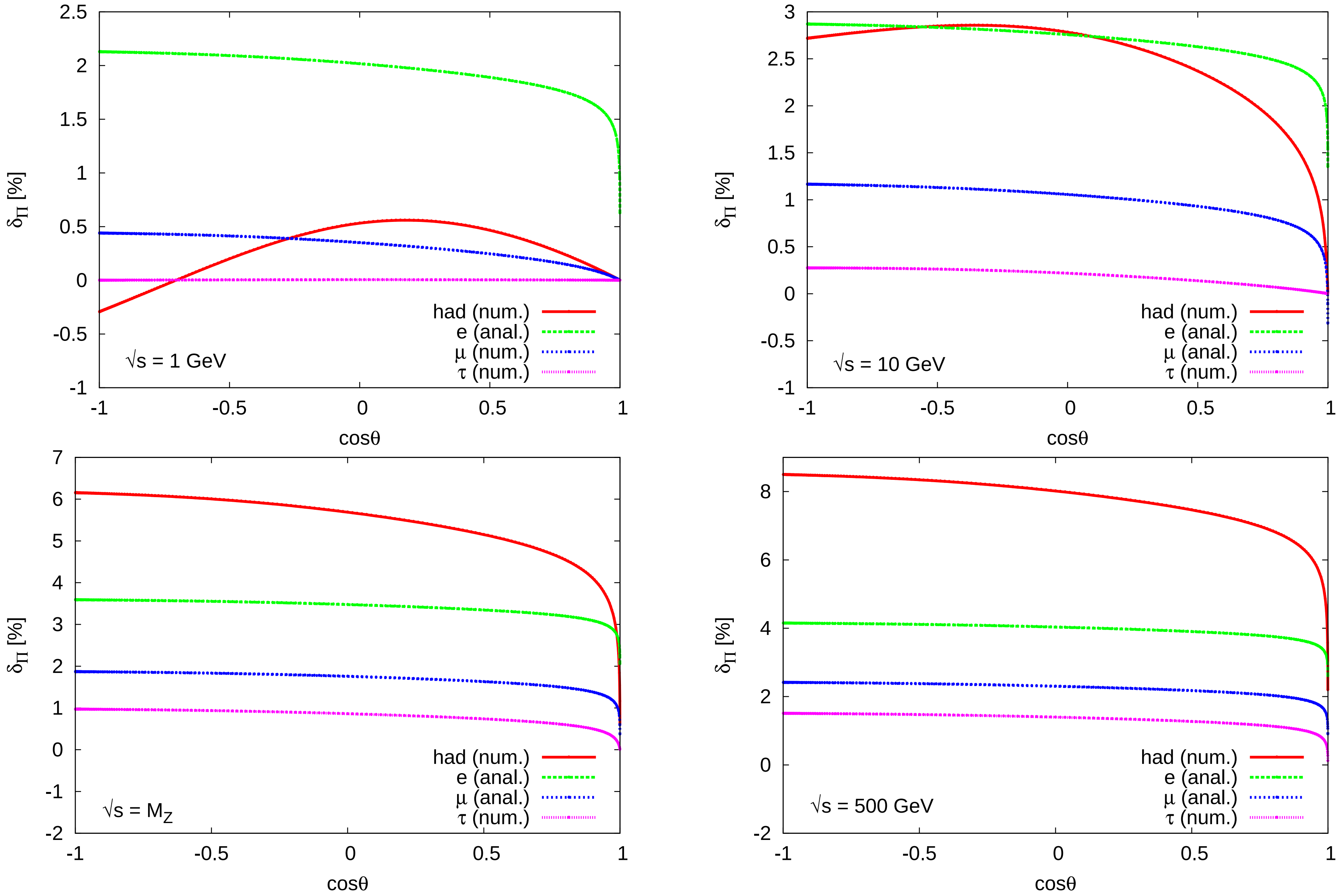
- **Evaluation of $\Pi(q^2)$**

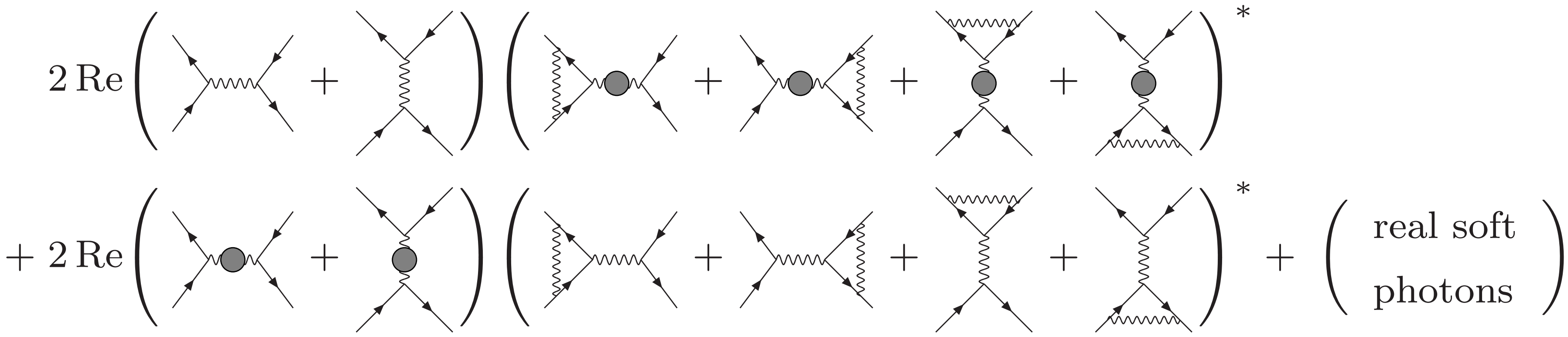
$$\Pi(t) = \frac{\alpha}{3\pi}\int_0^1 dy\,\frac{t}{yt-4m^2}\,R\left(\frac{4m^2}{y}\right),$$

$$\Pi(s) = \frac{\alpha}{3\pi}\left\{\ln\left(1-\frac{s}{4m^2-i\epsilon}\right)R(s) + \int_0^1 dy\,\frac{s}{ys-4m^2}\left[R\left(\frac{4m^2}{y}\right)-R(s)\right]\right\}$$

