

# Two-loop hadronic corrections to Bhabha scattering

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$$e^+ + e^- \rightarrow e^+ + e^-$$

### Important process to determine the luminosity:

- at high-energy colliders (**LEP,SLC**) in the **small-angle** region
- at flavor factories (**BABAR, BELLE, DAΦNE,...**) in the **large-angle** region
- at **ILC** in the **large-angle** region (luminosity spectrum)

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## Theoretical computations at two-loop in QED:

- Corrections with massless electrons (Bern-Dixon-Ghinculov '00)
- Photonic corrections with small electron mass (Penin '06, Becher-Melnikov '07)
- Corrections from light leptons (Actis-Czakon-Gluza-Riemann '07, Becher-Melnikov '07)
- Leptonic corrections (Bonciani-Ferrogli-Penin '07)
- Hadronic corrections (Actis-Czakon-Gluza-Riemann '07, Kühn-U. '08)

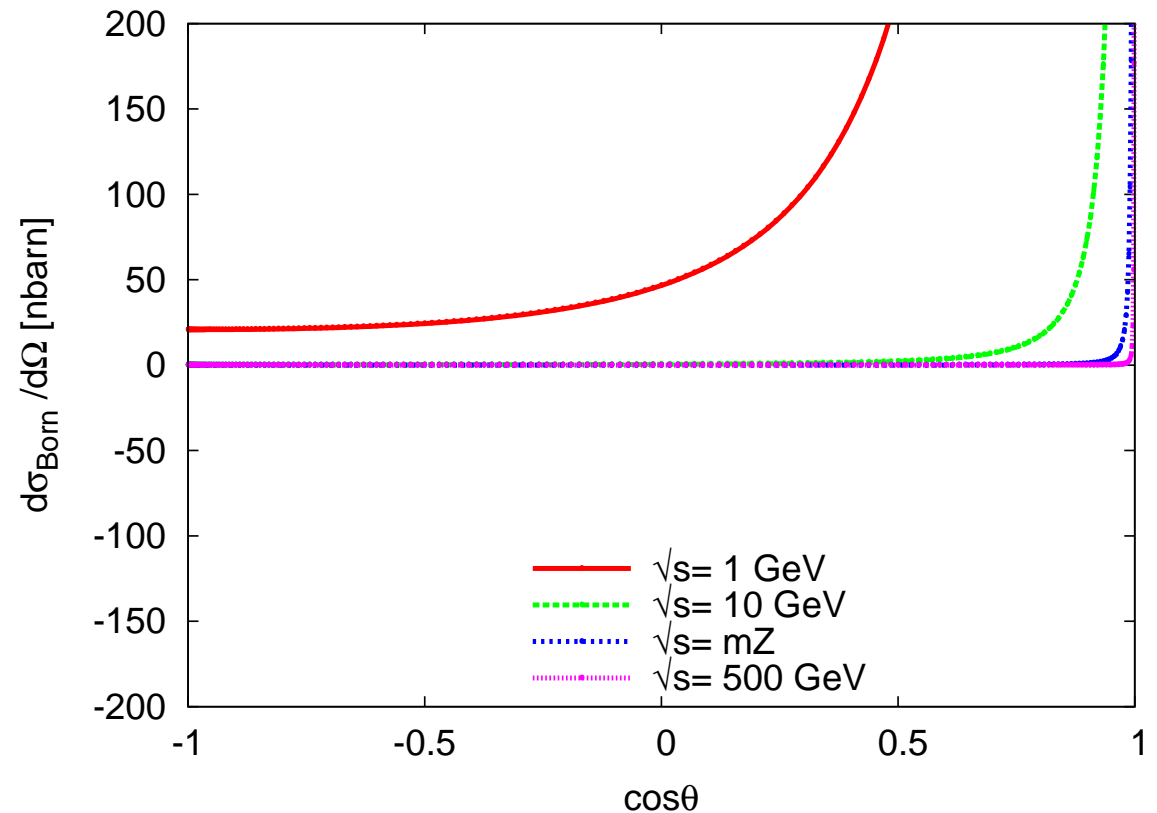
# The Born cross section

$$\mathcal{A}_{\text{Born}} = \text{[t-channel diagram]} + \text{[s-channel diagram]}$$

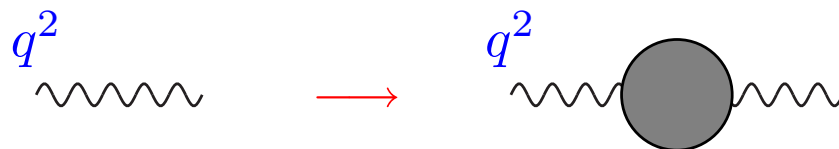
The diagram shows the Born amplitude  $\mathcal{A}_{\text{Born}}$  as the sum of two Feynman diagrams. The first diagram is a t-channel exchange of a photon between an incoming electron ( $e^-$ ) and positron ( $e^+$ ) pair, resulting in an outgoing electron and positron pair. The second diagram is an s-channel exchange of a photon between an incoming electron and positron pair, resulting in an outgoing electron and positron pair.

$$\frac{d\sigma_{\text{Born}}}{d\Omega} = \frac{\alpha^2}{s} \left( \frac{1 - x + x^2}{x} \right)^2$$

$$x = -\frac{t}{s} = \frac{1 - \cos\theta}{2}$$



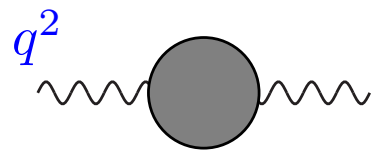
## Hadronic corrections



## Hadronic corrections

$$q^2 \text{ wavy line}$$

→



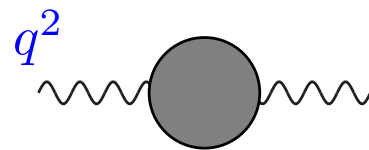
$$\frac{-ig_{\alpha\beta}}{q^2 + i\epsilon}$$

→

$$\frac{-ig_{\alpha\delta}}{q^2 + i\epsilon} i (q^2 g^{\delta\epsilon} - q^\delta q^\epsilon) \Pi(q^2) \frac{-ig_{\epsilon\beta}}{q^2 + i\epsilon}$$

## Hadronic corrections

$$q^2 \text{ wavy line}$$

 $\longrightarrow$ 


$$\frac{-ig_{\alpha\beta}}{q^2 + i\epsilon}$$

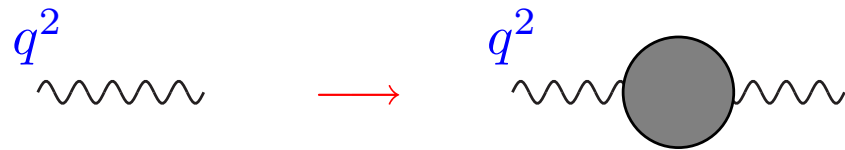
 $\longrightarrow$ 

$$\frac{-ig_{\alpha\delta}}{q^2 + i\epsilon} i (q^2 g^{\delta\epsilon} - q^\delta q^\epsilon) \Pi(q^2) \frac{-ig_{\epsilon\beta}}{q^2 + i\epsilon}$$

 $\longrightarrow$ 

$$\frac{-ig_{\alpha\beta}}{q^2 + i\epsilon} \Pi(q^2)$$

## Hadronic corrections




$$\frac{-ig_{\alpha\beta}}{q^2 + i\epsilon} \longrightarrow \frac{-ig_{\alpha\delta}}{q^2 + i\epsilon} i (q^2 g^{\delta\epsilon} - q^\delta q^\epsilon) \Pi(q^2) \frac{-ig_{\epsilon\beta}}{q^2 + i\epsilon} \longrightarrow \frac{-ig_{\alpha\beta}}{q^2 + i\epsilon} \Pi(q^2)$$

## Dispersion relation

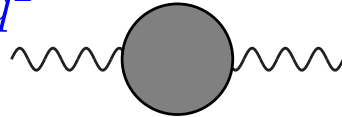
$$\Pi(q^2) = -\frac{q^2}{\pi} \int_{4m^2}^{\infty} \frac{dz}{z} \frac{\text{Im}\Pi(z)}{q^2 - z + i\epsilon}, \quad \text{Im}\Pi(z) = -\frac{\alpha}{3} R(z)$$



## Hadronic corrections

$$q^2$$




$$q^2$$


$$\frac{-ig_{\alpha\beta}}{q^2 + i\epsilon}$$



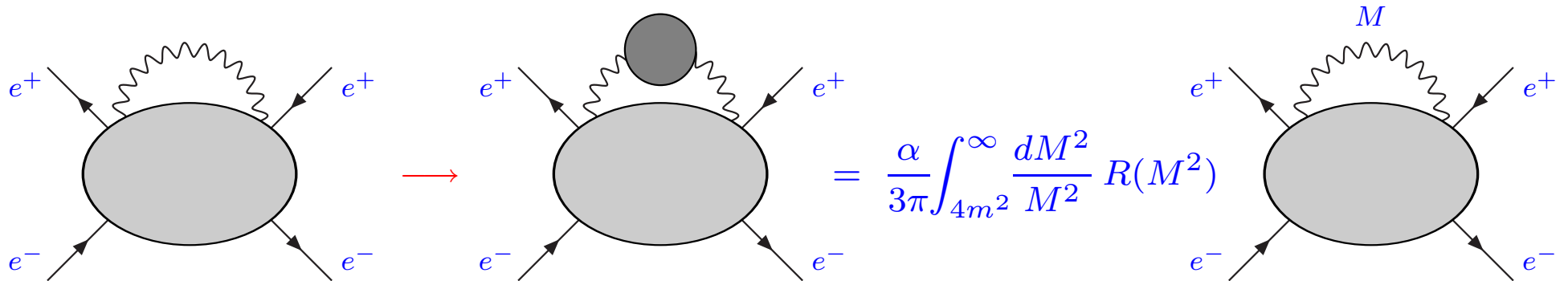
$$\frac{-ig_{\alpha\delta}}{q^2 + i\epsilon} i (q^2 g^{\delta\epsilon} - q^\delta q^\epsilon) \Pi(q^2) \frac{-ig_{\epsilon\beta}}{q^2 + i\epsilon}$$



$$\frac{-ig_{\alpha\beta}}{q^2 + i\epsilon} \Pi(q^2)$$

## Dispersion relation

$$\Pi(q^2) = \frac{\alpha}{3\pi} \int_{4m^2}^{\infty} dz \frac{q^2}{z} \frac{R(z)}{q^2 - z + i\epsilon},$$



