The Furry picture

In short: A picture to describe a quantum mechanical system in external fields: intermediate between the Heisenberg and the interaction representations

Outline:
- Introduction QM
- Schrödinger, Heisenberg and interaction representation
- Furry representation
  - → commutation relations
  - → gauge transformations
  - → charge conjugation
  - → vacuum polarization
  - → electron propagator
- Conclusions

1. Introduction QM:

a) state given by vector $|\psi\rangle$ $\quad$ ($\langle\psi|\psi\rangle = 1$ norm)

b) observables $\hat{=}$ hermitean operators A $\quad$ $A\,|a_i\rangle = \alpha_i\,|a_i\rangle$ with

α_i eigenvalues and $|\psi\rangle = \sum c_i\,|a_i\rangle$, $|a_i\rangle$ eigenstates

c) expectation values $\langle A\rangle = \langle\psi|A|\psi\rangle$ corresponds to mean value

$$= \sum_j \alpha_j\,|\langle\alpha_j|\psi\rangle|^2$$

d) if not full information on system: $\langle\langle A\rangle\rangle := \sum p_i \langle i|A|i\rangle = Tr(\rho\, A)$, where $\rho = \sum p_i |i\rangle\langle i|$ density operator and p_i probability to have state $|i\rangle$

e) commutation relation: two observables simultaneously measurable, if

$$[A,B] = 0 = AB - BA$$

f) correspondence principle: If $\sum_i \xrightarrow{i\to\infty}$ classical theory

Heisenbergs commutation relations: $\quad [X_i, X_j] = 0$

$$[P_i, P_j] = 0$$

$$[X_i, P_j] = i\hbar\,\delta_{ij}$$

g) uncertainty : $(\Delta A) = \sqrt{\langle (A - \langle A\rangle)^2\rangle}$ $\quad \Rightarrow \Delta A \cdot \Delta B \geq \frac{1}{2}\langle\psi|[A,B]|\psi\rangle$

2. Time-dependence in QM: Time development between two measurements

2.1 Schrödinger representation:

state vector t-dependent $\frac{d}{dt}|\psi_s\rangle = \frac{-i}{\hbar} H |\psi_s\rangle$, i.e. $|\psi_s(t)\rangle = U(t,t_0)|\psi_s(t_0)\rangle$

with unitary operator $U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt' H(t') U(t',t_0)$, $U(t_0,t_0)=1$

if closed system, d.h. $H \neq H(t)$: $U(t,t_0) = e^{-i/\hbar (t-t_0) H}$

energy eigenstates: $U(t,t_0)|E_n,t_0\rangle = e^{-i/\hbar (t-t_0) E_n}|E_n,t_0\rangle$

$\Rightarrow |\psi_s\rangle$ t-dependent, but observables A and basis vectors $|a_i\rangle$ constant

2.2 Heisenberg representation:

If $t=t_0 \Rightarrow A_H(t_0)=A_S$ and $|\psi_H\rangle = |\psi_s(t_0)\rangle$

observables, operators t-dependent: $\frac{dA}{dt} = \frac{[A,H]}{i\hbar} + \frac{\partial A}{\partial t}$ (cf. Hamilton theorie, Poisson br

$A_H(t) = U^+(t) A_S U(t)$ and $|\psi_H(t)\rangle = U^+(t)|\psi_s(t)\rangle = U^+(t) U(t)|\psi_s(t_0)\rangle = |\psi_s(t_0)\rangle$

basis vectors: $|a_i,t\rangle = U^+(t)|a_i\rangle$ $[\, U^+(t) A(t_0)|a_i\rangle = \underbrace{U^+ A U}_{A_H(t)} U^+|a_i\rangle = \alpha_i U^+(t)|$

$\Rightarrow A_H$ and basis vectors $|a_i\rangle$ t-dependent, but state vector $|\psi_H\rangle$ constant.

Mean values: $\langle\psi|A|\psi\rangle$ constant if $|\psi\rangle$ stationary ($|\psi_s\rangle$ t-indep.)

or A conserved quantity ($[A_H,H_H]=0$, i.e. $\frac{dA_H}{dt}=0$)

Advantage: Expectation values follow class. physics.