$$\delta_\Pi = \frac{d\sigma_\Pi - d\sigma_{\rm Born}}{d\sigma_{\rm Born}}$$



2. Reducible vertices ...



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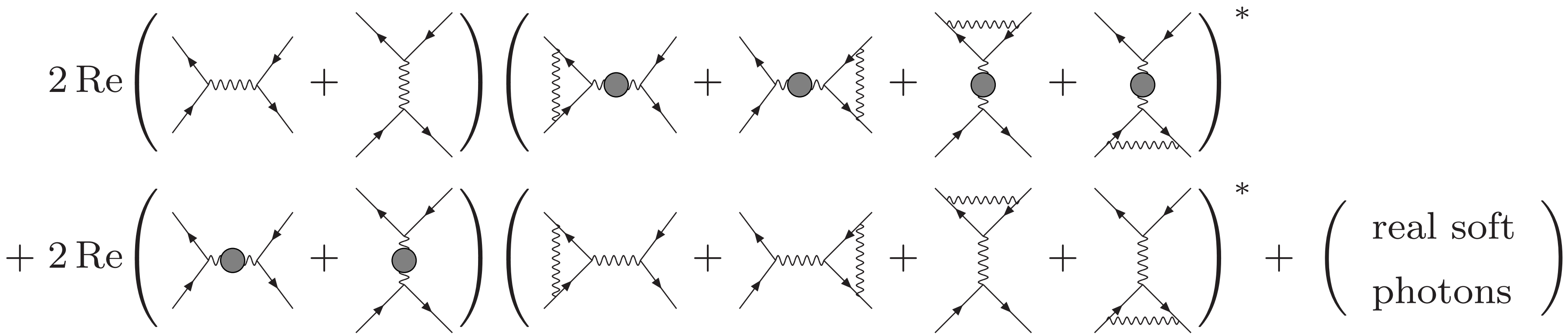


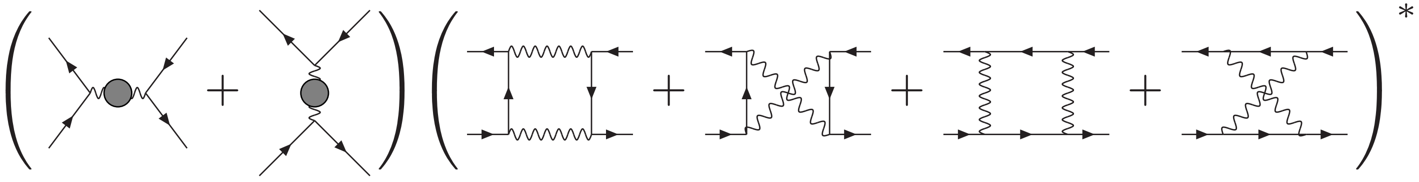
$$\frac{d\sigma_{\mathrm{red,V}}}{d\Omega} = \frac{\alpha^3}{s\pi}\left\{ \frac{1-2x+2x^2}{2}\left[4\,V_s^\gamma\,\mathrm{Re}\,\Pi(s)\right] + \frac{2-2x+x^2}{2\,x^2}\left[4\,V_t^\gamma\,\Pi(t)\right] \right.$$

$$\left. - \frac{(1-x)^2}{x}\left[(V_s^\gamma + V_t^\gamma)\left(\mathrm{Re}\,\Pi(s) + \Pi(t)\right) + \pi\left(\ln\frac{\lambda^2}{s} + \frac{3}{2}\right)\mathrm{Im}\,\Pi(s)\right]\right\}$$

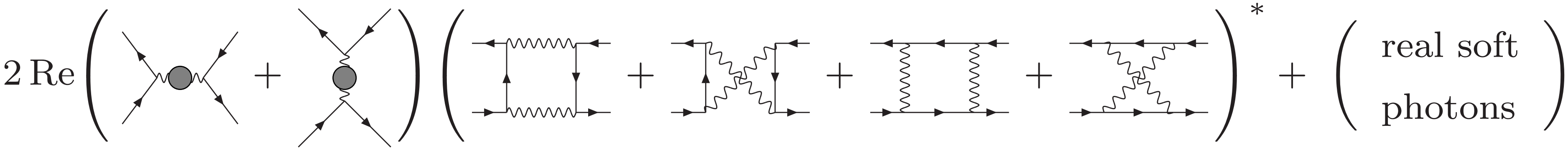
$$V_s^\gamma = 2\ln\frac{2\omega}{\sqrt{s}}\left(\ln\frac{s}{m_e^2} - 1\right) + \frac{3}{2}\ln\frac{s}{m_e^2} + 2\,\zeta(2) - 2$$

$$V_t^\gamma = 2\ln\frac{2\omega}{\sqrt{s}}\left(\ln\frac{-t}{m_e^2} - 1\right) + \frac{3}{2}\ln\frac{-t}{m_e^2} - \ln\frac{-t}{s}\,\ln\frac{-u}{s} - \mathrm{Li}_2\left(\frac{-t}{s}\right) - 2$$

... and reducible boxes

2 Re  + (real soft photons)

... and reducible boxes



$$\frac{d\sigma_{\rm red,B}}{d\Omega} = \frac{\alpha^3}{s\pi}\Bigg\{ \frac{1-2x+2x^2}{2}\left[2\,B_s^\gamma\,{\rm Re}\,\Pi(s) + 2\pi\ln\frac{t}{u}\,{\rm Im}\,\Pi(s)\right] + \frac{2-2x+x^2}{2\,x^2}\left[2\,B_t^\gamma\,\Pi(t)\right]$$

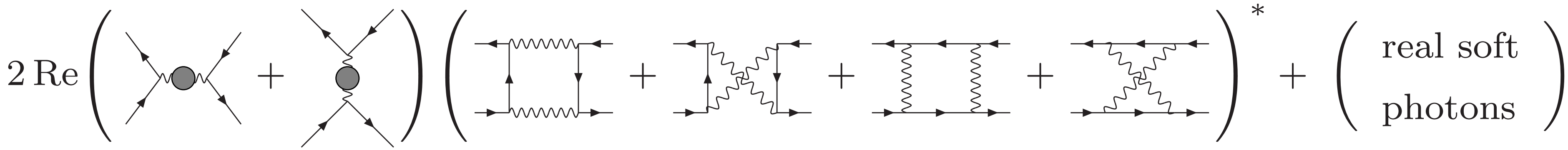
$$- \frac{(1-x)^2}{x}\left[B_t^\gamma{\rm Re}\Pi(s) + B_s^\gamma\Pi(t) - \pi\ln\frac{\lambda^2}{-t}\,{\rm Im}\Pi(s)\right]$$

$$+\ldots\Bigg\}$$

$$B_s^\gamma = 2\ln\frac{2\omega}{\sqrt{s}}\ln\frac{t}{u} + \frac{1}{2}\ln^2\frac{-t}{s} - \frac{1}{2}\ln^2\frac{-u}{s} - \ln\frac{-t}{s}\ln\frac{-u}{s} - 2\,{\rm Li}_2\left(\frac{-t}{s}\right) + \zeta(2),$$

$$B_t^\gamma = -2\ln\frac{2\omega}{\sqrt{s}}\ln\frac{-u}{s} - \frac{1}{2}\ln^2\frac{-u}{s} + \ln\frac{-t}{s}\ln\frac{-u}{s} - {\rm Li}_2\left(\frac{-t}{s}\right),$$

