Geometric and ultrarelativistic methods for accelerator physics

Jonathan Gratus

Physics Department Lancaster University and The Cockcroft Institute

Advanced QED methods for future accelerators 03-04 March 2009

Outline

- Advantages of the ultrarelativistic approach.
- The ultrarelativistic expansion for a cold fluid (hydrodynamic picture).
- The ultrarelativistic expansion for a charged particle.
- The ultrarelativistic expansion of the Maxwell-Vlasov equations.
- Unifying the three approaches.

Advantages of the ultrarelativistic approach.

- Classically the dynamics of particle beams due to external and self-fields is described by coupling Maxwell's equations with a model for the beam.
- The electromagnetic fields satisfy Maxwell's equations with the source given by the beam and the external magnets and RF fields.
- The beam satisfies the Lorentz force equation and may be modelled in a variety of ways.
    - A collection of point particles,
    - A one particle probability distribution satisfying the Vlasov or Boltzmann equation.
    - A fluid either warm or cold (the hydrodynamic picture).
    - ...
- It is then necessary to solve a nonlinear system of integral or algebraic partial differential equations.
- These are usually solved numerically, for example by PIC methods.

Advantages of the ultrarelativistic approach.

- In accelerators particles move at speeds close to the speed of light and therefore we can use an ultrarelativistic expansion.
- This replaces the nonlinear system of equation with a hierarchy of linear equations.
- There is usually a single step in the hierarchy which contains a nonlinear equation, but this can be solved using the method of characteristics. (Particle tracking, optics).
- The ultrarelativistic expansions are usually used implicitly, in statements like "Space charge can be ignored for high energy particles".
- Higher order terms in the expansion are used for more accurate modelling, especially for intense beams.

- This approach may enable one to perform quantum mechanics at one level of the hierarchy, using the preceding levels of the hierarchy as a fixed "classical" background.

The ultrarelativistic equations for a cold fluid (hydrodynamic picture).

- Maxwell's equations

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \qquad \nabla \cdot \boldsymbol{B} = 0,$$

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0} \gamma \rho, \qquad \nabla \times \boldsymbol{B} = \frac{1}{m\epsilon_0} \rho \boldsymbol{p} + \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t}$$

- Lorentz force for a charged fluid

$$\gamma \frac{\partial \boldsymbol{p}}{\partial t} + \left(\frac{\boldsymbol{p}}{m} \cdot \boldsymbol{\nabla}\right) \boldsymbol{p} = q\left(\gamma \boldsymbol{E} + \frac{1}{m} \boldsymbol{p} \times \boldsymbol{B}\right), \qquad m^2 c^2 \gamma^2 - \boldsymbol{p} \cdot \boldsymbol{p} = m^2 c^2,$$

- Conservation of charge

$$m \frac{\partial}{\partial t}(\gamma \rho) + \nabla \cdot (\rho \boldsymbol{p}) = 0$$

- $\boldsymbol{E}(t, \boldsymbol{x})$ and $\boldsymbol{B}(t, \boldsymbol{x})$ are the electric and magnetic fields.
- Here m and q are the particles' mass and charge.
- $\big(mc^2 \gamma(t, \boldsymbol{x}), \boldsymbol{p}(t, \boldsymbol{x})\big)$ is the energy and momentum fields of the electrons.
- $\rho(t, \boldsymbol{x})$ is the proper charge density.

Cold fluid

- In order to perform the ultrarelativistic expansion we introduce a parameter $\varepsilon > 0$ which we may think of as the reciprocal of the energy of the beam. I.e.

$$\varepsilon = \frac{1}{\gamma_{\text{design}}}$$

- We introduce the following expansions.

$$\begin{aligned} &\boldsymbol{E} = \varepsilon^{-1}\boldsymbol{E}_0 + \boldsymbol{E}_1 + \varepsilon\boldsymbol{E}_2 + \dots\,, &\qquad &\boldsymbol{B} = \varepsilon^{-1}\boldsymbol{B}_0 + \boldsymbol{B}_1 + \varepsilon\boldsymbol{B}_2 + \dots\,, \\ &\gamma = \varepsilon^{-1}\gamma_0 + \gamma_1 + \varepsilon\gamma_2 + \dots\,, &\qquad &\boldsymbol{p} = \varepsilon^{-1}\boldsymbol{p}_0 + \boldsymbol{p}_1 + \varepsilon\boldsymbol{p}_2 + \dots\,, \\ &\rho = \varepsilon\rho_0 + \varepsilon^2\rho_1 + \dots \end{aligned}$$

- The leading terms in $(m\gamma, \boldsymbol{p})$ must increase with γ_{design} and therefore we start the expansions of these at ε^{-1}.
- Observe that the leading terms in $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$ increase with the "energy of the beam". An example of these is the external bending and focusing magnets.
- The proper charge density starts at order ε so that the total current $\boldsymbol{J} = \rho\boldsymbol{p}$ remains finite.