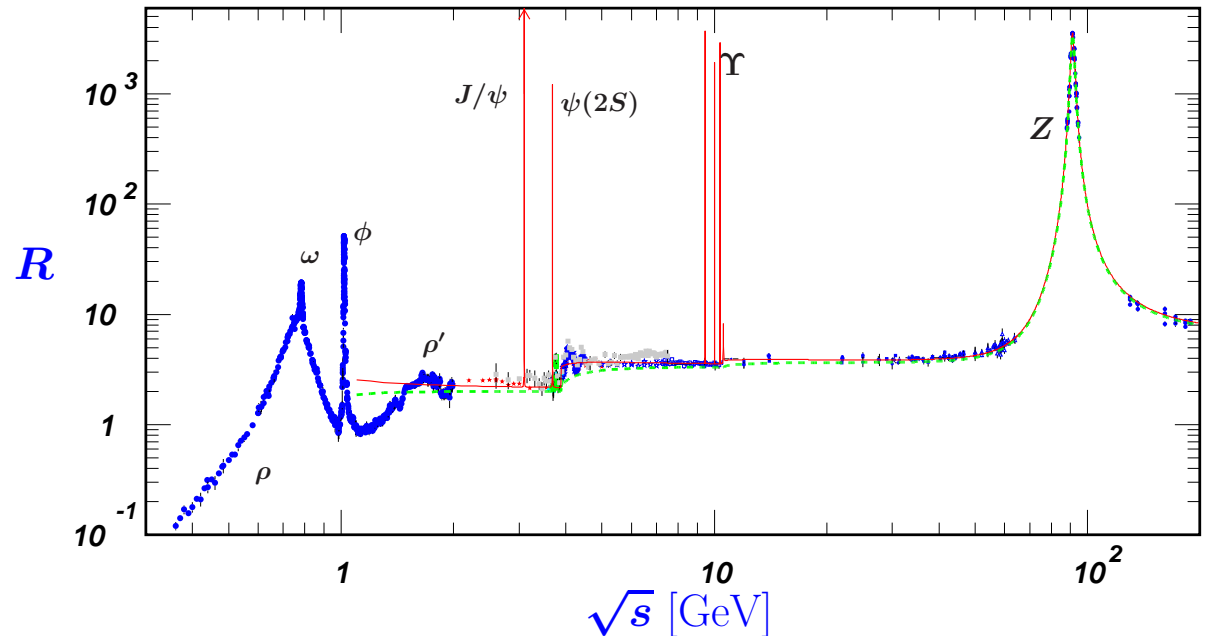
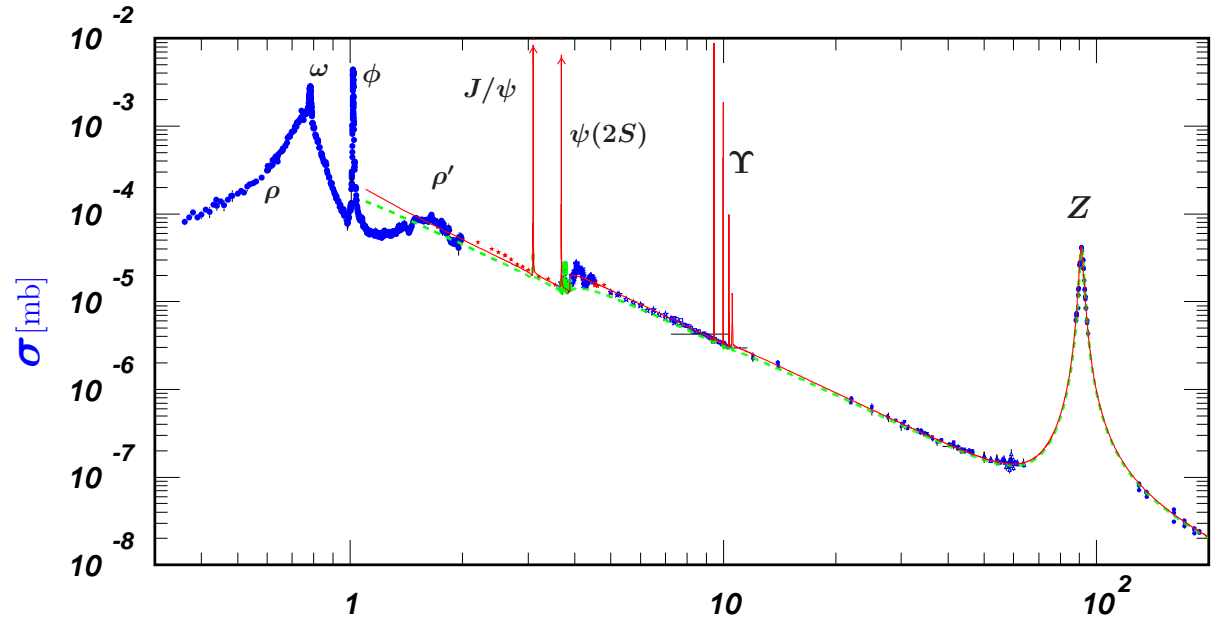
$$= \frac{\alpha}{3\pi} \int_{4m^2}^{\infty} \frac{dM^2}{M^2} R(M^2)$$

$$R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-, \text{QEDsp})}$$



leptonic case

$$R_l(s) = \left(1 + \frac{4m_l^2}{2s}\right) \sqrt{1 - \frac{4m_l^2}{s}}$$



## Three classes of contributions:

1. Vacuum polarization insertion
2. Reducible vertex and box corrections
3. Irreducible vertex and box corrections

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## Inclusion of real soft photons

$$2 \operatorname{Re} \left[ \text{Diagram 1} \right] + \frac{1}{(2\pi)^3} \int_{\omega} \frac{d^3k}{2k_0} \operatorname{Re} \left[ \text{Diagram 2} + \text{Diagram 3} \right]$$

$$k = (k_0, \vec{k}) \quad k_0^2 = \vec{k}^2 + \lambda^2 \quad |\vec{k}| < \omega$$

# 1. Vacuum polarization insertion

- Resummation:

The diagram shows the resummation of vacuum polarization insertions in a photon propagator. On the left, a wavy line representing a photon propagator with momentum  $q^2$  is shown. A red arrow points to the right, where the resummation is shown as a series of terms: a wavy line with momentum  $q^2$ , followed by a plus sign, a wavy line with momentum  $q^2$  and a single grey circular vacuum polarization insertion, followed by a plus sign, a wavy line with momentum  $q^2$  and two grey circular vacuum polarization insertions, followed by a plus sign and an ellipsis  $\dots$ .

# 1. Vacuum polarization insertion

## • Resummation:

$$\begin{array}{l}
 \begin{array}{c} q^2 \\ \text{wavy line} \end{array} \longrightarrow \begin{array}{c} q^2 \\ \text{wavy line} \end{array} + \begin{array}{c} q^2 \\ \text{wavy line} \end{array} \text{ (circle) } \begin{array}{c} \text{wavy line} \end{array} + \begin{array}{c} q^2 \\ \text{wavy line} \end{array} \text{ (circle) } \text{ (circle) } \begin{array}{c} \text{wavy line} \end{array} + \dots \\
 \frac{-ig_{\alpha\beta}}{q^2 + i\epsilon} \longrightarrow \frac{-ig_{\alpha\beta}}{q^2 + i\epsilon} \left[ 1 + \Pi(q^2) + \Pi(q^2)^2 + \dots \right]
 \end{array}$$

# 1. Vacuum polarization insertion

## • Resummation:

$$\begin{array}{c}
 q^2 \\
 \text{wavy line}
 \end{array}
 \longrightarrow
 \begin{array}{c}
 q^2 \\
 \text{wavy line}
 \end{array}
 +
 \begin{array}{c}
 q^2 \\
 \text{wavy line} \text{---} \text{circle} \text{---} \text{wavy line}
 \end{array}
 +
 \begin{array}{c}
 q^2 \\
 \text{wavy line} \text{---} \text{circle} \text{---} \text{circle} \text{---} \text{wavy line}
 \end{array}
 + \dots$$

$$\frac{-ig_{\alpha\beta}}{q^2 + i\epsilon}
 \longrightarrow
 \frac{-ig_{\alpha\beta}}{q^2 + i\epsilon}
 \left[ 1 + \Pi(q^2) + \Pi(q^2)^2 + \dots \right]
 =
 \frac{-ig_{\alpha\delta}}{q^2 + i\epsilon}
 \frac{1}{1 - \Pi(q^2)}$$