... and reducible boxes



$$\frac{d\sigma_{\rm red,B}}{d\Omega} = \frac{\alpha^3}{s\pi}\Bigg\{ \frac{1-2x+2x^2}{2}\left[2\,B_s^\gamma\,{\rm Re}\,\Pi(s) + 2\pi\,\ln\frac{t}{u}\,{\rm Im}\,\Pi(s)\right] + \frac{2-2x+x^2}{2\,x^2}\left[2\,B_t^\gamma\,\Pi(t)\right]$$

$$- \frac{(1-x)^2}{x}\left[B_t^\gamma{\rm Re}\Pi(s) + B_s^\gamma\Pi(t) - \pi\ln\frac{\lambda^2}{-t}\,{\rm Im}\Pi(s)\right]$$

$$-\,{\rm Re}\left[\Big(B^\gamma(s,t) - B^\gamma(s,u)\Big)\,\Pi^*(s)\right] - {\rm Re}\left[\Big(B^\gamma(t,s) - B^\gamma(t,u)\Big)\,\Pi(t)\right]$$

$$+{\rm Re}\left[xB^\gamma(t,s)\Pi^*(s) + \frac{1}{x}B^\gamma(s,t)\Pi(t)\right]\Bigg\}$$

$$B_s^\gamma = 2\ln\frac{2\omega}{\sqrt{s}}\ln\frac{t}{u} + \frac{1}{2}\ln^2\frac{-t}{s} - \frac{1}{2}\ln^2\frac{-u}{s} - \ln\frac{-t}{s}\ln\frac{-u}{s} - 2\,{\rm Li}_2\left(\frac{-t}{s}\right) + \zeta(2),$$

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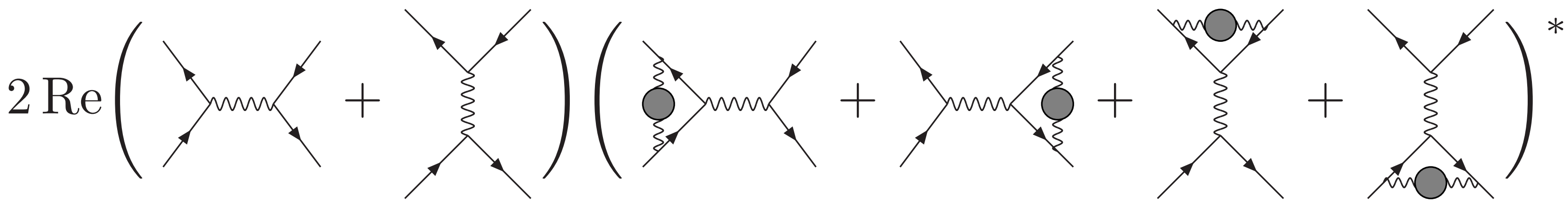
$$B^\gamma(a,b) = -\frac{a+b}{2\,a}\ln\frac{b}{a+i\,\epsilon} + \frac{a+2b}{4\,a}\left(\ln^2\frac{b}{a+i\,\epsilon} + \pi^2\right).$$

3. Irreducible vertices ...



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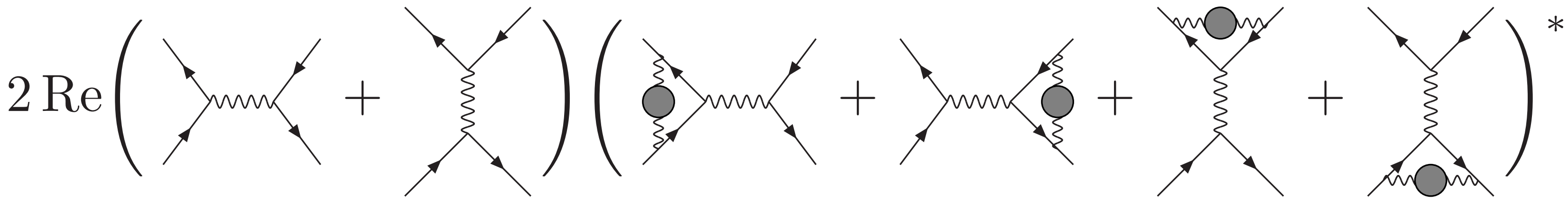

$$\frac{d\sigma_{\rm V}}{d\Omega} = \frac{\alpha^3}{s\,\pi}\left\{ \frac{1-2x+2x^2}{2}\left[4{\rm Re}V(s)\right] + \frac{2-2x+x^2}{2\,x^2}\left[4V(t)\right] - \frac{(1-x)^2}{x}\left[2V(t)+2{\rm Re}V(s)\right]\right\}$$

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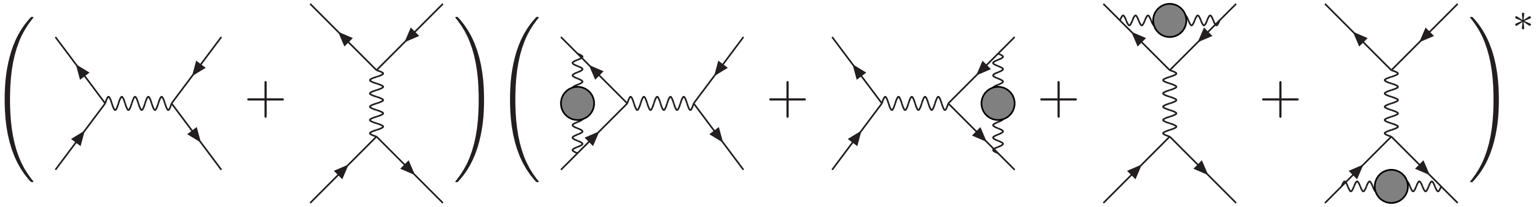
2 Re 

$$\frac{d\sigma_{\rm V}}{d\Omega} = \frac{\alpha^3}{s\,\pi}\left\{\frac{1-2x+2x^2}{2}\Big[4{\rm Re}V(s)\Big] + \frac{2-2x+x^2}{2\,x^2}\Big[4V(t)\Big] - \frac{(1-x)^2}{x}\Big[2V(t)+2{\rm Re}V(s)\Big]\right\}$$