Cold fluid: Hierarchy

- The external electric $\boldsymbol{E}_0$ and magnetic $\boldsymbol{B}_0$ fields satisfy the source-free Maxwell equations.

$$\nabla \cdot \boldsymbol{E}_0 = 0, \quad \nabla \times \boldsymbol{B}_0 = \frac{1}{c^2}\frac{\partial \boldsymbol{E}_0}{\partial t}, \quad \nabla \times \boldsymbol{E}_0 + \frac{\partial \boldsymbol{B}_0}{\partial t} = 0, \quad \nabla \cdot \boldsymbol{B}_0 = 0.$$

- The following non linear differential and algebraic equations are solved for γ_0 and $\boldsymbol{p}_0$.

$$\gamma_0 \frac{\partial \boldsymbol{p}_0}{\partial t} + \left(\frac{\boldsymbol{p}_0}{m} \cdot \boldsymbol{\nabla}\right)\boldsymbol{p}_0 = q\left(\gamma_0 \boldsymbol{E}_0 + \frac{1}{m}\boldsymbol{p}_0 \times \boldsymbol{B}_0\right), \quad m^2 c^2 \gamma_0^2 - \boldsymbol{p}_0 \cdot \boldsymbol{p}_0 = 0$$

These describe the motion of a lightlike fluid of charged particles undergoing the Lorentz force equations. Although these equations are nonlinear they can be solved using characteristic curves (particle tracking) techniques.

- The leading order equation describing charge conservation is

$$m \frac{\partial}{\partial t}(\gamma_0 \rho_0) + \nabla \cdot (\rho_0 \boldsymbol{p}_0) = 0$$

and is solved for ρ_0.

Cold fluid: Hierarchy

- Solve the Maxwell equations for $\boldsymbol{E}_1$ and $\boldsymbol{B}_1$ with the charge density $\frac{1}{\epsilon_0}\gamma_0\rho_0$ and 3-current $\frac{1}{m\epsilon_0}\rho_0\boldsymbol{p}_0$ as sources:

$$\nabla\cdot\boldsymbol{E}_1=\frac{1}{\epsilon_0}\gamma_0\rho_0,\quad \nabla\times\boldsymbol{B}_1=\frac{1}{m\epsilon_0}\rho_0\boldsymbol{p}_0+\frac{1}{c^2}\frac{\partial\boldsymbol{E}_1}{\partial t},$$
$$\nabla\times\boldsymbol{E}_1+\frac{\partial\boldsymbol{B}_1}{\partial t}=0,\quad \nabla\cdot\boldsymbol{B}_1=0.$$

- Solve

$$\gamma_0\frac{\partial\boldsymbol{p}_1}{\partial t}+\gamma_1\frac{\partial\boldsymbol{p}_0}{\partial t}+\left(\frac{\boldsymbol{p}_1}{m}\cdot\boldsymbol{\nabla}\right)\boldsymbol{p}_0+\left(\frac{\boldsymbol{p}_0}{m}\cdot\boldsymbol{\nabla}\right)\boldsymbol{p}_1$$
$$=q\Big(\gamma_0\boldsymbol{E}_1+\gamma_1\boldsymbol{E}_0\Big)+\frac{q}{m}\Big(\boldsymbol{p}_0\times\boldsymbol{B}_1+\boldsymbol{p}_1\times\boldsymbol{B}_0\Big),$$

and $$-\gamma_0\gamma_1+\frac{\boldsymbol{p}_0\cdot\boldsymbol{p}_1}{m_0^2c^2}=0$$

for γ_1 and $\boldsymbol{p}_1$. Use of the above algebraic equation to eliminate γ_0 leads to an inhomogeneous linear partial differential equation for $\boldsymbol{p}_1$.

- These equations give the correction to the motion of the beam.

Single particle

- Again we have the expansion for the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields.

$$\boldsymbol{E} = \varepsilon^{-1}\boldsymbol{E}_0 + \boldsymbol{E}_1 + \varepsilon\boldsymbol{E}_2 + \dots\,, \qquad \boldsymbol{B} = \varepsilon^{-1}\boldsymbol{B}_0 + \boldsymbol{B}_1 + \varepsilon\boldsymbol{B}_2 + \dots\,,$$