2.3 Interaction representation ("Dirac representation")

$\Rightarrow$ Any t-dependent unitary transformation of state vectors and observables is possible

Idea: $H = \underbrace{H_0}_{\text{free}} + \underbrace{V}_{\text{interaction}}$ ‖

Since H_0 t-independent $\Rightarrow U_0(t,t_0) = e^{-i/\hbar\,(t-t_0)\,H_0}$, H_0 with known eigenvalues/vectors

Assumption: if $H \approx H_0 \Rightarrow U \approx U_0$ for t sufficiently small

Dirac representation: $|\psi_D(t)\rangle = U_0^+(t)|\psi_S(t)\rangle = U_0^+(t)U(t)|\psi_H\rangle = U_D(t)\,|\psi_D(0)\rangle$

(with $|\psi_D(0)\rangle = |\psi_S(0)\rangle = |\psi_H\rangle$;) $= U_D(t)\,|\psi_H\rangle$

$$A_D = U_0^+ A_S U_0 = U_0^+ U A_H U^+ U_0$$

Explicit time dependence:

$$\frac{d}{dt}|\psi_D\rangle = \frac{d}{dt}\left[U_0^+(t)\,U(t)\,|\psi_H\rangle\right] = \frac{dU_0^+(t)}{dt}\left(U(t)\,|\psi_H\rangle\right) + U_0^+(t)\,\frac{dU(t)}{dt}\,|\psi_H\rangle + U_0^+ U \underbrace{\frac{d}{dt}|\psi_H\rangle}_{=0}$$

$$= \frac{i}{\hbar} H_0 U_0^+(t)U(t)|\psi_H\rangle + U_0^+\left(-\frac{i}{\hbar}H\right)U(t)|\psi_H\rangle = \frac{i}{\hbar}\left(H_0U_0^+ - U_0^+(t)\underbrace{H}_{U_0U_0^+}\right)U(t)\,|\psi_H\rangle$$

$$= \frac{i}{\hbar}\left(H_0 - U_0^+(t)\,H\,U_0(t)\right)|\psi_D\rangle$$

$\Rightarrow \quad \frac{d}{dt} |\psi_D\rangle = -\frac{i}{\hbar} V_D |\psi_D\rangle \quad (*) \quad$ with $\quad V_D = U_0^+(t)\, V\, U_0(t)$

and

$$\frac{d}{dt} A_D = \frac{i}{\hbar} [H_0, A_D]$$

$\Rightarrow |\psi_D\rangle$ t-dependent via V_D ; observables A_D time-dependent via H_0.

Since $|\psi_D(t)\rangle = U_D(t)\, |\psi_D(0)\rangle$ and with $(*)$:

$$\frac{d}{dt} U_D = -\frac{i}{\hbar} V_D(t)\, U_D(t) \qquad \text{chronology of operators crucial!}$$

<u>Ansatz:</u> $\int_{t_0}^{t}$

$U_D(t,t_0) - U_D(t_0,t_0) = \int_{t_0}^{t} dt' \frac{V_D(t',t_0) U_D(t',t_0)}{i\hbar} \Rightarrow$ time ordering gets crucial:

Since $U(t,t_0) = U_0(t,t_0)\, U_D(t,t_0)$ and using unitarity $U_0(t,t_0) U_0^+(t',t_0) = U_0(t,t')$

$$\Rightarrow U(t,t_0) = U_0(t,t_0) + \int_{t_0}^{t} dt' \frac{U_0(t,t') V U(t',t_0)}{i\hbar}$$

stepwise integration: $U(t,t_0) = \underbrace{U_0(t,t_0) + \int_{t_0}^{t} dt' \frac{U_0(t,t') V U_0(t',t_0)}{i\hbar}}_{\text{Born}} + \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \frac{U_0(t,t') V U_0(t',t'') V U_0(t'',t_0)}{(i\hbar)^2}$

$\Rightarrow$ <u>perturbation series</u>; higher orders step-by-step included

emember: basis is eigensystem of the unperturbed Hamiltonian, i.e. the free system.

Just as sketch: we need time operators from $t \to -\infty$ and $t \to \infty$, therefore set $x_0 = 0$ and discuss both cases $t \to \pm\infty$ separately.