- **Second dispersion integral:**

$$V(q^2) = \frac{\alpha}{3\pi}\int_{4m^2}^{\infty}\frac{dz}{z}R(z)\,\rho\,(q^2, z-i\epsilon)$$

$$\rho\,(q^2,z) = -\frac{7}{8} - \frac{z}{2q^2} + \frac{1}{2}\left(\frac{3}{2}+\frac{z}{q^2}\right)\ln\frac{-z}{q^2} + \frac{1}{2}\left(1+\frac{z}{q^2}\right)^2\left[\zeta(2) - {\rm Li}_2\left(1+\frac{z}{q^2}\right)\right].$$

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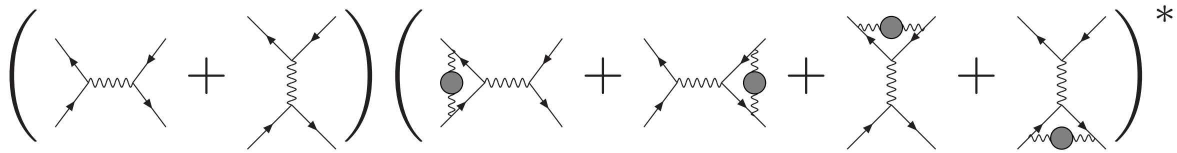
2 Re 

$$\frac{d\sigma_{\rm V}}{d\Omega} = \frac{\alpha^3}{s\,\pi}\left\{\frac{1-2x+2x^2}{2}\Big[4{\rm Re}V(s)\Big] + \frac{2-2x+x^2}{2\,x^2}\Big[4V(t)\Big] - \frac{(1-x)^2}{x}\Big[2V(t)+2{\rm Re}V(s)\Big]\right\}$$

- **Second dispersion integral:** $V(q^2) = \dfrac{\alpha}{3\pi}\displaystyle\int_{4m^2}^{\infty}\frac{dz}{z}R(z)\,\rho\,(q^2, z-i\epsilon)$

$$\rho\,(q^2,z) = -\frac{7}{8} - \frac{z}{2q^2} + \frac{1}{2}\left(\frac{3}{2}+\frac{z}{q^2}\right)\ln\frac{-z}{q^2} + \frac{1}{2}\left(1+\frac{z}{q^2}\right)^2\left[\zeta(2) - {\rm Li}_2\left(1+\frac{z}{q^2}\right)\right].$$

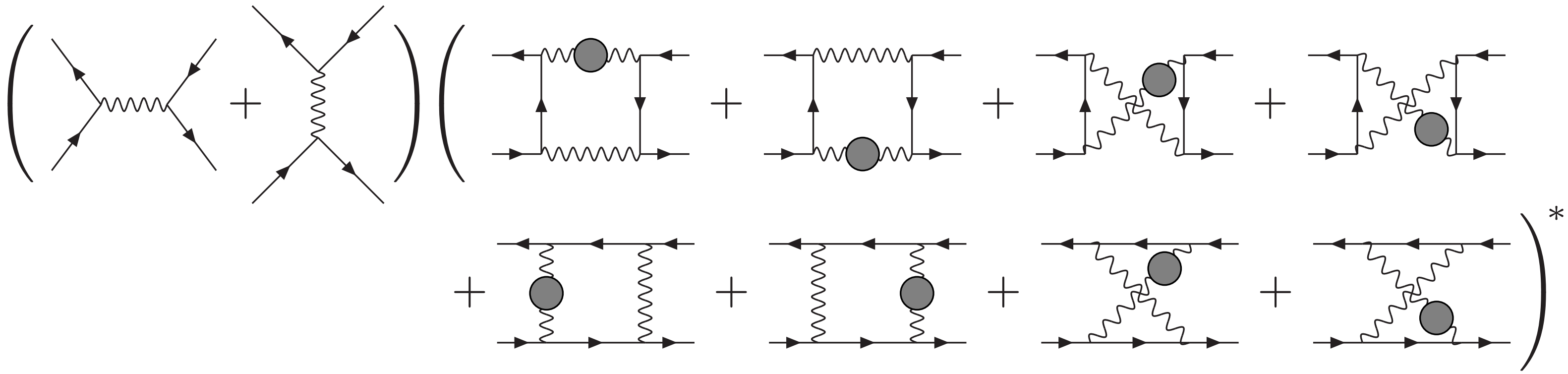
- **Evaluation of $V(q^2)$**

$$V(q^2) = \frac{\alpha}{3\pi}\left\{R(\infty)\int_0^1\frac{dy}{y}\rho\left(q^2,\frac{4m^2}{y}\right) + \int_0^1\frac{dy}{y}\rho\left(q^2,\frac{4m^2}{y}\right)\left[R\left(\frac{4m^2}{y}\right) - R(\infty)\right]\right\}$$

$$\int_0^1\frac{dy}{y}\rho\left(q^2,\frac{4m^2}{y}\right) = -\frac{1}{12}\ln^3(-r) - \ln(-r)\left[\zeta(2)+\frac{7}{8}+\frac{1}{4r}+\frac{1}{2}{\rm Li}_2\left(-\frac{1}{r}\right)\right]$$
$$+\left(\frac{3}{4}+\frac{1}{r}+\frac{1}{4r^2}\right)\Big[\zeta(2) - {\rm Li}_2(1+r)\Big] + \frac{15}{16} + \frac{1}{4r} - {\rm Li}_3\left(-\frac{1}{r}\right) \qquad r = \frac{q^2}{4m^2}$$

... and irreducible boxes

2 Re



+ (real soft photons)