# 1. Vacuum polarization insertion

## • Resummation:

$$q^2 \text{ wavy line} \longrightarrow q^2 \text{ wavy line} + q^2 \text{ wavy line} \text{ (with one loop)} + q^2 \text{ wavy line} \text{ (with two loops)} + \dots$$

$$\frac{-ig_{\alpha\beta}}{q^2 + i\epsilon} \longrightarrow \frac{-ig_{\alpha\beta}}{q^2 + i\epsilon} \left[ 1 + \Pi(q^2) + \Pi(q^2)^2 + \dots \right] = \frac{-ig_{\alpha\beta}}{q^2 + i\epsilon} \frac{1}{1 - \Pi(q^2)}$$

$$\frac{d\sigma_{\Pi}}{d\Omega} = \frac{\alpha^2}{s} \left\{ \frac{1-2x+2x^2}{2} \left| \frac{1}{1-\Pi(s)} \right|^2 + \frac{2-2x+x^2}{2x^2} \left| \frac{1}{1-\Pi(t)} \right|^2 - \frac{(1-x)^2}{x} \operatorname{Re} \frac{1}{[1-\Pi(s)][1-\Pi(t)]} \right\}$$



# 1. Vacuum polarization insertion

## • Resummation:

$$q^2 \text{ wavy line} \longrightarrow q^2 \text{ wavy line} + q^2 \text{ wavy line} \text{ with one loop} + q^2 \text{ wavy line} \text{ with two loops} + \dots$$

$$\frac{-ig_{\alpha\beta}}{q^2 + i\epsilon} \longrightarrow \frac{-ig_{\alpha\beta}}{q^2 + i\epsilon} \left[ 1 + \Pi(q^2) + \Pi(q^2)^2 + \dots \right] = \frac{-ig_{\alpha\delta}}{q^2 + i\epsilon} \frac{1}{1 - \Pi(q^2)}$$

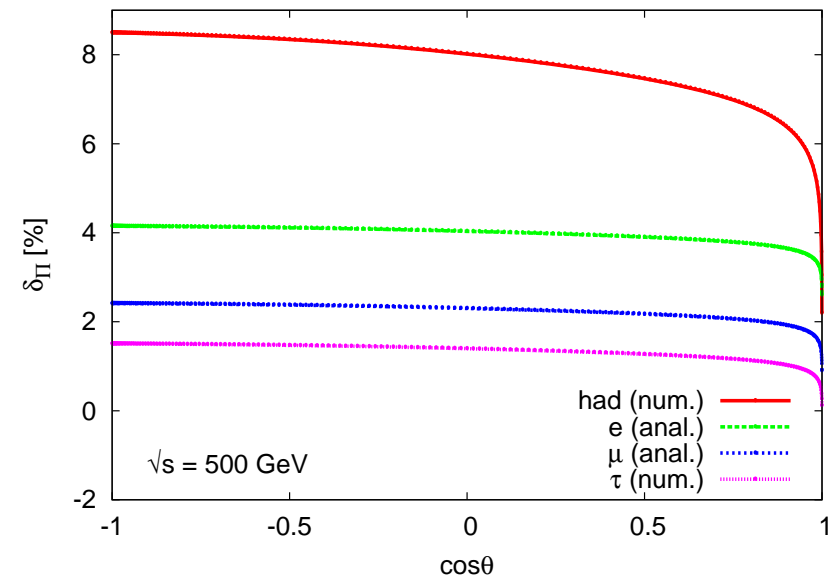
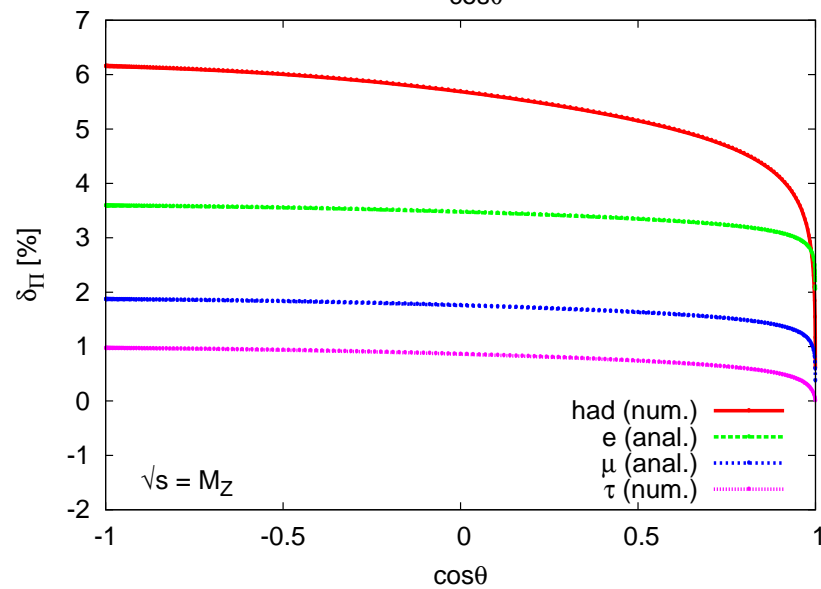
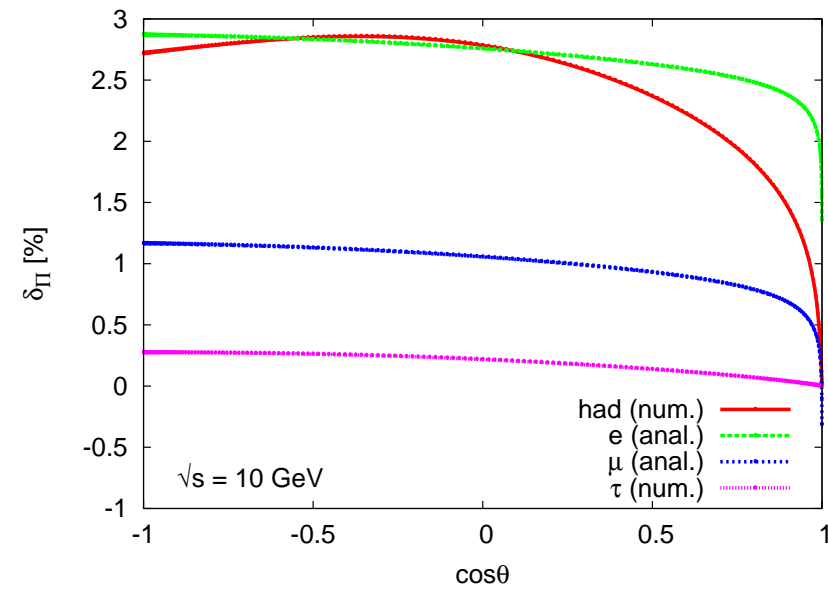
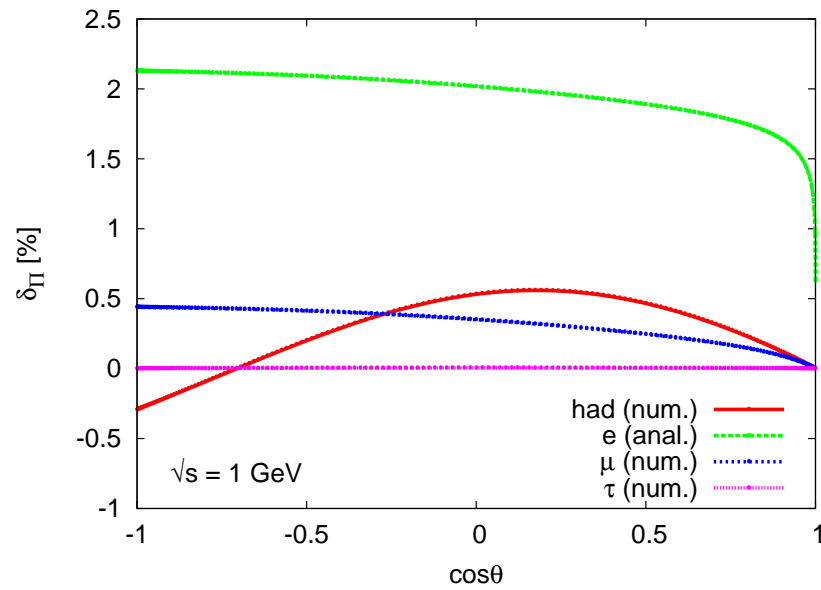
$$\frac{d\sigma_{\Pi}}{d\Omega} = \frac{\alpha^2}{s} \left\{ \frac{1-2x+2x^2}{2} \left| \frac{1}{1-\Pi(s)} \right|^2 + \frac{2-2x+x^2}{2x^2} \left| \frac{1}{1-\Pi(t)} \right|^2 - \frac{(1-x)^2}{x} \operatorname{Re} \frac{1}{[1-\Pi(s)][1-\Pi(t)]} \right\}$$

## • Evaluation of $\Pi(q^2)$

$$\Pi(t) = \frac{\alpha}{3\pi} \int_0^1 dy \frac{t}{yt - 4m^2} R\left(\frac{4m^2}{y}\right),$$

$$\Pi(s) = \frac{\alpha}{3\pi} \left\{ \ln\left(1 - \frac{s}{4m^2 - i\epsilon}\right) R(s) + \int_0^1 dy \frac{s}{ys - 4m^2} \left[ R\left(\frac{4m^2}{y}\right) - R(s) \right] \right\}$$

$$\delta_{\Pi} = \frac{d\sigma_{\Pi} - d\sigma_{\text{Born}}}{d\sigma_{\text{Born}}}$$



