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A detector-independent analytical study of the contribution of multiple scattering to the momentum error in barrel detectors, and comparison with an exact Kalman filter

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A detector independent analytical study of the contribution of multiple scattering to the momentum error in barrel detectors

and comparison with an exact Kalman filter

Introduction

- Gluckstern's formulae [1]
  - Frequently used
  - Often stressed far beyond their limits
  - Assumptions: constant magnetic field, track in the symmetry plane (perpendicular to the magnetic field)
- Generalization of Gluckstern's formulae [2]
  - Different resolutions, material budgets
  - Quite large incident angles, high curvatures
  - Only in symmetry plane of barrel detector (see above)

Introduction

- Rossi: multiple scattering in homogeneous detectors (diagonal elements)

$$\mathbf{V}_{MS} = \langle \vartheta^2 \rangle \cdot \begin{pmatrix} L & \frac{L^2}{2} \\ \frac{L^2}{2} & \frac{L^3}{3} \end{pmatrix}$$

- Gluckstern: First attempt to deal with multiple scattering in discrete detectors, based on earlier publication by Bruno Rossi

$$(\mathbf{V}_{MS})_{kl} = \left\langle \frac{\theta^2}{2} \right\rangle_{det} \sum_{j}^{\min(k,l)-1} (z_k - z_j)(z_l - z_j)$$
$$= \frac{1}{3} \cdot \left\langle \frac{\vartheta^2}{2} \right\rangle \cdot z_k^2 (3z_l - z_k), \qquad z_l \geq z_k$$

homogeneous material, equidistant measurements

Introduction

- Multiple scattering, general formulae
  - General case applied at the Split Field Magnet (SFM) at the first high energy pp collider (CERN Intersecting Storage Rings ISR)[3]
  - Only restriction: validity of local linear expansion
- Aim of this study:
  - Complement generalized Gluckstern formulae for realistic dip angle λ range in the barrel region
  - Method independent of detector, mathematically exact
  - Detector optimization: needs 9 coefficients per detector setup instead of simulation program
  - Prerequisites: detector rotational symmetric, invariant w.r.t. translations parallel to magnetic field (no z dependent resolution in e.g. a TPC)

Sample detector for simulation

- Example silicon detector
  - B = 4 [T], solenoid
  - 11 cylinder layers
  - 10 mm ≤ R ≤ 1010 mm equidistant
  - Thickness of each layer: $X = 0.01\ X_0$
  - Point resolutions σ(RΦ) = σ(z) = 5μm
  - Reference surface in front of innermost detector layer
- Simulation with LiC Detector Toy 2.0 [4]
  - Parameters: λ, φ, $\kappa = 1/R_H$

The global formula

- In plane z = 0: $\sigma(\Delta p_t/p_t)=\sigma(\Delta p/p)$
- Global formula (extension to $\lambda \neq 0$):

$$\sigma^2\left(\frac{\Delta p}{p}\right) = \sigma^2\left(\frac{\Delta p_t}{p_t}\right) + \tan^2\lambda\cdot\sigma^2(\lambda) + 2\tan\lambda\cdot\rho(\kappa,\lambda)\cdot\sigma\left(\frac{\Delta p_t}{p_t}\right)\sigma(\lambda)$$

- Without MS:
  Follows behavior of detector errors (see below)
- With MS:
  σ(λ) and ρ(λ,κ) have to be studied more extensively

Multiple scattering covariance matrix

- Assumptions:
  - multiple scattering in λ and in φ independent
  - > covariance matrix block diagonal

$$\mathbf{V}_{MS} \approx \begin{pmatrix} \mathrm{var}\left(R\vec{\Phi}\right) & 0 \\ 0 & \mathrm{var}\left(\vec{z}\right) \end{pmatrix}$$