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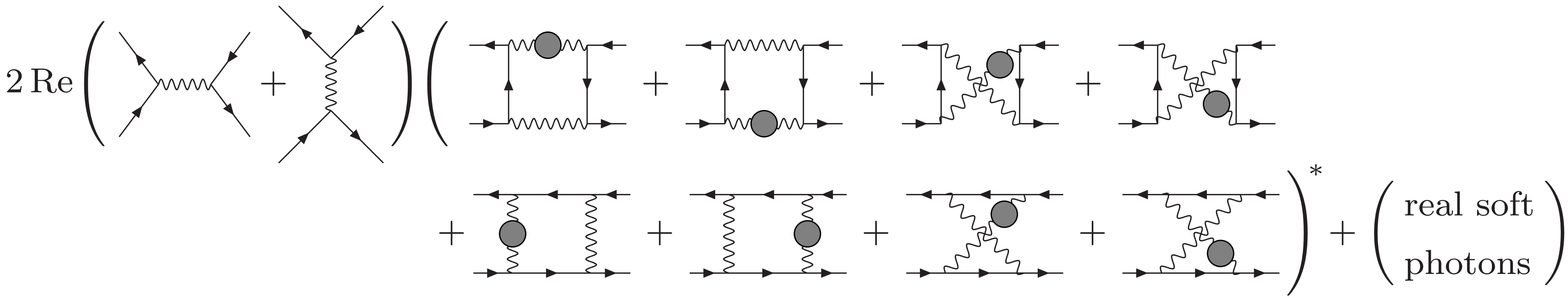
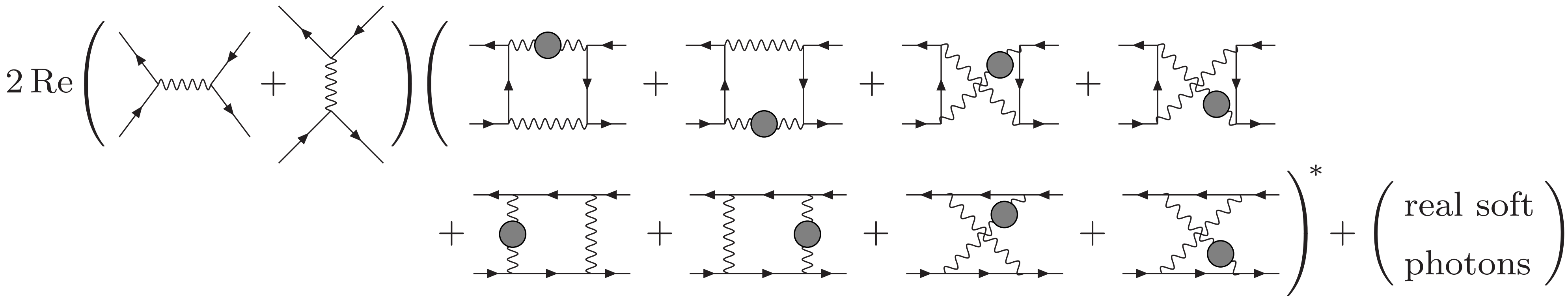


$$\frac{d\sigma_{\rm B}}{d\Omega} = \frac{\alpha^3}{s\pi}\left\{ \frac{1-2x+2x^2}{2}\Big[2B_s{\rm Re}\Pi(s)\Big] + \frac{2-2x+x^2}{2\,x^2}\Big[2B_t\Pi(t)\Big] - \frac{(1-x)^2}{x}\Big[B_t{\rm Re}\Pi(s)+B_s\Pi(t)\Big] \right.$$
$$\left. + \dots \right\}$$

$$B_s = 2\ln\frac{2\omega}{\sqrt{s}}\,\ln\frac{t}{u} - \ln\frac{-t}{s}\,\ln\frac{-u}{s} - 2\,{\rm Li}_2\left(\frac{-t}{s}\right) + \zeta(2) \qquad B_t = -2\ln\frac{2\omega}{\sqrt{s}}\,\ln\frac{-u}{s} - {\rm Li}_2\left(\frac{-t}{s}\right) + 3\,\zeta(2)$$

... and irreducible boxes



$$\frac{d\sigma_{\rm B}}{d\Omega} = \frac{\alpha^3}{s\pi}\Bigg\{ \frac{1-2x+2x^2}{2}\Big[2B_s{\rm Re}\Pi(s)\Big] + \frac{2-2x+x^2}{2\,x^2}\Big[2B_t\Pi(t)\Big] - \frac{(1-x)^2}{x}\Big[B_t{\rm Re}\Pi(s)+B_s\Pi(t)\Big]$$

$$- {\rm Re}\Big[B(s,t,u)-B(s,u,t)\Big] - {\rm Re}\Big[B(t,s,u)-B(t,u,s)\Big] + {\rm Re}\Big[xB(t,s,u)+\frac{1}{x}B(s,t,u)\Big]\Bigg\}$$

$$B_s = 2\ln\frac{2\omega}{\sqrt{s}}\ln\frac{t}{u} - \ln\frac{-t}{s}\ln\frac{-u}{s} - 2\,{\rm Li}_2\left(\frac{-t}{s}\right) + \zeta(2) \qquad B_t = -2\ln\frac{2\omega}{\sqrt{s}}\ln\frac{-u}{s} - {\rm Li}_2\left(\frac{-t}{s}\right) + 3\,\zeta(2)$$

- **Third dispersion integral**

$$B(a,b,c) = \frac{\alpha}{3\pi}\int_{4m^2}^{\infty}\frac{dz}{z}R(z)\left[\xi_A(a,b,c,z-i\epsilon)+\xi_B(a,b,c,z-i\epsilon)\right]$$

$$\xi_A(a,b,c,z) = \frac{c^2}{a\,(z-a)}\left[2\ln\frac{c}{b+i\epsilon}\ln\left(1-\frac{a}{z}\right)-\mathrm{Li}_2\left(1+\frac{b}{z}\right)+\mathrm{Li}_2\left(1+\frac{c}{z}\right)\right]$$

$$\xi_B(a,b,c,z) = \frac{c}{a}\left[\left(\frac{z}{a}-1\right)\ln\left(1-\frac{a}{z}\right)+\ln\frac{-b}{z}\right] + \frac{c-b-z}{a}\left[\ln\frac{b+i\epsilon}{-a}\ln\left(1-\frac{a}{z}\right)-\mathrm{Li}_2\left(1-\frac{a}{z}\right)+\mathrm{Li}_2\left(1+\frac{b}{z}\right)\right]$$