## 2. Reducible vertices ...

$$\begin{aligned}
 & 2 \operatorname{Re} \left( \begin{array}{c} \text{diagram 1} \\ + \\ \text{diagram 2} \end{array} \right) \left( \begin{array}{c} \text{diagram 3} \\ + \\ \text{diagram 4} \\ + \\ \text{diagram 5} \\ + \\ \text{diagram 6} \end{array} \right)^* \\
 & + 2 \operatorname{Re} \left( \begin{array}{c} \text{diagram 7} \\ + \\ \text{diagram 8} \end{array} \right) \left( \begin{array}{c} \text{diagram 9} \\ + \\ \text{diagram 10} \\ + \\ \text{diagram 11} \\ + \\ \text{diagram 12} \end{array} \right)^* + \left( \begin{array}{c} \text{real soft} \\ \text{photons} \end{array} \right)
 \end{aligned}$$

## 2. Reducible vertices ...

$$\begin{aligned}
 & 2 \operatorname{Re} \left( \text{diagram 1} + \text{diagram 2} \right) \left( \text{diagram 3} + \text{diagram 4} + \text{diagram 5} + \text{diagram 6} \right)^* \\
 & + 2 \operatorname{Re} \left( \text{diagram 7} + \text{diagram 8} \right) \left( \text{diagram 9} + \text{diagram 10} + \text{diagram 11} + \text{diagram 12} \right)^* + \left( \text{real soft photons} \right)
 \end{aligned}$$

$$\begin{aligned}
 \frac{d\sigma_{\text{red},V}}{d\Omega} = \frac{\alpha^3}{s\pi} \left\{ \right. & \frac{1-2x+2x^2}{2} \left[ 4 V_s^\gamma \operatorname{Re} \Pi(s) \right] + \frac{2-2x+x^2}{2x^2} \left[ 4 V_t^\gamma \Pi(t) \right] \\
 & \left. - \frac{(1-x)^2}{x} \left[ (V_s^\gamma + V_t^\gamma) \left( \operatorname{Re} \Pi(s) + \Pi(t) \right) + \pi \left( \ln \frac{\lambda^2}{s} + \frac{3}{2} \right) \operatorname{Im} \Pi(s) \right] \right\}
 \end{aligned}$$

$$V_s^\gamma = 2 \ln \frac{2\omega}{\sqrt{s}} \left( \ln \frac{s}{m_e^2} - 1 \right) + \frac{3}{2} \ln \frac{s}{m_e^2} + 2 \zeta(2) - 2$$

$$V_t^\gamma = 2 \ln \frac{2\omega}{\sqrt{s}} \left( \ln \frac{-t}{m_e^2} - 1 \right) + \frac{3}{2} \ln \frac{-t}{m_e^2} - \ln \frac{-t}{s} \ln \frac{-u}{s} - \operatorname{Li}_2 \left( \frac{-t}{s} \right) - 2$$

## ... and reducible boxes

$$2 \operatorname{Re} \left( \begin{array}{c} \text{diagram 1} \\ + \\ \text{diagram 2} \end{array} \right) \left( \begin{array}{c} \text{diagram 3} \\ + \\ \text{diagram 4} \\ + \\ \text{diagram 5} \\ + \\ \text{diagram 6} \end{array} \right)^* + \left( \begin{array}{c} \text{real soft} \\ \text{photons} \end{array} \right)$$

## ... and reducible boxes

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$$\frac{d\sigma_{\text{red,B}}}{d\Omega} = \frac{\alpha^3}{s\pi} \left\{ \frac{1-2x+2x^2}{2} \left[ 2 B_s^\gamma \operatorname{Re} \Pi(s) + 2\pi \ln \frac{t}{u} \operatorname{Im} \Pi(s) \right] + \frac{2-2x+x^2}{2x^2} \left[ 2 B_t^\gamma \Pi(t) \right] \right. \\ \left. - \frac{(1-x)^2}{x} \left[ B_t^\gamma \operatorname{Re} \Pi(s) + B_s^\gamma \Pi(t) - \pi \ln \frac{\lambda^2}{-t} \operatorname{Im} \Pi(s) \right] \right. \\ \left. + \dots \right\}$$

$$B_s^\gamma = 2 \ln \frac{2\omega}{\sqrt{s}} \ln \frac{t}{u} + \frac{1}{2} \ln^2 \frac{-t}{s} - \frac{1}{2} \ln^2 \frac{-u}{s} - \ln \frac{-t}{s} \ln \frac{-u}{s} - 2 \operatorname{Li}_2 \left( \frac{-t}{s} \right) + \zeta(2),$$

$$B_t^\gamma = -2 \ln \frac{2\omega}{\sqrt{s}} \ln \frac{-u}{s} - \frac{1}{2} \ln^2 \frac{-u}{s} + \ln \frac{-t}{s} \ln \frac{-u}{s} - \operatorname{Li}_2 \left( \frac{-t}{s} \right),$$

## ... and reducible boxes

$$2 \operatorname{Re} \left( \begin{array}{c} \text{diagram 1} \\ + \\ \text{diagram 2} \end{array} \right) \left( \begin{array}{c} \text{diagram 3} \\ + \\ \text{diagram 4} \\ + \\ \text{diagram 5} \\ + \\ \text{diagram 6} \end{array} \right)^* + \left( \begin{array}{c} \text{real soft} \\ \text{photons} \end{array} \right)$$

$$\begin{aligned} \frac{d\sigma_{\text{red,B}}}{d\Omega} = & \frac{\alpha^3}{s\pi} \left\{ \frac{1-2x+2x^2}{2} \left[ 2 B_s^\gamma \operatorname{Re} \Pi(s) + 2\pi \ln \frac{t}{u} \operatorname{Im} \Pi(s) \right] + \frac{2-2x+x^2}{2x^2} \left[ 2 B_t^\gamma \Pi(t) \right] \right. \\ & - \frac{(1-x)^2}{x} \left[ B_t^\gamma \operatorname{Re} \Pi(s) + B_s^\gamma \Pi(t) - \pi \ln \frac{\lambda^2}{-t} \operatorname{Im} \Pi(s) \right] \\ & - \operatorname{Re} \left[ \left( B^\gamma(s,t) - B^\gamma(s,u) \right) \Pi^*(s) \right] - \operatorname{Re} \left[ \left( B^\gamma(t,s) - B^\gamma(t,u) \right) \Pi(t) \right] \\ & \left. + \operatorname{Re} \left[ x B^\gamma(t,s) \Pi^*(s) + \frac{1}{x} B^\gamma(s,t) \Pi(t) \right] \right\} \end{aligned}$$

$$B_s^\gamma = 2 \ln \frac{2\omega}{\sqrt{s}} \ln \frac{t}{u} + \frac{1}{2} \ln^2 \frac{-t}{s} - \frac{1}{2} \ln^2 \frac{-u}{s} - \ln \frac{-t}{s} \ln \frac{-u}{s} - 2 \operatorname{Li}_2 \left( \frac{-t}{s} \right) + \zeta(2),$$

$$B_t^\gamma = -2 \ln \frac{2\omega}{\sqrt{s}} \ln \frac{-u}{s} - \frac{1}{2} \ln^2 \frac{-u}{s} + \ln \frac{-t}{s} \ln \frac{-u}{s} - \operatorname{Li}_2 \left( \frac{-t}{s} \right),$$

$$B^\gamma(a,b) = -\frac{a+b}{2a} \ln \frac{b}{a+i\epsilon} + \frac{a+2b}{4a} \left( \ln^2 \frac{b}{a+i\epsilon} + \pi^2 \right).$$

### 3. Irreducible vertices ...

$$2 \operatorname{Re} \left( \begin{array}{c} \text{Diagram 1} \\ + \\ \text{Diagram 2} \end{array} \right) \left( \begin{array}{c} \text{Diagram 3} \\ + \\ \text{Diagram 4} \\ + \\ \text{Diagram 5} \\ + \\ \text{Diagram 6} \end{array} \right)^*$$

The diagram shows a mathematical expression for the real part of a sum of Feynman diagrams. The expression is  $2 \operatorname{Re} \left( \dots \right) \left( \dots \right)^*$ . The first part is a sum of two diagrams enclosed in large parentheses. The second part is a sum of six diagrams enclosed in large parentheses with an asterisk. The diagrams are:

- Diagram 1: A four-point vertex with a wavy line in the middle. Two incoming lines from the left and two outgoing lines to the right.
- Diagram 2: A four-point vertex with a wavy line in the middle. Two incoming lines from the top and two outgoing lines to the bottom.
- Diagram 3: A four-point vertex with a wavy line in the middle. Two incoming lines from the left and two outgoing lines to the right. A grey circle is attached to the top-left wavy line.
- Diagram 4: A four-point vertex with a wavy line in the middle. Two incoming lines from the left and two outgoing lines to the right. A grey circle is attached to the top-right wavy line.
- Diagram 5: A four-point vertex with a wavy line in the middle. Two incoming lines from the left and two outgoing lines to the right. A grey circle is attached to the top wavy line.
- Diagram 6: A four-point vertex with a wavy line in the middle. Two incoming lines from the left and two outgoing lines to the right. A grey circle is attached to the bottom wavy line.