- General covariance matrix for discrete layers:

$$(\mathbf{V}_{MS})_{kl} = \sum_{i}^{\min(k,l)-1} \left\langle \frac{\theta^2}{2} \right\rangle \left(\frac{\partial f_k}{\partial \theta_{1,i}} \frac{\partial f_l}{\partial \theta_{1,i}} + \frac{\partial f_k}{\partial \theta_{2,i}} \frac{\partial f_l}{\partial \theta_{2,i}} \right)$$

Multiple scattering covariance matrix

$$(\mathbf{V}_{MS})_{kl} = \sum_{i}^{\min(k,l)-1} \left\langle \frac{\theta^2}{2} \right\rangle \left(\frac{\partial f_k}{\partial \theta_{1,i}} \frac{\partial f_l}{\partial \theta_{1,i}} + \frac{\partial f_k}{\partial \theta_{2,i}} \frac{\partial f_l}{\partial \theta_{2,i}} \right)$$

- Rossi-Greisen: $\left\langle \frac{\theta^2}{2} \right\rangle \propto \frac{d}{d_0} \cdot \frac{1}{p^2}$
- Extension to $\lambda \neq 0$: $p = \frac{p_t}{\cos\lambda}$ $d = \frac{d_0}{\cos\lambda}$ $\Rightarrow \left\langle \frac{\theta^2}{2} \right\rangle \propto \cos\lambda$
- Least squares method: $\tilde{\mathbf{C}} = \left(\mathbf{D}^T \mathbf{V}_{tot}^{-1} \mathbf{D}\right)^{-1}$ $\mathbf{V}_{tot} = \mathbf{V}_{MS} + \mathbf{V}_{det}$
- Rigorous procedure:
  - $\mathbf{V}_{MS}$ dominates at low p_t, but is singular
  - $\mathbf{V}_{det}$ from detector errors, asymptotic values
  - keep $\mathbf{V}_{det}$ for inversion, after inversion limit $\mathbf{V}_{det} \to 0$

Multiple scattering in λ

- Covariance matrix: $$(\mathbf{V}_{MS(\lambda)})_{kl} = \sum_{i}^{\min(k,l)-1} \left\langle \frac{\theta^2}{2} \right\rangle \frac{\partial z_k}{\partial \lambda_i} \frac{\partial z_l}{\partial \lambda_i}$$
  - With $\frac{\partial z_k}{\partial \lambda_i} \propto \frac{1}{\cos^2 \lambda}$

$$\mathbf{V}_{MS(\lambda)}^{\lambda \neq 0} = \mathbf{V}_{MS(\lambda)}^{\lambda = 0} \cdot \frac{1}{\cos^3 \lambda}$$

- Derivative matrix D_z for LSM:
  - Dimension: $N_{\text{Coordinates}}$ x 2

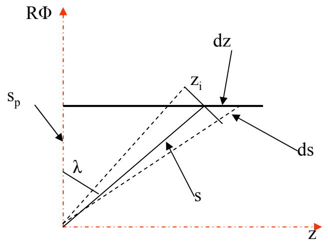
$$\mathbf{D}_z = \begin{pmatrix} \frac{\partial z_1}{\partial \lambda} & \frac{\partial z_1}{\partial \kappa} \\ \vdots & \vdots \end{pmatrix}$$

Multiple scattering in λ

- Derivatives from geometry:

$$\frac{z_i}{R_i} = \tan\lambda \longrightarrow \frac{\partial z_i}{\partial \lambda} \propto \frac{1}{\cos^2 \lambda}$$

$$\frac{\partial zi}{\partial \kappa} = \underbrace{\frac{\partial zi}{\partial s}}_{\propto \frac{1}{\sin\lambda}} \underbrace{\frac{\partial s}{\partial s_p}}_{\frac{1}{\cos\lambda}} \underbrace{\frac{\partial s_p}{\partial \kappa}}_{\text{const}} \propto \frac{1}{\sin\lambda \cos\lambda}$$



