Kinematic Cusps[1] and Algebraic Singularity Method[2] for Missing Energy Measurement

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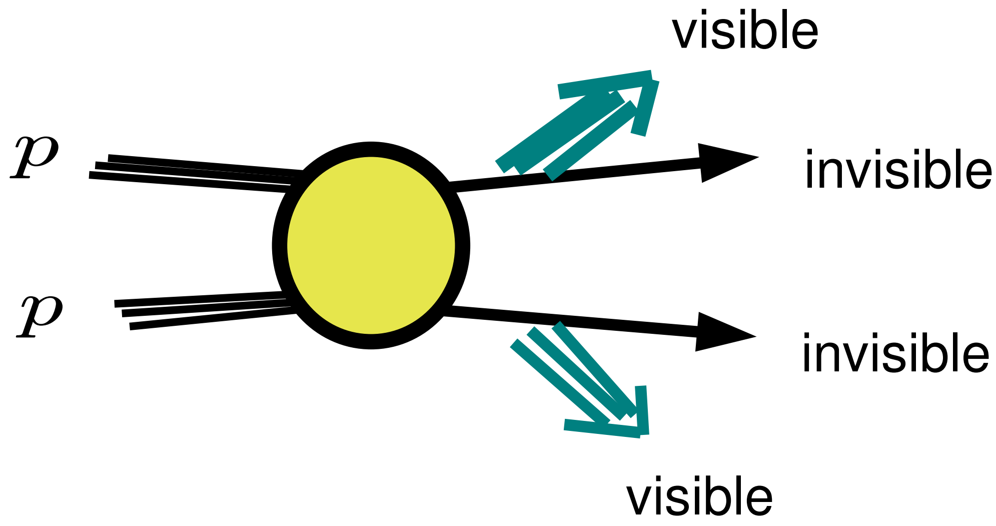
[1] T. Han, IWK, J.Song : arXiv:0906.5009

[2] IWK : work in progress

Linear Collider Workshop Univ. of New Mexico Sep 30, 2009



Reconstructable event : Solve mass shell equation for each event

Nojiri, Polesello, Tovey (2003)

Cheng, Gunion, Han, Marandella, McElrath (2007)

Nonreconstructable event :

Use end point or cusps of kinematic variable

Hinchliffe, Paige, Shapiro, Soderqvist, Yao (1997)

Han, IWK, Song (2009)

Transverse mass variables

Lester, Summers (1999)

Cho, Choi, Kim, Park (2007)

Bar, Gripaios, Lester (2007)

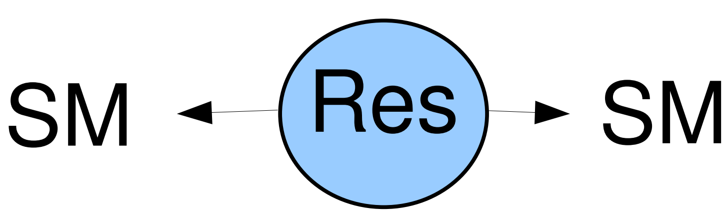
Use heavy resonant particle.