- Assume for the moment that these are prescribed.
- $\boldsymbol{E}_0$, $\boldsymbol{B}_0$ satisfy the source free Maxwell equations
- $\boldsymbol{E}_1$, $\boldsymbol{B}_1$ satisfy Maxwell's equations with the charge and current sources given by the motion of "other particles". For example in wakefield analysis, the other particles are the particles ahead of the bunch.
- Let $\big(t(\tau), \boldsymbol{x}(\tau)\big)$ be the laboratory time and position of a particle at proper time τ. We also set

$$\boldsymbol{p}(\tau) = m\frac{d\boldsymbol{x}}{d\tau}\,, \qquad \gamma(\tau) = \sqrt{1 + \frac{\boldsymbol{p}\cdot\boldsymbol{p}}{m^2c^2}} = \frac{dt}{d\tau}$$

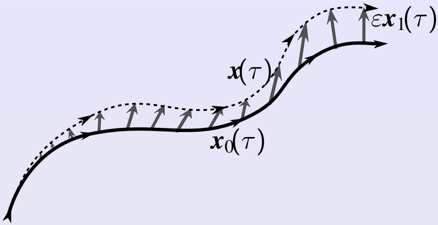
- The expansions of these variables are given by

$$\boldsymbol{x}(\tau) = \boldsymbol{x}_0(\tau) + \varepsilon\boldsymbol{x}_1(\tau) + \cdots\,, \quad t(\tau) = t_0(\tau) + \varepsilon t_1(\tau) + \cdots\,,$$

$$\gamma(\tau) = \frac{\gamma_0(\tau)}{\varepsilon} + \gamma_1(\tau) + \cdots \quad \text{and} \quad \boldsymbol{p}(\tau) = \frac{\boldsymbol{p}_0(\tau)}{\varepsilon} + \boldsymbol{p}_1(\tau) + \cdots$$

since $\boldsymbol{x}(\tau)$ and $t(\tau)$ are bounded as $\varepsilon \to 0$ but $\gamma(\tau)$ and $\boldsymbol{p}(\tau)$ diverge as $\varepsilon \to 0$.

Single particle, Hierarchy



- The leading order motion for the unperturbed particle is given by

$$\frac{d\boldsymbol{p}_0}{d\hat{\tau}} = q\left(\gamma_0 \boldsymbol{E}_0 + \frac{\boldsymbol{p}_0}{m} \times \boldsymbol{B}_0\right) \qquad \text{and} \qquad m^2c^2\frac{d\gamma_0}{d\hat{\tau}} = q\boldsymbol{p}_0 \cdot \boldsymbol{E}_0$$

where $\hat{\tau} = \varepsilon/\tau$, subject to the constraint

$$m^2c^2\gamma_0^2 - \boldsymbol{p}_0 \cdot \boldsymbol{p}_0 = 0$$

- These describe the motion of a charged particle moving at the speed of light and undergoing the Lorentz force equation.
- The reparametrisation of τ with $\hat{\tau}$ is because particles moving at lightspeed do not have a proper time.

Single particle, Hierarchy

- The first order corrections to the motion of the affected particle, and hence the wakefield kick formula, are given by

$$\frac{d\boldsymbol{p}_1}{d\hat{\tau}} = q\left(\gamma_0\left(t_1\frac{\partial \boldsymbol{E}_0}{\partial t} + (\boldsymbol{x}_1\cdot\nabla)\boldsymbol{E}_0\right) + \frac{\boldsymbol{p}_0}{m}\times\left(t_1\frac{\partial \boldsymbol{B}_0}{\partial t} + (\boldsymbol{x}_1\cdot\nabla)\boldsymbol{B}_0\right)\right.$$
$$\left. + \gamma_1\boldsymbol{E}_0 + \frac{\boldsymbol{p}_1}{m}\times\boldsymbol{B}_0 + \gamma_0\boldsymbol{E}_1 + \frac{\boldsymbol{p}_0}{m}\times\boldsymbol{B}_1\right)$$

  and

$$m^2c^2\frac{d\gamma_1}{d\hat{\tau}} = q\left(\boldsymbol{p}_0\cdot\left(t_1\frac{\partial \boldsymbol{E}_0}{\partial t} + (\boldsymbol{x}_1\cdot\nabla)\boldsymbol{E}_0\right) + \boldsymbol{p}_1\cdot\boldsymbol{E}_0 + \boldsymbol{p}_0\cdot\boldsymbol{E}_1\right)$$

  subject to the constraint

$$m^2c^2\gamma_0\gamma_1 = \boldsymbol{p}_0\cdot\boldsymbol{p}_1$$

- When considering the particles passing through wakefields, it is usual to apply the rigid bunches approximation. I.e. ignoring all the terms not containing $\boldsymbol{E}_1$ or $\boldsymbol{B}_1$:

$$\frac{d\boldsymbol{p}_1}{d\hat{\tau}} = \gamma_0\boldsymbol{E}_1 + \frac{\boldsymbol{p}_0}{m}\times\boldsymbol{B}_1 \qquad \text{and} \qquad m^2c^2\frac{d\gamma_1}{d\hat{\tau}} = \boldsymbol{p}_0\cdot\boldsymbol{E}_1$$