LSM

$$\mathbf{C}(\lambda,\kappa)_{MS(\lambda)} = \begin{pmatrix} a_{1,1} \cdot \cos\lambda & a_{1,2} \cdot \sin\lambda \\ a_{1,2} \cdot \sin\lambda & a_{2,2} \cdot \frac{\sin^2\lambda}{\cos\lambda} \end{pmatrix}$$

$a_{1,2}$ and $a_{2,2}$ *not* straight forward determinable at $\lambda = 0$!

$$\rho(\kappa,\lambda) = -1 = \frac{a_{1,2} \cdot \sin\lambda}{\sqrt{a_{1,1} a_{2,2} \cdot \sin^2\lambda}} \qquad a_{i,j} = a_{i,j}(\lambda = 0, \kappa)$$

$$\Rightarrow a_{1,2} = -\sqrt{a_{1,1} a_{2,2}}$$

Multiple scattering in φ

- Covariance matrix: $(\mathrm{V}_{MS(\varphi)})_{kl} = \sum_{i}^{\min(k,l)-1} \underbrace{\left\langle \frac{\theta^2}{2} \right\rangle}_{\propto \cos\lambda} \underbrace{\frac{\partial R\Phi_k}{\partial \varphi_i}}_{\text{const.}} \underbrace{\frac{\partial R\Phi_l}{\partial \varphi_i}}_{\text{const.}}$
  - Projection to plane ⊥ B
    → additional factor $\frac{1}{\cos^2\lambda}$

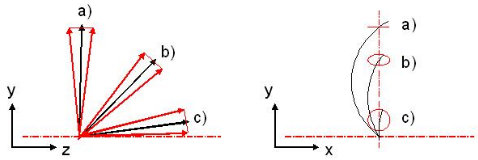
$$\mathrm{V}_{MS(\varphi)}^{\lambda\neq 0} = \mathrm{V}_{MS(\varphi)}^{\lambda=0} \cdot \frac{1}{\cos\lambda}$$

- Derivative matrix $D_{R\Phi}$ for LSM:
  - Dimension: $N_{\text{Coordinates}}$ x 2

$$\mathrm{D}_{R\Phi} = \begin{pmatrix} \frac{\partial R\Phi_1}{\partial \lambda} & \frac{\partial R\Phi_1}{\partial \kappa} \\ \vdots & \vdots \end{pmatrix}$$

Multiple scattering in φ

- $\lambda \neq 0$, low p_t: Correlations between λ and φ (b)



- Calculated derivatives checked by simulation

$$\frac{\partial R\Phi}{\partial \kappa} = \text{const.} \qquad \frac{\partial R\Phi}{\partial \lambda} \propto \frac{1}{\sin\lambda\cos\lambda}$$

$b_{1,1}$ and $b_{1,2}$ *not* straight forward determinable at λ = 0!

$$\mathrm{C}(\lambda,\kappa)_{MS(\varphi)} = \begin{pmatrix} b_{1,1}\cdot\sin^2\lambda\cos\lambda & b_{1,2}\cdot\sin\lambda \\ b_{1,2}\cdot\sin\lambda & b_{2,2}\cdot\frac{1}{\cos\lambda} \end{pmatrix} \qquad b_{i,j} = b_{i,j}(\lambda=0,\kappa)$$

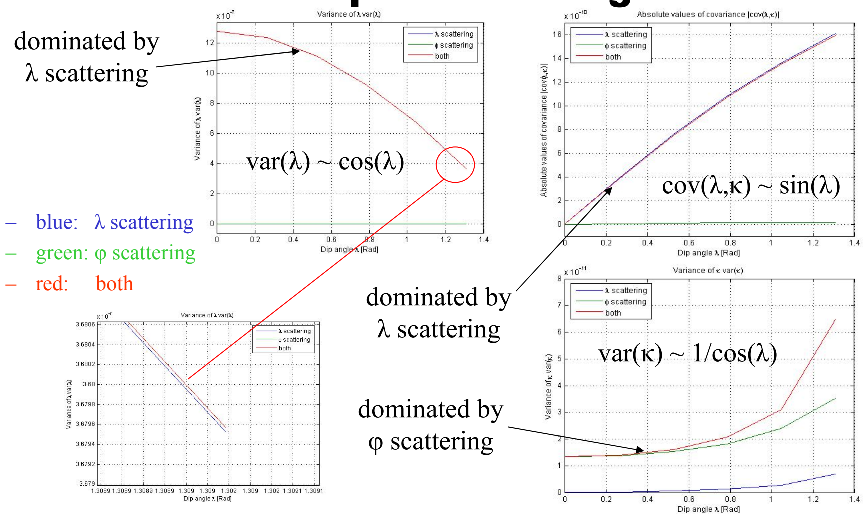
Total multiple scattering covariance matrix