Third dispersion integral

$$B(a,b,c) = \frac{\alpha}{3\pi}\int_{4m^2}^{\infty}\frac{dz}{z}R(z)\left[\xi_A(a,b,c,z-i\epsilon)+\xi_B(a,b,c,z-i\epsilon)\right]$$

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Evaluation of $B(a,b,c)$

$$B_A(t,b,c) = \frac{\alpha}{3\pi}\int_0^1\frac{dy}{y}\,\xi_A\left(t,b,c,\frac{4m^2}{y}\right)R\left(\frac{4m^2}{y}\right)$$

$$B_A(s,b,c) = \frac{\alpha}{3\pi}\left\{R(s)\int_0^1\frac{dy}{y}\xi_A\left(s,b,c,\frac{4m^2}{y}\right) + \int_0^1\frac{dy}{y}\xi_A\left(s,b,c,\frac{4m^2}{y}\right)\left[R\left(\frac{4m^2}{y}\right) - R(s)\right]\right\}$$

$$B_B(a,b,c) = \frac{\alpha}{3\pi}\left\{R(\infty)\int_0^1\frac{dy}{y}\xi_B\left(a,b,c,\frac{4m^2}{y}\right) + \int_0^1\frac{dy}{y}\xi_B\left(a,b,c,\frac{4m^2}{y}\right)\left[R\left(\frac{4m^2}{y}\right) - R(\infty)\right]\right\}$$

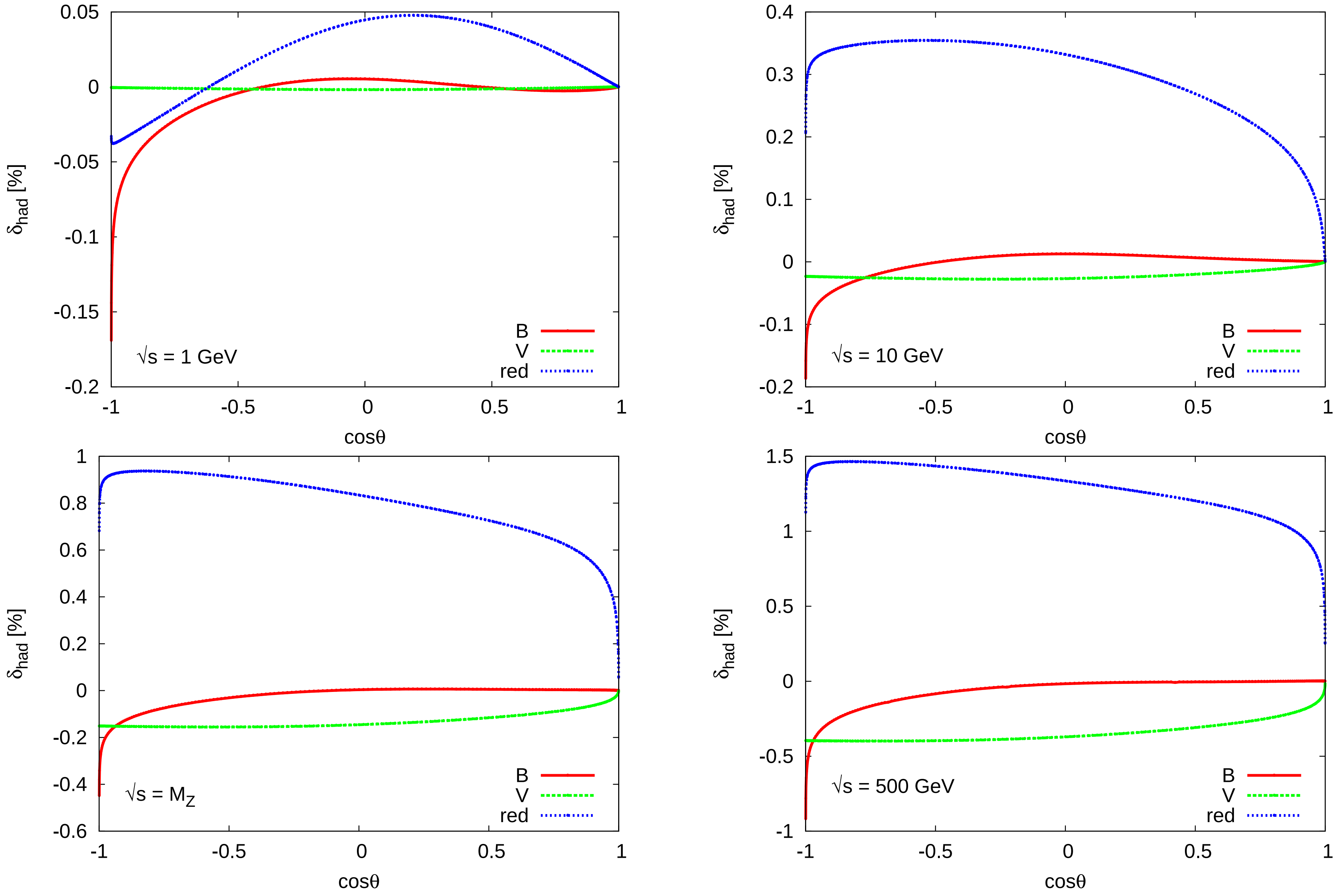
$$\int_0^1 \frac{dy}{y}\xi_A\left(s,b,c,\frac{4m^2}{y}\right) = \frac{c^2}{s^2}\left[\ \ln\frac{c}{b+i\epsilon}\ \ln^2\frac{4m^2-s}{4m^2} + J_A\left(-\frac{b}{s}\right) - J_A\left(-\frac{c}{s}\right)\right]$$

$$J_A(x) = \frac{1}{6}\ln^3(-xr) + \frac{1}{6}\ln^3\frac{xr}{-\bar{r}} - \frac{1}{2}\ln x\ln^2(-xr) - \frac{1}{2}\ln(\bar{x}r)\ln^2(1-\bar{x}r) + \frac{1}{2}\ln\frac{1-\bar{x}r}{x}\ln^2\frac{xr}{-\bar{r}}$$
$$-\ln\bar{r}\,\mathrm{Li}_2(1-\bar{x}r) + \ln\frac{xr}{-\bar{r}}\,\mathrm{Li}_2\left(\frac{\bar{r}\bar{x}}{-x}\right) + \ln(\bar{x}r)\left[\mathrm{Li}_2(r)-\mathrm{Li}_2(\bar{x}r)-\mathrm{Li}_2\left(\frac{xr}{1-\bar{x}r}\right)\right]$$
$$+\ln(-xr)\left[\mathrm{Li}_2(r)-\mathrm{Li}_2(\bar{x}r)-\mathrm{Li}_2\left(\frac{-\bar{x}}{x}\right)\right] - \mathrm{Li}_3\left(\frac{-\bar{x}}{x}\right) + \mathrm{Li}_3(\bar{x}r) + \mathrm{Li}_3\left(\frac{\bar{r}\bar{x}}{-x}\right) + S_{12}\left(\frac{1-\bar{x}r}{\bar{r}}\right).$$