With step-function $\varepsilon(x) \equiv \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$ (derivative of ε is δ-distribution!)

$\Rightarrow$ introduce Green functions via $G^{\pm}(t) := \dfrac{\varepsilon(\pm t)\, U(t,0)}{\pm i\hbar}$, $G_0^{\pm}(t) := \dfrac{\varepsilon(\pm t)\, U_0(t,0)}{\pm i\hbar}$

solutions of: $\left(i\hbar \frac{\partial}{\partial t} - H\right) G^{\pm}(t) = \delta(t)$ and $\left(i\hbar\frac{\partial}{\partial t} - H_0\right) G_0^{\pm}(t) = \delta(t)$

These Green functions $\underline{G^{\pm}(t-t_0)}$ provide the propagators.

Example: $\mathcal{L}$ in QED

$$\mathcal{L}_{QED} = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi}_{\text{free terms}} \underbrace{- e\bar{\psi}\not{A}\psi}_{\text{interaction}} \left(\underbrace{-\mathcal{L}_{GF}}_{\text{gauge fixing}} \right)$$

$\Rightarrow$ electromagnetic interaction given by: $-e\bar{\psi}\gamma_\mu A^\mu \psi = -e j^\mu A_\mu$ where current density fulfilles continuity $\partial_\mu j^\mu = 0$

$$\Rightarrow \quad V \mathrel{\hat{=}} -e j_\mu A^\mu$$

3. Furry representation (Phys. Rev. 81 (1951) 115)

- Kind of interaction representation

difference : regard the system in a bound state instead of using free-particle states.

Applicable if external fields contribute.

$\Rightarrow$ External field "causes" the bound states

Example : QED Lagrangian

$$\mathcal{L}_{\text{QED, external field}} = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\text{free term}} \underbrace{- e \bar{\psi} \gamma_\mu A^\mu \psi}_{\text{interaction term}} + \underbrace{\bar{\psi}\left[(i\not{\partial} - e \not{A}^{ext}) - m \right] \psi}_{\substack{\text{free Dirac + external field} \\ \to \text{ bound "Dirac field"}}}$$

3.1 Commutation relations in Furry representation

Transition :

$$\text{QM} \longrightarrow \text{QFT}$$

QM:

$$[x_i, p_j] = i\hbar\,\delta_{ij}$$
$$[x_i, x_j] = 0$$
$$[p_i, p_j] = 0$$

QFT: canonical quantization
scalar, spinor + gauge fields

$$[\dot\phi(\vec{x},t), \phi(\vec{y},t)] = -i\delta^{(3)}(\vec{x}-\vec{y})$$
$$\{\psi^+_\alpha(\vec{x},t), \psi_\beta(\vec{y},t)\} = -i\delta_{\alpha\beta}\,\delta^{(3)}(\vec{x}-\vec{y})$$
$$[\dot A_\mu(\vec{x},t), A_\nu(\vec{y},t)] = -i\, g_{\mu\nu}\,\delta^{(3)}(\vec{x}-\vec{y})$$

- In Furry representation : effects of external field A_μ^{ext} included in wave functions ("bound states")

$\Rightarrow$ commutation relations for $\{\psi, \psi^+\}_{Furry} \neq \{\psi, \psi^+\}_{Dirac}$

(Aside: "ψ^+" = adjoint $\Rightarrow$ denotation Furry)

$$\{\psi^+_\alpha(x), \psi_\beta(x')\} = \sum_i \psi^+_{(i)\alpha}(x)\, \psi_{(i)\beta}(x') \xrightarrow{A_\mu^{ext}\to 0} -i\delta_{\alpha\beta}\,\delta^{(3)}(x-x$$

- There exist a canonical transformation between the field operators in the interaction picture and in the Furry picture.