### 3. Irreducible vertices ...

$$2 \operatorname{Re} \left( \begin{array}{c} \text{diagram 1} \\ + \\ \text{diagram 2} \end{array} \right) \left( \begin{array}{c} \text{diagram 3} \\ + \\ \text{diagram 4} \\ + \\ \text{diagram 5} \\ + \\ \text{diagram 6} \end{array} \right)^*$$

$$\frac{d\sigma_V}{d\Omega} = \frac{\alpha^3}{s\pi} \left\{ \frac{1-2x+2x^2}{2} \left[ 4\operatorname{Re}V(s) \right] + \frac{2-2x+x^2}{2x^2} \left[ 4V(t) \right] - \frac{(1-x)^2}{x} \left[ 2V(t) + 2\operatorname{Re}V(s) \right] \right\}$$

### 3. Irreducible vertices ...

$$2 \operatorname{Re} \left( \begin{array}{c} \text{Diagram 1} \\ + \\ \text{Diagram 2} \end{array} \right) \left( \begin{array}{c} \text{Diagram 3} \\ + \\ \text{Diagram 4} \\ + \\ \text{Diagram 5} \\ + \\ \text{Diagram 6} \end{array} \right)^*$$

$$\frac{d\sigma_V}{d\Omega} = \frac{\alpha^3}{s\pi} \left\{ \frac{1-2x+2x^2}{2} \left[ 4\operatorname{Re}V(s) \right] + \frac{2-2x+x^2}{2x^2} \left[ 4V(t) \right] - \frac{(1-x)^2}{x} \left[ 2V(t) + 2\operatorname{Re}V(s) \right] \right\}$$

#### • Second dispersion integral:

$$V(q^2) = \frac{\alpha}{3\pi} \int_{4m^2}^{\infty} \frac{dz}{z} R(z) \rho(q^2, z - i\epsilon)$$

$$\rho(q^2, z) = -\frac{7}{8} - \frac{z}{2q^2} + \frac{1}{2} \left( \frac{3}{2} + \frac{z}{q^2} \right) \ln \frac{-z}{q^2} + \frac{1}{2} \left( 1 + \frac{z}{q^2} \right)^2 \left[ \zeta(2) - \operatorname{Li}_2 \left( 1 + \frac{z}{q^2} \right) \right].$$

### 3. Irreducible vertices ...

$$2 \operatorname{Re} \left( \begin{array}{c} \text{diagram 1} \\ + \\ \text{diagram 2} \end{array} \right) \left( \begin{array}{c} \text{diagram 3} \\ + \\ \text{diagram 4} \\ + \\ \text{diagram 5} \\ + \\ \text{diagram 6} \end{array} \right)^*$$

$$\frac{d\sigma_V}{d\Omega} = \frac{\alpha^3}{s\pi} \left\{ \frac{1-2x+2x^2}{2} [4\operatorname{Re}V(s)] + \frac{2-2x+x^2}{2x^2} [4V(t)] - \frac{(1-x)^2}{x} [2V(t) + 2\operatorname{Re}V(s)] \right\}$$

• **Second dispersion integral:**  $V(q^2) = \frac{\alpha}{3\pi} \int_{4m^2}^{\infty} \frac{dz}{z} R(z) \rho(q^2, z - i\epsilon)$

$$\rho(q^2, z) = -\frac{7}{8} - \frac{z}{2q^2} + \frac{1}{2} \left( \frac{3}{2} + \frac{z}{q^2} \right) \ln \frac{-z}{q^2} + \frac{1}{2} \left( 1 + \frac{z}{q^2} \right)^2 \left[ \zeta(2) - \operatorname{Li}_2 \left( 1 + \frac{z}{q^2} \right) \right].$$

• **Evaluation of  $V(q^2)$**

$$V(q^2) = \frac{\alpha}{3\pi} \left\{ R(\infty) \int_0^1 \frac{dy}{y} \rho \left( q^2, \frac{4m^2}{y} \right) + \int_0^1 \frac{dy}{y} \rho \left( q^2, \frac{4m^2}{y} \right) \left[ R \left( \frac{4m^2}{y} \right) - R(\infty) \right] \right\}$$

$$\int_0^1 \frac{dy}{y} \rho \left( q^2, \frac{4m^2}{y} \right) = -\frac{1}{12} \ln^3(-r) - \ln(-r) \left[ \zeta(2) + \frac{7}{8} + \frac{1}{4r} + \frac{1}{2} \operatorname{Li}_2 \left( -\frac{1}{r} \right) \right]$$

$$+ \left( \frac{3}{4} + \frac{1}{r} + \frac{1}{4r^2} \right) \left[ \zeta(2) - \operatorname{Li}_2(1+r) \right] + \frac{15}{16} + \frac{1}{4r} - \operatorname{Li}_3 \left( -\frac{1}{r} \right) \quad r = \frac{q^2}{4m^2}$$

## ... and irreducible boxes

$$2 \operatorname{Re} \left( \begin{array}{c} \text{diagram 1} \\ + \\ \text{diagram 2} \end{array} \right) \left( \begin{array}{c} \text{diagram 3} + \text{diagram 4} + \text{diagram 5} + \text{diagram 6} \\ + \\ \text{diagram 7} + \text{diagram 8} + \text{diagram 9} + \text{diagram 10} \end{array} \right)^* + \left( \begin{array}{c} \text{real soft} \\ \text{photons} \end{array} \right)$$

The diagram shows the real part of the amplitude,  $2 \operatorname{Re}$ , applied to a sum of two tree-level diagrams (a t-channel and an s-channel exchange of a photon). This is multiplied by the complex conjugate of a sum of ten one-loop diagrams. The first four diagrams are box diagrams with a photon loop and a fermion loop, with a grey circle representing a vertex correction. The next four diagrams are box diagrams with a fermion loop and a photon loop, also with a grey circle. The final two diagrams are box diagrams with a fermion loop and a photon loop, with a grey circle. The entire expression is then added to the contribution from real soft photons.

## ... and irreducible boxes

$$2 \operatorname{Re} \left( \begin{array}{c} \text{diagram 1} \\ + \\ \text{diagram 2} \end{array} \right) \left( \begin{array}{c} \text{diagram 3} + \text{diagram 4} + \text{diagram 5} + \text{diagram 6} \\ + \\ \text{diagram 7} + \text{diagram 8} + \text{diagram 9} + \text{diagram 10} \end{array} \right)^* + \left( \begin{array}{c} \text{real soft} \\ \text{photons} \end{array} \right)$$

$$\frac{d\sigma_B}{d\Omega} = \frac{\alpha^3}{s\pi} \left\{ \frac{1-2x+2x^2}{2} [2B_s \operatorname{Re}\Pi(s)] + \frac{2-2x+x^2}{2x^2} [2B_t \Pi(t)] - \frac{(1-x)^2}{x} [B_t \operatorname{Re}\Pi(s) + B_s \Pi(t)] + \dots \right\}$$

$$B_s = 2 \ln \frac{2\omega}{\sqrt{s}} \ln \frac{t}{u} - \ln \frac{-t}{s} \ln \frac{-u}{s} - 2 \operatorname{Li}_2 \left( \frac{-t}{s} \right) + \zeta(2) \quad B_t = -2 \ln \frac{2\omega}{\sqrt{s}} \ln \frac{-u}{s} - \operatorname{Li}_2 \left( \frac{-t}{s} \right) + 3 \zeta(2)$$

## ... and irreducible boxes

$$2 \operatorname{Re} \left( \begin{array}{c} \text{diagram 1} \\ + \\ \text{diagram 2} \end{array} \right) \left( \begin{array}{c} \text{diagram 3} + \text{diagram 4} + \text{diagram 5} + \text{diagram 6} \\ + \\ \text{diagram 7} + \text{diagram 8} + \text{diagram 9} + \text{diagram 10} \end{array} \right)^* + \left( \begin{array}{c} \text{real soft} \\ \text{photons} \end{array} \right)$$

$$\frac{d\sigma_B}{d\Omega} = \frac{\alpha^3}{s\pi} \left\{ \frac{1-2x+2x^2}{2} \left[ 2B_s \operatorname{Re}\Pi(s) \right] + \frac{2-2x+x^2}{2x^2} \left[ 2B_t \Pi(t) \right] - \frac{(1-x)^2}{x} \left[ B_t \operatorname{Re}\Pi(s) + B_s \Pi(t) \right] \right. \\ \left. - \operatorname{Re} \left[ B(s, t, u) - B(s, u, t) \right] - \operatorname{Re} \left[ B(t, s, u) - B(t, u, s) \right] + \operatorname{Re} \left[ xB(t, s, u) + \frac{1}{x} B(s, t, u) \right] \right\}$$

$$B_s = 2 \ln \frac{2\omega}{\sqrt{s}} \ln \frac{t}{u} - \ln \frac{-t}{s} \ln \frac{-u}{s} - 2 \operatorname{Li}_2 \left( \frac{-t}{s} \right) + \zeta(2) \quad B_t = -2 \ln \frac{2\omega}{\sqrt{s}} \ln \frac{-u}{s} - \operatorname{Li}_2 \left( \frac{-t}{s} \right) + 3 \zeta(2)$$