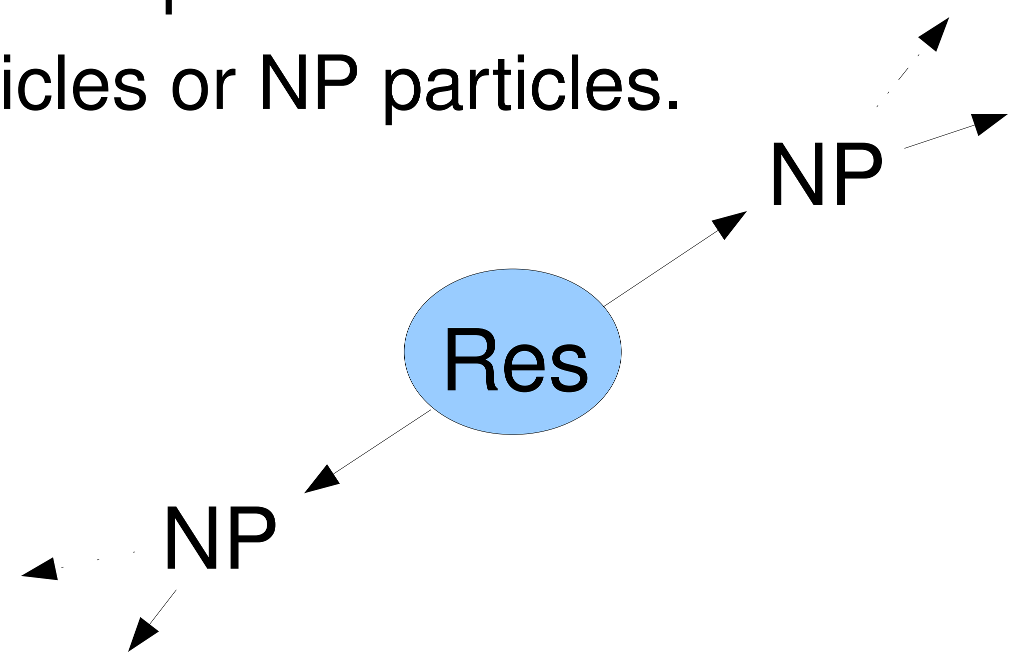
Z' new neutral gauge boson

$V_\mu^{(2)}$ higher KK modes

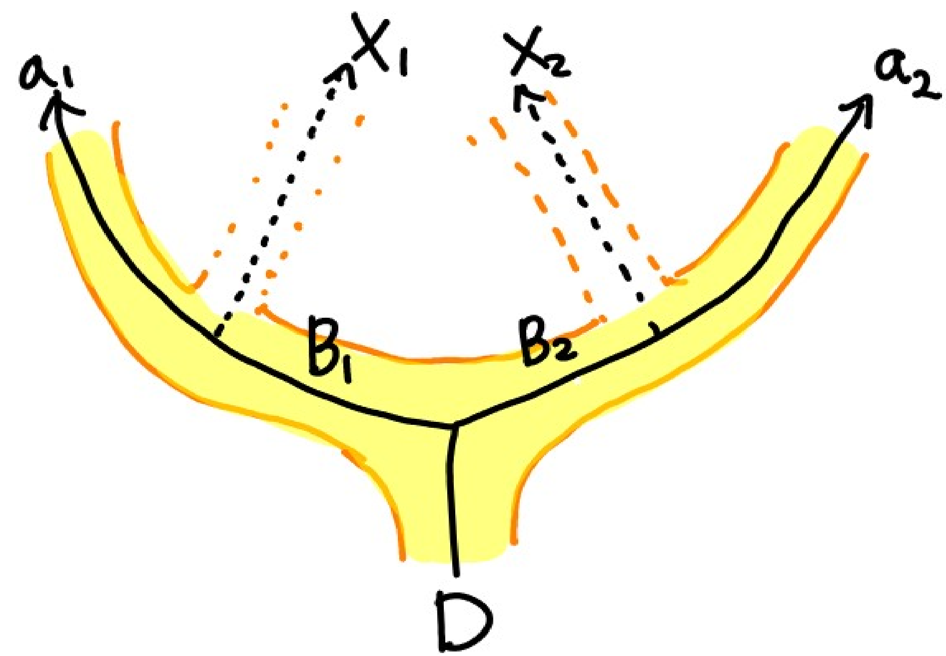
H Heavy Higgs

Many models have resonance particle that can decay to SM particles or NP particles.