$$C(\lambda,\kappa)_{MS}^{tot} = \overbrace{\begin{pmatrix} a_{1,1}\cdot\cos\lambda & a_{1,2}\cdot\sin\lambda \\ a_{1,2}\cdot\sin\lambda & a_{2,2}\cdot\frac{\sin^2\lambda}{\cos\lambda}\end{pmatrix}}^{MS(\lambda)} + \overbrace{\begin{pmatrix} b_{1,1}\cdot\sin^2\lambda\cos\lambda & b_{1,2}\cdot\sin\lambda \\ b_{1,2}\cdot\sin\lambda & b_{2,2}\cdot\frac{1}{\cos\lambda}\end{pmatrix}}^{MS(\varphi)}$$

$$C(\lambda,\kappa)_{MS}^{tot} \approx \begin{pmatrix} a_{1,1}\cdot\cos\lambda & a_{1,2}\cdot\sin\lambda \\ a_{1,2}\cdot\sin\lambda & b_{2,2}\cdot\frac{1}{\cos\lambda}\end{pmatrix}$$

- Comparison of magnitude by simulation yields:
  - $a_{1,1} >> b_{1,1}$ (var(λ) dominated by λ scattering)
    $b_{1,1}$ at 1 GeV/c comparable to the corresponding value due to detector errors, but several orders of magnitude smaller than $a_{1,1}$
  - $a_{1,2} >> b_{1,2}$ (cov(λ,κ) dominated by λ scattering)
  - $b_{2,2} >> a_{2,2}$ (var(κ) dominated by φ scattering)

Total multiple scattering matrix



The covariance in the global formula

$$\sigma^2\left(\frac{\Delta p}{p}\right) = \sigma^2\left(\frac{\Delta p_t}{p_t}\right) + \tan^2\lambda \cdot \sigma^2(\lambda) + 2\tan\lambda \, \rho(\kappa,\lambda)\cdot\sigma\left(\frac{\Delta p_t}{p_t}\right)\sigma(\lambda)$$

= cov(λ,κ)

λ	$\sigma(\lambda)$	$\rho(\kappa,\lambda)$	$\sigma\left(\frac{\Delta p_t}{p_t}\right)$	$\sigma\left(\frac{\Delta p}{p}\right)_{sim}$	$\sigma\left(\frac{\Delta p}{p}\right)_{calc}$
0°	$1.181 \cdot 10^{-3}$	0	$3.004 \cdot 10^{-3}$	$3.004 \cdot 10^{-3}$	$3.004 \cdot 10^{-3}$
15°	$1.159 \cdot 10^{-3}$	- 0.095	$3.072 \cdot 10^{-3}$	$3.060 \cdot 10^{-3}$	$3.058 \cdot 10^{-3}$
30°	$1.101 \cdot 10^{-3}$	- 0.181	$3.297 \cdot 10^{-3}$	$3.245 \cdot 10^{-3}$	$3.243 \cdot 10^{-3}$
45°	$1.002 \cdot 10^{-3}$	- 0.252	$3.741 \cdot 10^{-3}$	$3.620 \cdot 10^{-3}$	$3.621 \cdot 10^{-3}$

only small difference

excellent agreement, even for the worst case of 0.75 m projected helix radius (1GeV/c @ 4T)