## • Third dispersion integral

$$B(a, b, c) = \frac{\alpha}{3\pi} \int_{4m^2}^{\infty} \frac{dz}{z} R(z) [\xi_A(a, b, c, z - i\epsilon) + \xi_B(a, b, c, z - i\epsilon)]$$

$$\xi_A(a, b, c, z) = \frac{c^2}{a(z-a)} \left[ 2 \ln \frac{c}{b+i\epsilon} \ln \left( 1 - \frac{a}{z} \right) - \text{Li}_2 \left( 1 + \frac{b}{z} \right) + \text{Li}_2 \left( 1 + \frac{c}{z} \right) \right]$$

$$\begin{aligned} \xi_B(a, b, c, z) = & \frac{c}{a} \left[ \left( \frac{z}{a} - 1 \right) \ln \left( 1 - \frac{a}{z} \right) + \ln \frac{-b}{z} \right] \\ & + \frac{c-b-z}{a} \left[ \ln \frac{b+i\epsilon}{-a} \ln \left( 1 - \frac{a}{z} \right) - \text{Li}_2 \left( 1 - \frac{a}{z} \right) + \text{Li}_2 \left( 1 + \frac{b}{z} \right) \right] \end{aligned}$$

## • Third dispersion integral

$$B(a, b, c) = \frac{\alpha}{3\pi} \int_{4m^2}^{\infty} \frac{dz}{z} R(z) \left[ \xi_A(a, b, c, z - i\epsilon) + \xi_B(a, b, c, z - i\epsilon) \right]$$

$$\xi_A(a, b, c, z) = \frac{c^2}{a(z-a)} \left[ 2 \ln \frac{c}{b+i\epsilon} \ln \left( 1 - \frac{a}{z} \right) - \text{Li}_2 \left( 1 + \frac{b}{z} \right) + \text{Li}_2 \left( 1 + \frac{c}{z} \right) \right]$$

$$\begin{aligned} \xi_B(a, b, c, z) &= \frac{c}{a} \left[ \left( \frac{z}{a} - 1 \right) \ln \left( 1 - \frac{a}{z} \right) + \ln \frac{-b}{z} \right] \\ &\quad + \frac{c-b-z}{a} \left[ \ln \frac{b+i\epsilon}{-a} \ln \left( 1 - \frac{a}{z} \right) - \text{Li}_2 \left( 1 - \frac{a}{z} \right) + \text{Li}_2 \left( 1 + \frac{b}{z} \right) \right] \end{aligned}$$

## • Evaluation of $B(a, b, c)$

$$B_A(t, b, c) = \frac{\alpha}{3\pi} \int_0^1 \frac{dy}{y} \xi_A \left( t, b, c, \frac{4m^2}{y} \right) R \left( \frac{4m^2}{y} \right)$$

$$B_A(s, b, c) = \frac{\alpha}{3\pi} \left\{ R(s) \int_0^1 \frac{dy}{y} \xi_A \left( s, b, c, \frac{4m^2}{y} \right) + \int_0^1 \frac{dy}{y} \xi_A \left( s, b, c, \frac{4m^2}{y} \right) \left[ R \left( \frac{4m^2}{y} \right) - R(s) \right] \right\}$$

$$B_B(a, b, c) = \frac{\alpha}{3\pi} \left\{ R(\infty) \int_0^1 \frac{dy}{y} \xi_B \left( a, b, c, \frac{4m^2}{y} \right) + \int_0^1 \frac{dy}{y} \xi_B \left( a, b, c, \frac{4m^2}{y} \right) \left[ R \left( \frac{4m^2}{y} \right) - R(\infty) \right] \right\}$$



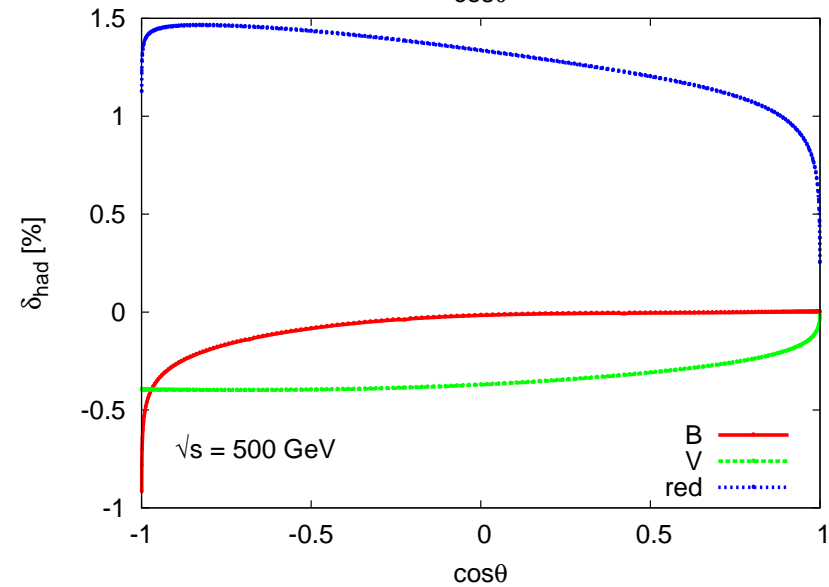
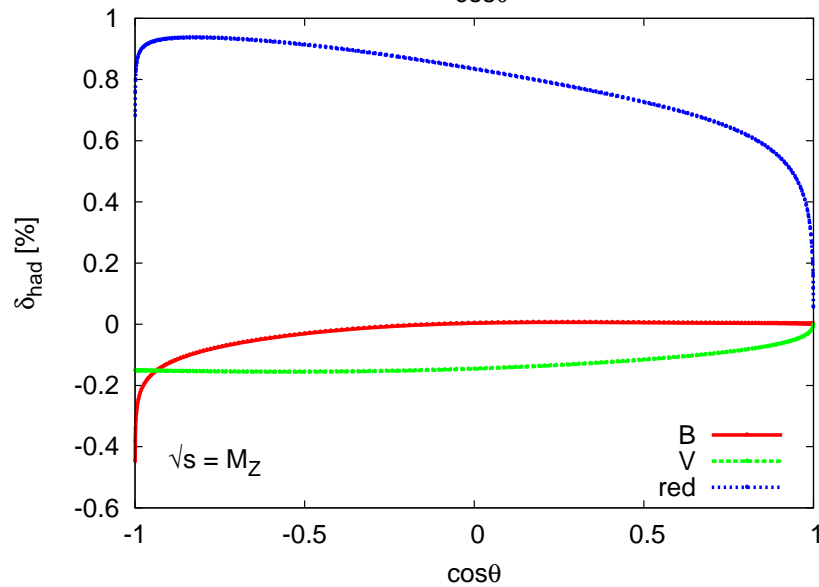
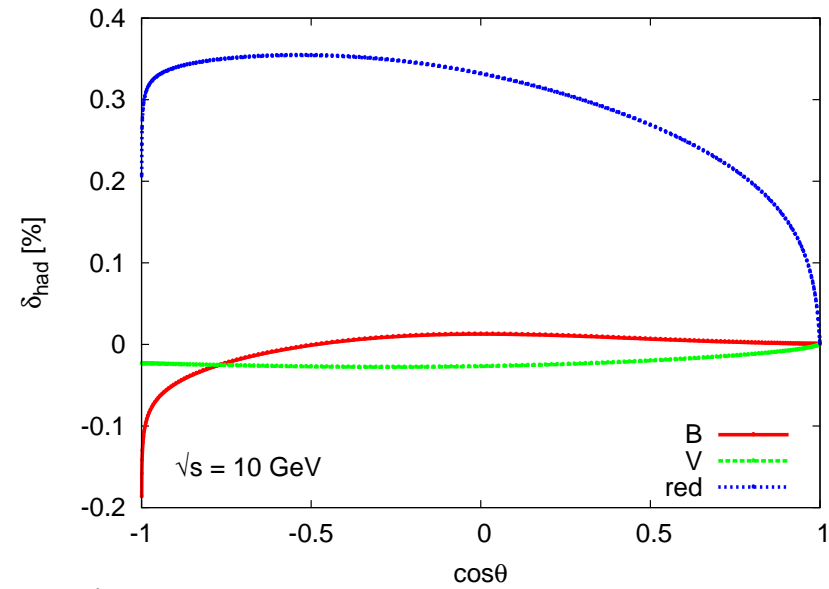
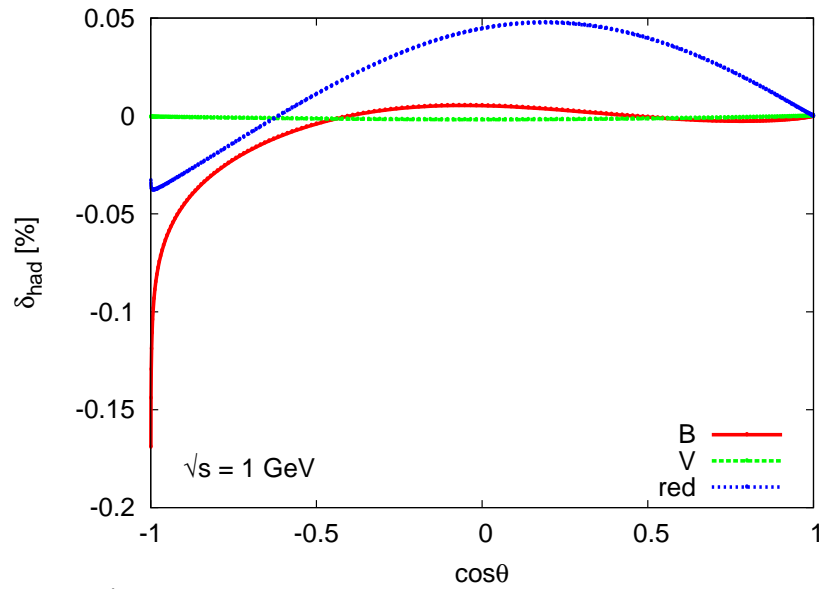
$$\int_0^1 \frac{dy}{y} \xi_A \left( s, b, c, \frac{4m^2}{y} \right) = \frac{c^2}{s^2} \left[ \ln \frac{c}{b+i\epsilon} \ln^2 \frac{4m^2-s}{4m^2} + J_A \left( -\frac{b}{s} \right) - J_A \left( -\frac{c}{s} \right) \right]$$