3.2 Gauge transformations in Furry picture

QFT = "gauge theory", ie. invariant under gauge transformations:

- $A_\mu(x) \to A_\mu(x) - \frac{\partial \Lambda(x)}{\partial x_\mu}$, $\psi(x) \to e^{-ie\Lambda(x)}\psi(x)$, $\psi^+(x) \to e^{ie\Lambda(x)}\psi^+(x)$,

where $\Lambda(x)$ is scalar function $\Rightarrow \frac{\partial^2 \Lambda(x)}{\partial x_\mu^2} = \Box^2 \Lambda(x) = 0$

In Furry picture:

$$A_\mu^{ext} \to A_\mu^{ext} - \frac{\partial \Lambda^{ext}}{\partial x_\mu} \text{ and } \psi \to e^{-ie\Lambda^{ext}}\psi,\ \psi^+ \to e^{ie\Lambda^{ext}}\psi^+ \text{ and } A_\mu \to A_\mu - \frac{\partial \Lambda}{\partial x_\mu}$$

3.3 Charge conjugation in Furry picture

Usually: $\psi^c(x) = C\psi^+(x)$ and $\psi^{+c}(x) = C^{-1}\psi(x)$ charge-conjugated wave functions

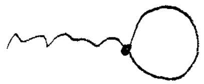
with $C^{-1}\gamma_\mu C = -\gamma_\mu^T$

In Furry picture: since C operator commutes with canonical transformation Dirac <-> Furry

$\Rightarrow$ resulting $\left\{ \begin{array}{l} \text{equations of motion} \\ \text{commutation relations} \end{array} \right\}$ of charge-conjugated function differs by sign in "e"

$\Rightarrow$ lack of absolute symmetry between ψ and ψ^c due to external field leads to physical consequences (vacuum polarization)

3.4 Vacuum polarization



Diagrams with self-closed electron lines can in "normal" QED be rejected
→ association with "vacuum current"

a) vacuum expectation value of current $j^\mu = \psi^+ \gamma^\mu \psi$ has to vanish due to Lorentz invariance

b) also because of $C j^\mu(x) C^+ = -j^\mu(x) \Rightarrow \langle \Omega | j^\mu | \Omega \rangle \xrightarrow{C} -\langle \Omega | j^\mu | \Omega \rangle \equiv 0$

In Furry picture: external field causes vacuum polarization

$\Rightarrow$ such diagrams have to be included

Also remember: different behaviour of ψ and ψ^c !

Problem: $\langle j_\mu(x) \rangle_0 \sim \left[\sum_s \psi_s^+ \gamma_\mu \psi_s - \sum_\sigma \psi_\sigma^+ \gamma_\mu \psi_\sigma \right]$ (*) $\psi_{s/\sigma} \,\hat{=}$ positive/negative energy states of e^- in field

$\Rightarrow \sum$ are divergent

- (*) contains whole vacuum polarization up to order e^2
- in higher orders: also contributions from interaction between ψ, A_μ, ψ^+
- included in (*): higher-order contributions from $A_\mu^{ext} \sim e^4 (A_\mu^{ext})^3, \ldots$ non-linear

→ divergences completely removable via regularization etc.

- still problem with Born approximation exists

  replace $\underset{\text{"bound"}}{(\Psi)_{Furry}} \to \underset{\text{"free"}}{(\Psi)_{Dirac}}$

  → log. divergence removed via renormalization

$$\langle j_\mu \rangle_0 = \underbrace{J_\mu^{\log}}_{\substack{\text{removed} \\ \to \text{renormal.}}} + J_\mu'$$

- contribution of J_μ' depends on $\begin{cases} \text{wavelength} \to \text{if } \lambda = \text{large} \to \text{negligible} \\ \text{Coulomb field} \to \text{vanishes if spherically symme[tric]} \end{cases}$

.5 Electron Propagator

If external field → space and time no longer homogeneous

⇒ Green function $\mathcal{G}(x,x')$ depends on x and x', not only on difference $(x-x')$!

$$\mathcal{G}(x,x') = -i\langle 0|T\,\psi_\ell^{ext}(x)\,\psi_k^{+ext}(x')|0\rangle$$ = zero-order approximation

. Conclusion

- Furry representation: intermediate between Heisenberg and Dirac representation
- eigenstates of bound system instead of free-particle system
- effects of external field adapted in Dirac function of electron
- due to this bound state, adaption/inclusion of
  - commutation relations
  - gauge transformations
  - charge conjugation operation
  - contribution of self-closed diagrams in vacuum polarization
  - electron propagator $\mathcal{G}(x,x') \neq \mathcal{G}(x-x')$