Discussion of the covariance

$$C(\lambda,\kappa)^{tot}_{MS} \approx \begin{pmatrix} a_{1,1}\cdot\cos\lambda & a_{1,2}\cdot\sin\lambda \\ a_{1,2}\cdot\sin\lambda & b_{2,2}\cdot\frac{1}{\cos\lambda} \end{pmatrix}$$

- Problem: can’t get $a_{1,2}$ from e.g. Gluckstern’s formulae, because cov(λ,κ) = 0 in the symmetry plane z = 0
- Neglect small difference between $\sigma(\Delta p_t/p_t)$ and $\sigma(\Delta p/p)$
- Large external lever arm and traversal of much passive material: assume $\rho(\lambda,\kappa) \to -1$
- Determine $a_{1,2}$ using simulation at $\lambda \neq 0$

A. Einstein: ***“It’s better to be roughly right than to be precisely wrong.”***

Covariance matrix at higher energy

- Summation of covariance matrices for MS (which dominates at small p_t) and for detector errors:

$$\mathrm{C}(\lambda,\kappa)^{tot} = \mathrm{C}(\lambda,\kappa)^{tot}_{MS} + \mathrm{C}(\lambda,\kappa)_{det}$$

- Dependence on p_t:

$$\mathrm{C}(\lambda,\kappa)^{tot}_{MS} \approx \gamma^{-2} \cdot \begin{pmatrix} a'_{1,1} \cdot \cos\lambda & a'_{1,2} \cdot \gamma^{-1} \cdot \sin\lambda \\ a'_{1,2} \cdot \gamma^{-1} \cdot \sin\lambda & b'_{2,2} \cdot \frac{1}{\cos\lambda} \end{pmatrix}$$

$$a'_{i,j} = a_{i,j}(\lambda = 0, \kappa = \kappa_0) \qquad \gamma = \frac{p_t}{p_t^{ref}}$$
$$b'_{i,j} = b_{i,j}(\lambda = 0, \kappa = \kappa_0)$$

Covariance matrix at higher energy

- Covariance matrix due to detector errors:

$$\mathrm{C}(\lambda,\kappa)_{det} \approx \begin{pmatrix} c_{1,1} \cdot \cos^4\lambda & 0 \\ 0 & c_{2,2} \end{pmatrix}$$

  - var(κ) can be assumed to be constant w.r.t. λ, and cov(λ,κ) can be neglected
  - All terms are constant w.r.t. p_t down to 5 GeV/c, where multiple scattering dominates by an order of magnitude

Inclusion of azimuthal angle φ

- MS in λ:

$$\mathbf{C}(\lambda,\kappa)_{MS(\lambda)} = \left(\begin{array}{c|c} a_{1,1}\cdot\cos\lambda & a_{1,2}\cdot\sin\lambda \\ a_{1,2}\cdot\sin\lambda & a_{2,2}\cdot\frac{\sin^2\lambda}{\cos\lambda}\end{array}\right) \Rightarrow$$

$$\mathbf{C}(\lambda,\varphi,\kappa)_{MS(\lambda)} = \begin{pmatrix} a_{1,1}\cdot\cos\lambda & 0 & a_{1,2}\cdot\sin\lambda \\ 0 & a_{3,3}\cdot f(\lambda) & a_{3,2}\cdot f(\lambda) \\ a_{1,2}\cdot\sin\lambda & a_{3,2}\cdot f(\lambda) & a_{2,2}\cdot\frac{\sin^2\lambda}{\cos\lambda}\end{pmatrix}$$