$$\begin{aligned} J_A(x) &= \frac{1}{6} \ln^3(-xr) + \frac{1}{6} \ln^3 \frac{xr}{-\bar{r}} - \frac{1}{2} \ln x \ln^2(-xr) - \frac{1}{2} \ln(\bar{x}r) \ln^2(1-\bar{x}r) + \frac{1}{2} \ln \frac{1-\bar{x}r}{x} \ln^2 \frac{xr}{-\bar{r}} \\ &\quad - \ln \bar{r} \operatorname{Li}_2(1-\bar{x}r) + \ln \frac{xr}{-\bar{r}} \operatorname{Li}_2 \left( \frac{\bar{r}\bar{x}}{-x} \right) + \ln(\bar{x}r) \left[ \operatorname{Li}_2(r) - \operatorname{Li}_2(\bar{x}r) - \operatorname{Li}_2 \left( \frac{xr}{1-\bar{x}r} \right) \right] \\ &\quad + \ln(-xr) \left[ \operatorname{Li}_2(r) - \operatorname{Li}_2(\bar{x}r) - \operatorname{Li}_2 \left( \frac{-\bar{x}}{x} \right) \right] - \operatorname{Li}_3 \left( \frac{-\bar{x}}{x} \right) + \operatorname{Li}_3(\bar{x}r) + \operatorname{Li}_3 \left( \frac{\bar{r}\bar{x}}{-x} \right) + S_{12} \left( \frac{1-\bar{x}r}{\bar{r}} \right). \end{aligned}$$

$$\bar{x} = 1 - x \qquad r = \frac{s}{4m^2}$$

$$\begin{aligned} \int_0^1 \frac{dy}{y} \xi_B \left( a, b, c, \frac{4m^2}{y} \right) &= + \frac{4m^2-a}{a} \left[ \left( \ln \frac{-b}{4m^2} - \frac{c}{a} \right) \ln \frac{4m^2-a}{4m^2} + \operatorname{Li}_2 \left( \frac{a}{4m^2} \right) \right] \\ &\quad - \frac{4m^2+b}{a} \left[ \ln \frac{-b}{4m^2} \ln \frac{4m^2+b}{4m^2} + \operatorname{Li}_2 \left( \frac{-b}{4m^2} \right) \right] + \frac{c}{a} \left[ \operatorname{Li}_2 \left( \frac{a}{4m^2} \right) - \ln \frac{-b}{4m^2} \right] \\ &\quad + \frac{c-b}{a} \left\{ \ln \frac{-b}{4m^2} \left[ \operatorname{Li}_2 \left( \frac{-b}{4m^2} \right) - \operatorname{Li}_2 \left( \frac{a}{4m^2} \right) \right] + 2 \operatorname{Li}_3 \left( \frac{a}{4m^2} \right) - 2 \operatorname{Li}_3 \left( \frac{-b}{4m^2} \right) \right\} \end{aligned}$$

$$\delta_{\text{had}} = \frac{d\sigma_{\{B,V,\text{red}\}}}{d\sigma_{\text{Born}}}$$



**Light leptons contribution** ( $m_l^2 \ll s, |t|, |u|$ )

$$m \rightarrow m_l \quad R \rightarrow R_l(s) = \left(1 + \frac{4m_l^2}{2s}\right) \sqrt{1 - \frac{4m_l^2}{s}}$$

The dispersion integrals are computed analytically

# Light leptons contribution ( $m_l^2 \ll s, |t|, |u|$ )

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The dispersion integrals are computed analytically:

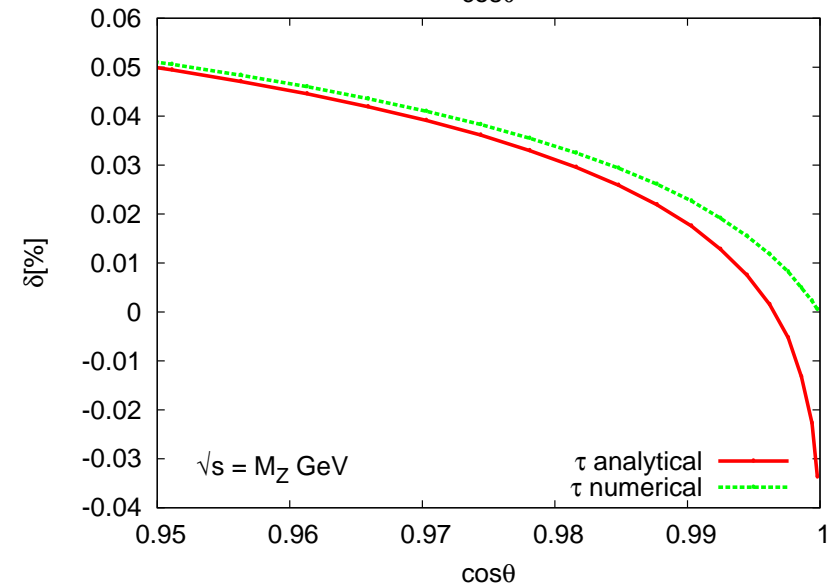
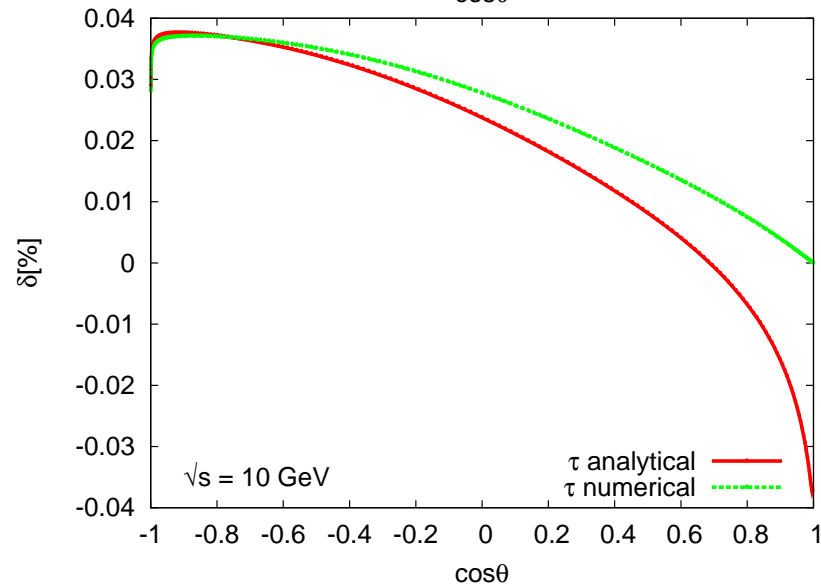
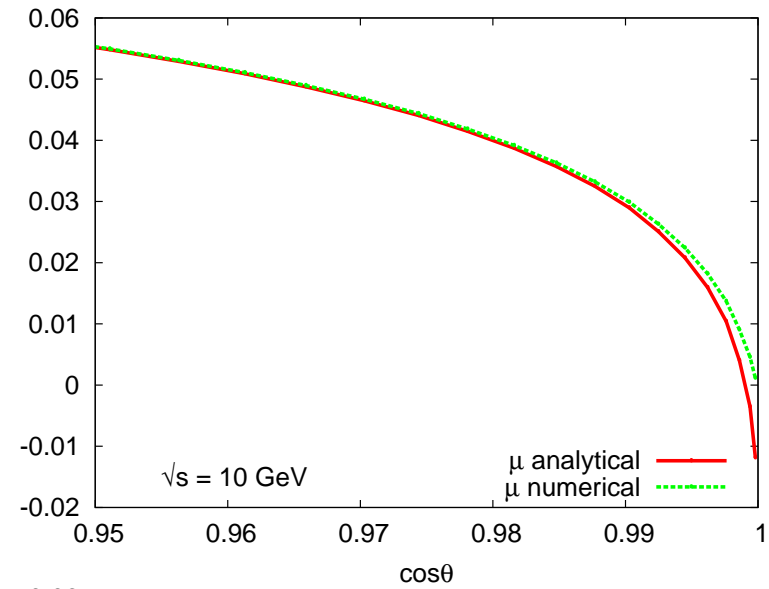
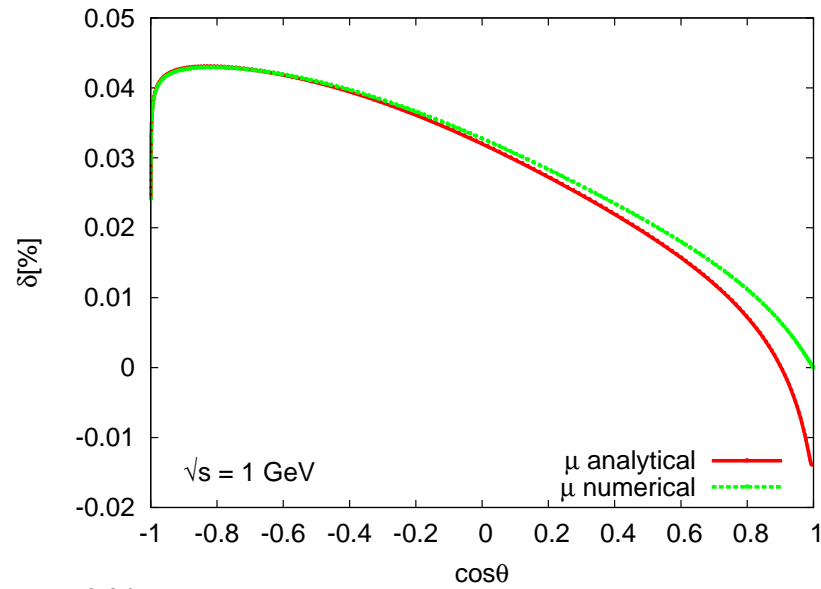
$$\Pi_l(q^2) = -\frac{\alpha}{3\pi} \left( \ln \frac{-q^2}{m_l^2} - \frac{5}{3} \right)$$

$$V_l(q^2) = -\frac{\alpha}{3\pi} \left[ \frac{1}{12} \ln^3 \frac{-q^2}{m_l^2} - \frac{19}{24} \ln^2 \frac{-q^2}{m_l^2} + \frac{1}{2} \left( \zeta(2) + \frac{265}{72} \right) \ln \frac{-q^2}{m_l^2} + \zeta(3) - \frac{19}{12} \zeta(2) - \frac{3355}{432} \right]$$