$$\bar{x} = 1-x \qquad\qquad r = \frac{s}{4m^2}$$

$$\int_0^1\frac{dy}{y}\xi_B(a,b,c,\frac{4m^2}{y}) = +\frac{4m^2-a}{a}\left[\left(\ln\frac{-b}{4m^2}-\frac{c}{a}\right)\ln\frac{4m^2-a}{4m^2}+\mathrm{Li}_2\left(\frac{a}{4m^2}\right)\right]$$
$$-\frac{4m^2+b}{a}\left[\ln\frac{-b}{4m^2}\ln\frac{4m^2+b}{4m^2}+\mathrm{Li}_2\left(\frac{-b}{4m^2}\right)\right] + \frac{c}{a}\left[\mathrm{Li}_2\left(\frac{a}{4m^2}\right) - \ln\frac{-b}{4m^2}\right]$$
$$+\frac{c-b}{a}\left\{\ln\frac{-b}{4m^2}\left[\mathrm{Li}_2\left(\frac{-b}{4m^2}\right) - \mathrm{Li}_2\left(\frac{a}{4m^2}\right)\right] + 2\,\mathrm{Li}_3\left(\frac{a}{4m^2}\right) - 2\,\mathrm{Li}_3\left(\frac{-b}{4m^2}\right)\right\}$$

$$\delta_{\rm had} = \frac{d\sigma_{\{B,V,\rm red\}}}{d\sigma_{\rm Born}}$$



Light leptons contribution $(m_l^2 \ll s, |t|, |u|)$

$$m \quad \rightarrow \quad m_l \qquad\qquad R \quad \rightarrow \quad R_l(s) = \left(1 + \frac{4m_l^2}{2\,s}\right)\sqrt{1 - \frac{4m_l^2}{s}}$$

The dispersion integrals are computed analytically

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The dispersion integrals are computed analytically:

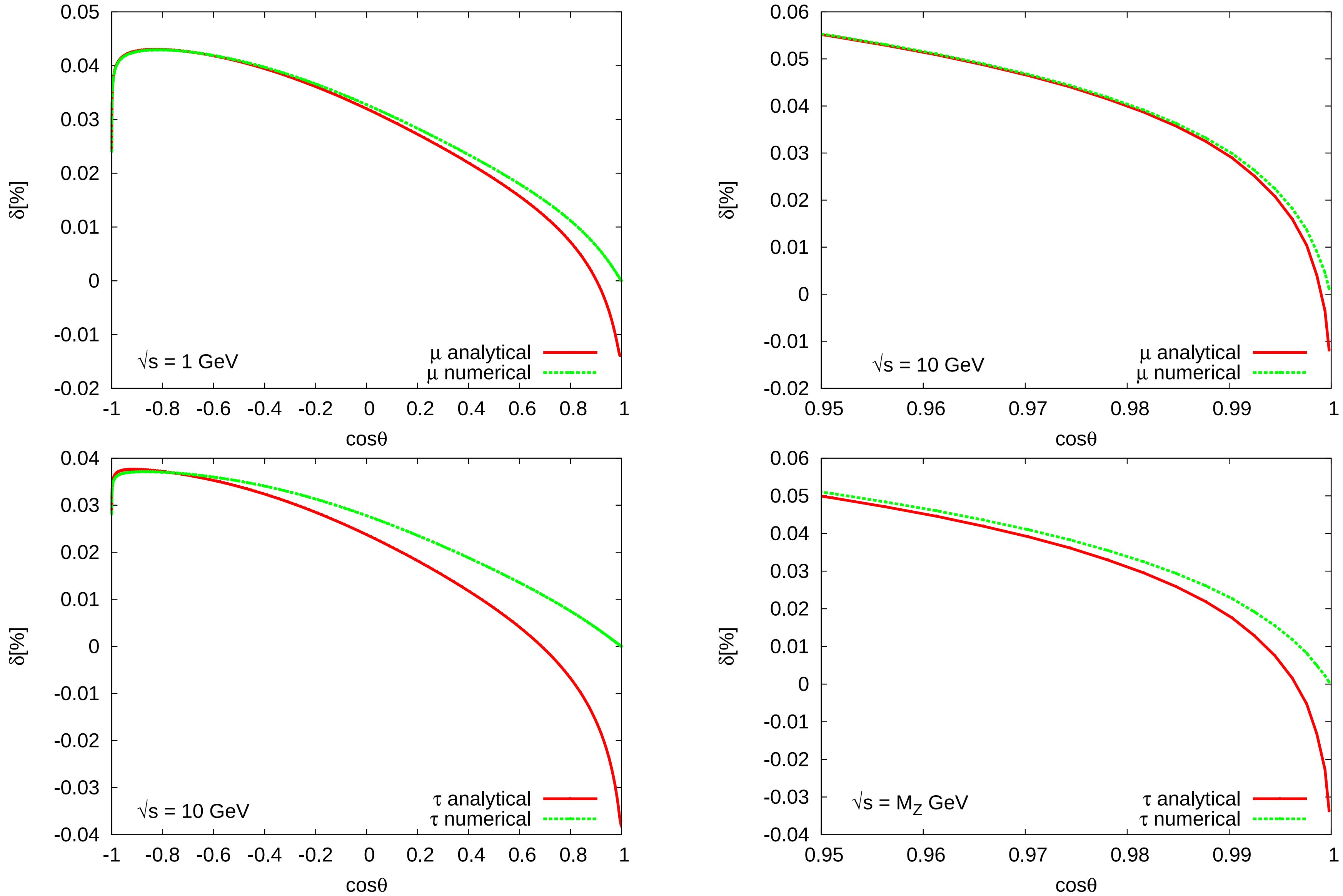
$$\Pi_l(q^2) = -\frac{\alpha}{3\pi}\left(\ln\frac{-q^2}{m_l^2} - \frac{5}{3}\right)$$