Single particle: Lorentz Dirac

- Up to now we have considered $\boldsymbol{E}_1$ and $\boldsymbol{B}_1$ to be prescribed fields.
- We would like $\boldsymbol{E}_1$ and $\boldsymbol{B}_1$ to be derived from the motion of the unperturbed particle $\boldsymbol{x}_0(\hat{\tau})$.
- However along the world-line of the unperturbed particle, the resulting electric and magnetic fields diverge.
- We therefore need to do a regularisation, similar to the Lorentz Dirac regularisation.
- This would give a new equation for the back reaction due to the emission of electromagnetic radiation.
- The new equation would be second degree, not third degree, and will therefore not suffer from the pathological nature of the Lorentz Dirac equation.
- This is a natural setting for looking at quantum corrections, where $\boldsymbol{E}_0, \boldsymbol{B}_0$ and $\boldsymbol{x}_0$ are the classical background.

Vlasov Field

- Final example: consider the ultrarelativistic Maxwell-Vlasov equations.
- We expand out the one particle probability distribution $f = f(t, \boldsymbol{x}_0, \boldsymbol{p}_0)$

$$f(t, \boldsymbol{x}_0, \boldsymbol{p}_0) = f_0(t, \boldsymbol{x}_0, \boldsymbol{p}_0) + \varepsilon f_1(t, \boldsymbol{x}_0, \boldsymbol{p}_0) + \ldots$$

- Solve the Vlasov equation for fixed electromagnetic fields.

$$\left(\gamma_0 \frac{\partial}{\partial t} + \frac{\boldsymbol{p}_0}{m} \cdot \frac{\partial}{\partial \boldsymbol{x}_0} + q(\gamma_0 \boldsymbol{E}_0 + \frac{\boldsymbol{p}_0}{m} \times \boldsymbol{B}_0) \cdot \frac{\partial}{\partial \boldsymbol{p}_0}\right) f_0 = 0$$

Vlasov Field

- Solve the correction to the electromagnetic fields,

$$\nabla \cdot \boldsymbol{B}_1 = 0\,, \qquad \nabla \times \boldsymbol{E}_1 + \frac{\partial \boldsymbol{B}_1}{\partial t} = 0\,,$$
$$\nabla \cdot \boldsymbol{E}_1 = \frac{q}{\epsilon_0} \int f(t, \boldsymbol{x}_0, \boldsymbol{p}_0) d^3\boldsymbol{p}_0 \quad \text{and}$$
$$\nabla \times \boldsymbol{B}_1 - \frac{1}{c^2}\frac{\partial \boldsymbol{E}_1}{\partial t} = \mu_0 q \int \frac{\boldsymbol{p}_0}{\gamma_0 m} f(t, \boldsymbol{x}_0, \boldsymbol{p}_0) d^3\boldsymbol{p}_0$$

- Calculate the correction $f_1(t, \boldsymbol{x}_0, \boldsymbol{p}_0)$ to the one particle probability distribution.

$$\left(\gamma_0 \frac{\partial}{\partial t} + \frac{\boldsymbol{p}_0}{m} \cdot \frac{\partial}{\partial \boldsymbol{x}_0} + q(\gamma_0 \boldsymbol{E}_0 + \frac{\boldsymbol{p}_0}{m} \times \boldsymbol{B}_0) \cdot \frac{\partial}{\partial \boldsymbol{p}_0}\right) f_1 = q(\gamma_0 \boldsymbol{E}_1 + \frac{\boldsymbol{p}_0}{m} \times \boldsymbol{B}_1) \cdot \frac{\partial f_0}{\partial \boldsymbol{p}_0}$$

Unifying the three models of ultrarelativistic charge: Some thoughts from differential geometry

- The three models are all examples to the **"Distributional Ultrarelativistic Maxwell-Vlasov equations"**
- We look at distributions on $T\mathcal{M}$ where $\mathcal{M}$ is spacetime.
- The distributions are similar in nature to δ', the derivative of Dirac δ-functions, only in higher dimensions.
- They have support on a submanifold and are differentiated with respect to vector field transverse to the manifold.
- This method has a number of advantages:
  - It unifies all three approaches, together with the ultrarelativistic KV distribution and the ultrarelativistic multicurrent distribution.
  - It gives alternative ways to write the ultrarelativistic expansion.
  - It leads to new models of ultrarelativistic charge.
  - The equations are "prettier".
  - It naturally contains the effects of gravity.
- Is this a good formulation to perform QED for intense beams?