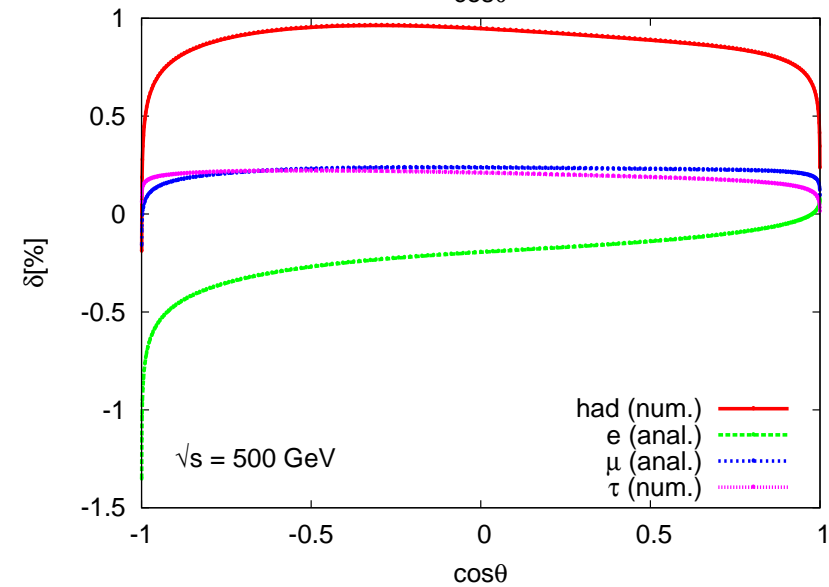
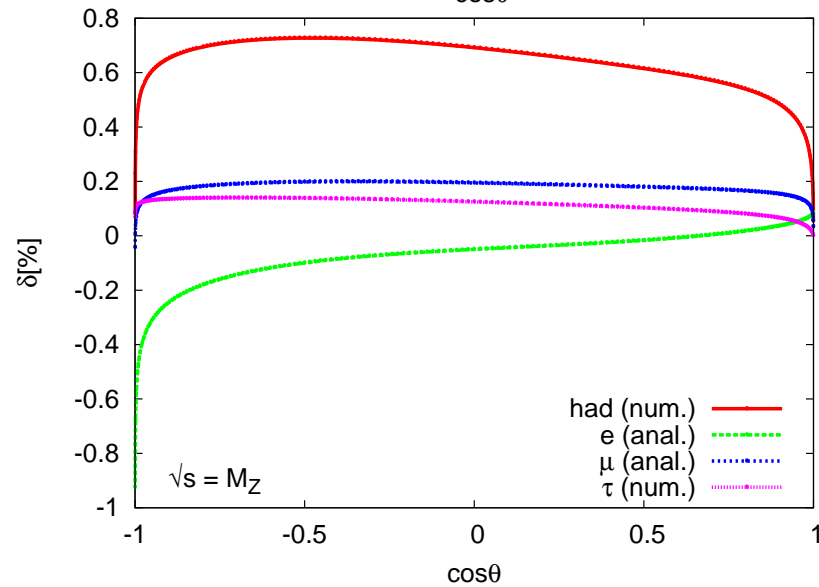
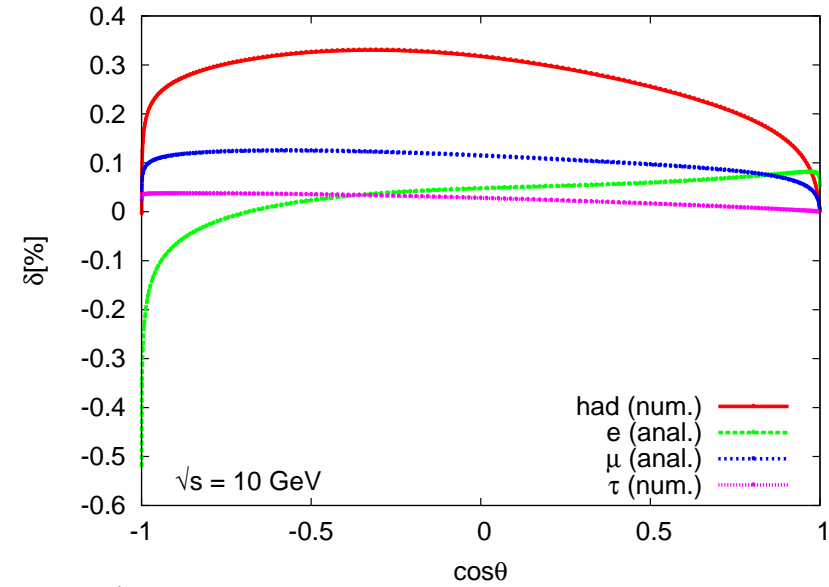
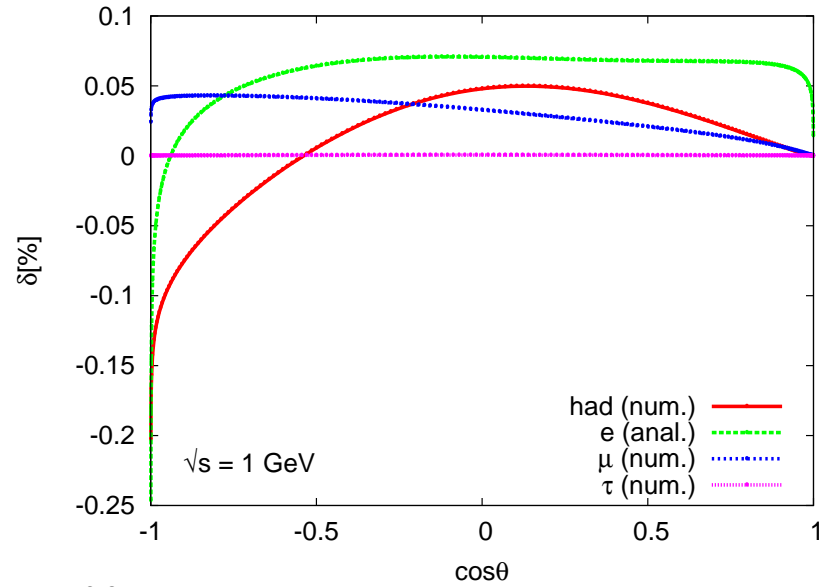
$$B_l(a,b,c) = -\frac{\alpha}{3\pi} \left\{ \frac{c^2}{a^2} \left[ \beta_l(a,b,c) - \beta_l(a,c,b) \right] + \frac{c}{a} \left[ \frac{3}{2} \ln^2 \frac{b+i\epsilon}{a} - \ln \frac{b+i\epsilon}{a} \left( \ln \frac{-b}{m_l^2} - \frac{8}{3} \right) + 5 \zeta(2) \right] \right. \\ \left. - \frac{c-b}{a} \left[ \frac{1}{3} \ln^3 \frac{b+i\epsilon}{a} - \frac{1}{2} \ln^2 \frac{b+i\epsilon}{a} \left( \ln \frac{-b}{m_l^2} - \frac{8}{3} \right) + \zeta(2) \left( 2 \ln \frac{b+i\epsilon}{a} - 3 \ln \frac{-b}{m_l^2} + 8 \right) \right] \right\}$$

$$\beta_l(a,b,c) = S_{12} \left( \frac{-b}{c-i\epsilon} \right) + i\pi \operatorname{Li}_2 \left( \frac{-b}{c-i\epsilon} \right) \\ + \ln \frac{-c}{m_l^2} \left[ \frac{3}{2} \ln^2 \frac{-a}{m_l^2} - \frac{1}{2} \ln \frac{-a}{m_l^2} \ln \frac{-c}{m_l^2} + \frac{5}{6} \ln \frac{-c}{m_l^2} - \frac{10}{3} \ln \frac{-a}{m_l^2} + \frac{28}{9} \right]$$

$$\delta = \frac{d\sigma_{B+V+red}}{d\sigma_{Born}}$$



$$\delta = \frac{d\sigma_{B+V+red}}{d\sigma_{Born}}$$



## Summary

- Full QED corrections under control
- **Hadronic corrections** are of the same order of the leptonic ones or bigger (0.1% – 1%)
- **Dispersion relation is the simplest tool** to deal with fermionic corrections