- MS in φ:

$$\mathbf{C}(\lambda,\kappa)_{MS(\varphi)} = \left(\begin{array}{c|c} b_{1,1}\cdot\sin^2\lambda\cos\lambda & b_{1,2}\cdot\sin\lambda \\ b_{1,2}\cdot\sin\lambda & b_{2,2}\cdot\frac{1}{\cos\lambda}\end{array}\right) \Rightarrow$$

$$\mathbf{C}(\lambda,\varphi,\kappa)_{MS(\lambda)} = \begin{pmatrix} b_{1,1}\cdot\sin^2\lambda\cos\lambda & 0 & b_{1,2}\cdot\sin\lambda \\ 0 & b_{3,3}\cdot\frac{1}{\cos\lambda} & b_{3,2}\cdot\frac{1}{\cos\lambda} \\ b_{1,2}\cdot\sin\lambda & b_{3,2}\cdot\frac{1}{\cos\lambda} & b_{2,2}\cdot\frac{1}{\cos\lambda}\end{pmatrix}$$

- Derivatives not λ dependent: var(φ) and cov(φ,κ) show same λ dependence as

$$\mathbf{V}^{\lambda\neq 0}_{MS(\varphi)} = \mathbf{V}^{\lambda=0}_{MS(\varphi)}\cdot\frac{1}{\cos\lambda}$$

- cov(λ,φ) can be neglected

Inclusion of azimuthal angle φ

- var(φ) and cov(φ,κ) strongly dominated by MS in φ
- Total MS covariance matrix of kinematic terms (including p_t dependence):

$$\mathbf{C}(\lambda,\varphi,\kappa)^{tot}_{MS} \approx \gamma^{-2}\cdot\begin{pmatrix} a'_{1,1}\cdot\cos\lambda & 0 & a'_{1,2}\cdot\gamma^{-1}\cdot\sin\lambda \\ 0 & b'_{3,3}\cdot\frac{1}{\cos\lambda} & b'_{2,3}\cdot\frac{1}{\cos\lambda} \\ a'_{1,2}\cdot\gamma^{-1}\cdot\sin\lambda & b'_{2,3}\cdot\frac{1}{\cos\lambda} & b'_{2,2}\cdot\frac{1}{\cos\lambda}\end{pmatrix}$$

- Covariance matrix for detector errors:

$$\mathbf{C}(\lambda,\kappa)_{det} \approx \left(\begin{array}{c|c} c_{1,1}\cdot\cos^4\lambda & 0 \\ 0 & c_{2,2}\end{array}\right) \Rightarrow \mathbf{C}(\lambda,\varphi,\kappa)_{det} \approx \begin{pmatrix} c_{1,1}\cdot\cos^4\lambda & 0 & 0 \\ 0 & c_{3,3} & c_{3,2} \\ 0 & c_{3,2} & c_{2,2}\end{pmatrix}$$

Summary

- Method needs 9 coefficients per detector setup for detector optimization
  - 5 to build covariance matrix of multiple scattering
  - 4 to build covariance matrix of detector errors
- Coefficients determinable using only two single tracks
  - One low energetic track yielding the coefficients of the multiple scattering matrix
  - One high energetic track yielding the coefficients of the detector error matrix
  - Both starting at x = 0, y = 0 → *independent of* φ
  - Eventually expansion to Perigee parameter and z

Summary

- Optimization carried out in plane perpendicular to the magnetic field only
  - Leave this plane using the λ dependencies
  - Desired momentum using the p_t dependencies
- Prerequisites:
  - rotational symmetry
  - invariance w.r.t. translations parallel to magnetic field (no z dependent resolution in e.g. a TPC)

Summary

- Step 1: Calculate coefficients of C_{MS}^{tot} at $\lambda = 0$ and $p_t = p_t^{ref}$, e.g. from [1]
- Step 2: Calculate coefficients of C_{det} at $\lambda = 0$, e.g. from [2]
- Step 3: Use λ dependencies to leave the plane perpendicular to the magnetic field
- Step 4: Use p_t dependencies for desired momentum
- Step 5: Add C_{MS} and C_{Det}
- Step 6: Use the global formula to calculate $\sigma(\Delta p/p)$
  - neglecting the correction terms or
  - assuming $\rho = -1$ for a long lever arm incl. material
  - taking $a_{1,2}$ from simulation at $\lambda \neq 0$

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