$$V_l(q^2) = -\frac{\alpha}{3\pi}\left[\frac{1}{12}\ln^3\frac{-q^2}{m_l^2} - \frac{19}{24}\ln^2\frac{-q^2}{m_l^2} + \frac{1}{2}\left(\zeta(2) + \frac{265}{72}\right)\ln\frac{-q^2}{m_l^2} + \zeta(3) - \frac{19}{12}\zeta(2) - \frac{3355}{432}\right]$$

$$\begin{aligned}B_l(a,b,c) = -\frac{\alpha}{3\pi}\Bigg\{&\frac{c^2}{a^2}\Big[\beta_l(a,b,c) - \beta_l(a,c,b)\Big] + \frac{c}{a}\left[\frac{3}{2}\ln^2\frac{b+i\epsilon}{a} - \ln\frac{b+i\epsilon}{a}\left(\ln\frac{-b}{m_l^2} - \frac{8}{3}\right) + 5\,\zeta(2)\right]\\ &- \frac{c-b}{a}\left[\frac{1}{3}\ln^3\frac{b+i\epsilon}{a} - \frac{1}{2}\ln^2\frac{b+i\epsilon}{a}\left(\ln\frac{-b}{m_l^2} - \frac{8}{3}\right) + \zeta(2)\left(2\ln\frac{b+i\epsilon}{a} - 3\ln\frac{-b}{m_l^2} + 8\right)\right]\Bigg\}\end{aligned}$$

$$\begin{aligned}\beta_l(a,b,c) \;=\; & S_{12}\left(\frac{-b}{c-i\epsilon}\right) + i\pi\,\mathrm{Li}_2\left(\frac{-b}{c-i\epsilon}\right)\\ &+ \ln\frac{-c}{m_l^2}\left[\frac{3}{2}\ln^2\frac{-a}{m_l^2} - \frac{1}{2}\ln\frac{-a}{m_l^2}\ln\frac{-c}{m_l^2} + \frac{5}{6}\ln\frac{-c}{m_l^2} - \frac{10}{3}\ln\frac{-a}{m_l^2} + \frac{28}{9}\right]\end{aligned}$$

$$\delta = \frac{d\sigma_{\mathrm{B+V+red}}}{d\sigma_{\mathrm{Born}}}$$



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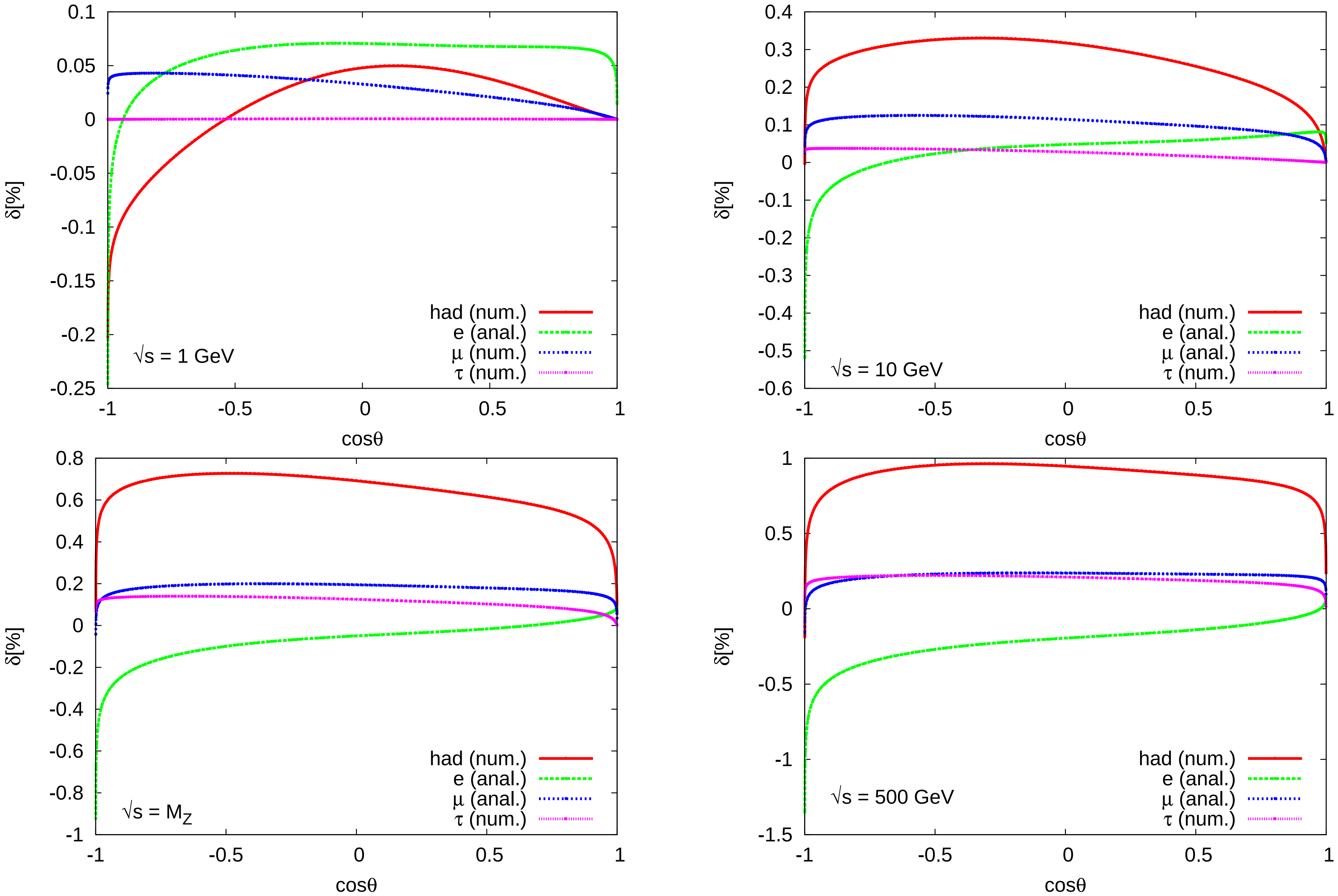

Summary

- Full QED corrections under control
- Hadronic corrections are of the same order of the leptonic ones or bigger ($0.1\% - 1\%$)
- Dispersion relation is the simplest tool to deal with fermionic corrections