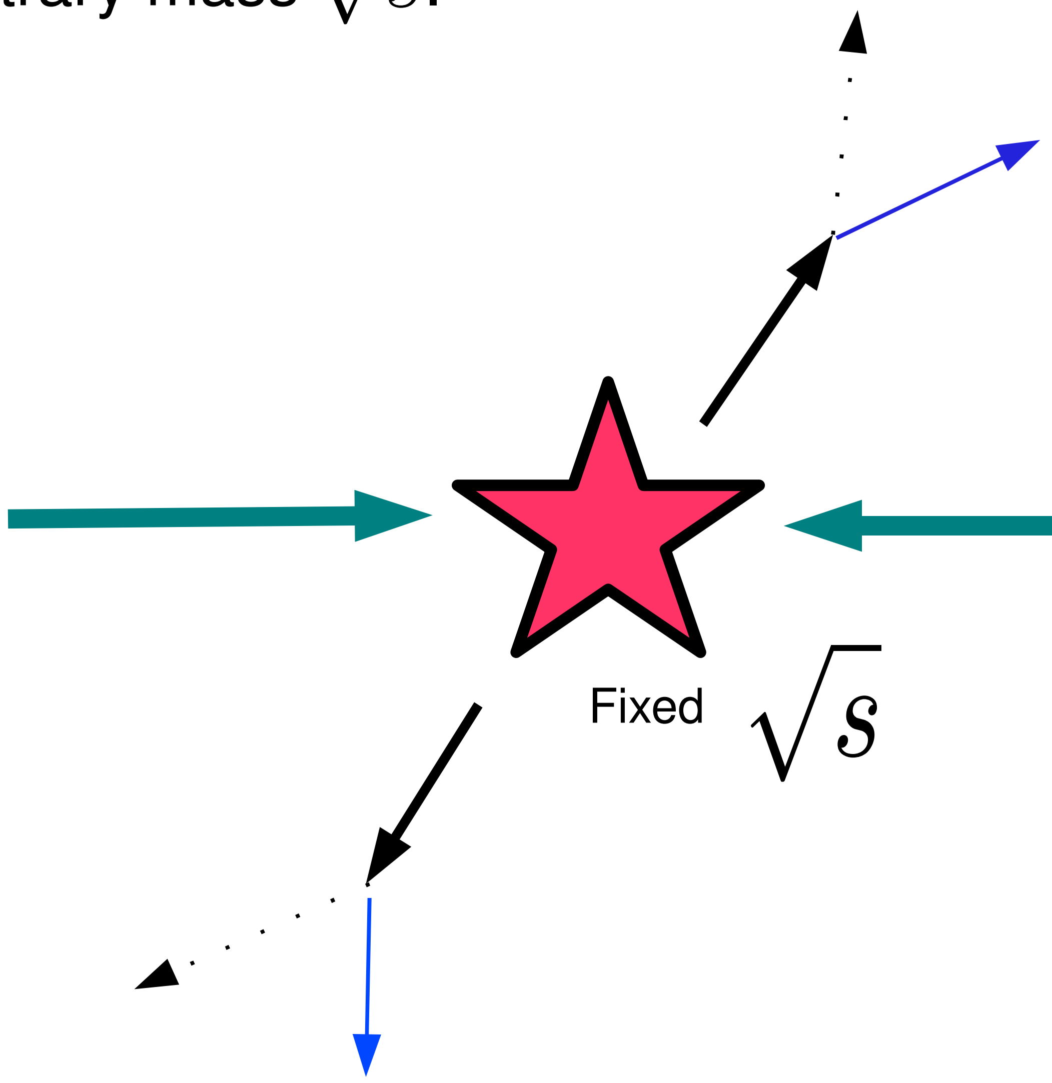


Antler decay

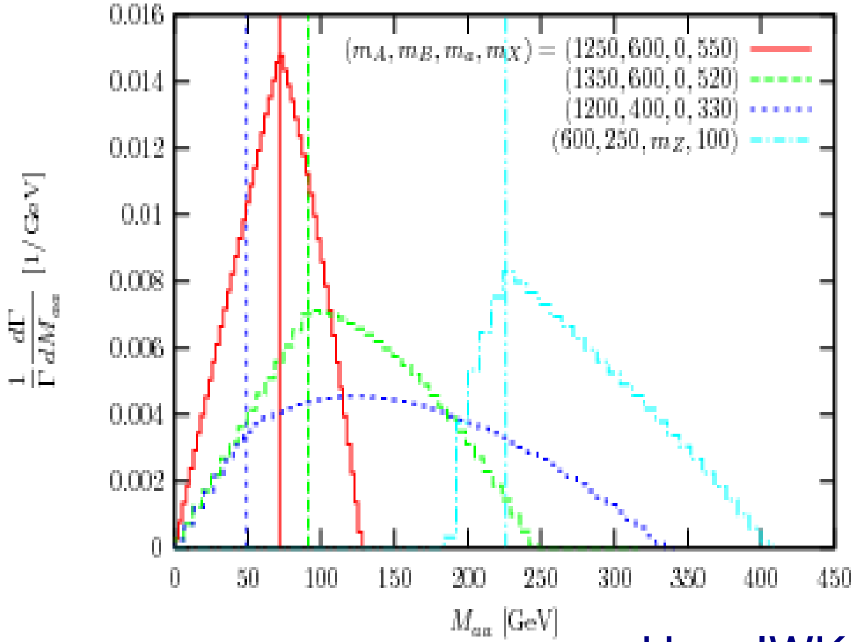


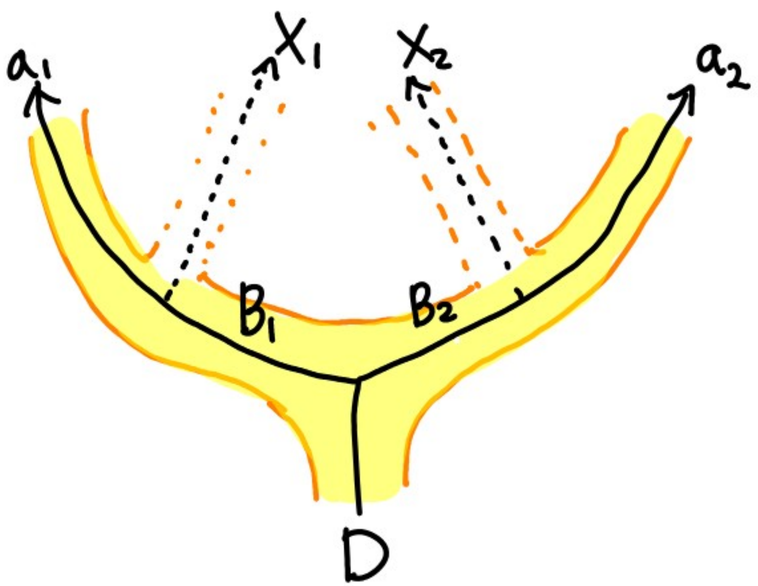
- Z' SUSY: $Z' \to l^+ l^- \to l^+ \chi_1^0 l^- \chi_1^0$
- MSSM: $H \to \chi_2^0 \chi_2^0 \to Z \chi_1^0 Z \chi_1^0$
- UED: $Z_\mu^{(2)} \to L^{(1)} L^{(1)} \to l^+ \gamma^{(1)} l^- \gamma^{(1)}$
- LHwT: $H \to T\bar{T} \to tA\bar{t}A$

In the ILC experiment, we can even make a virtual resonance with arbitrary mass $\sqrt{s}$.



Cusp and End point singularity in invariant mass distribution





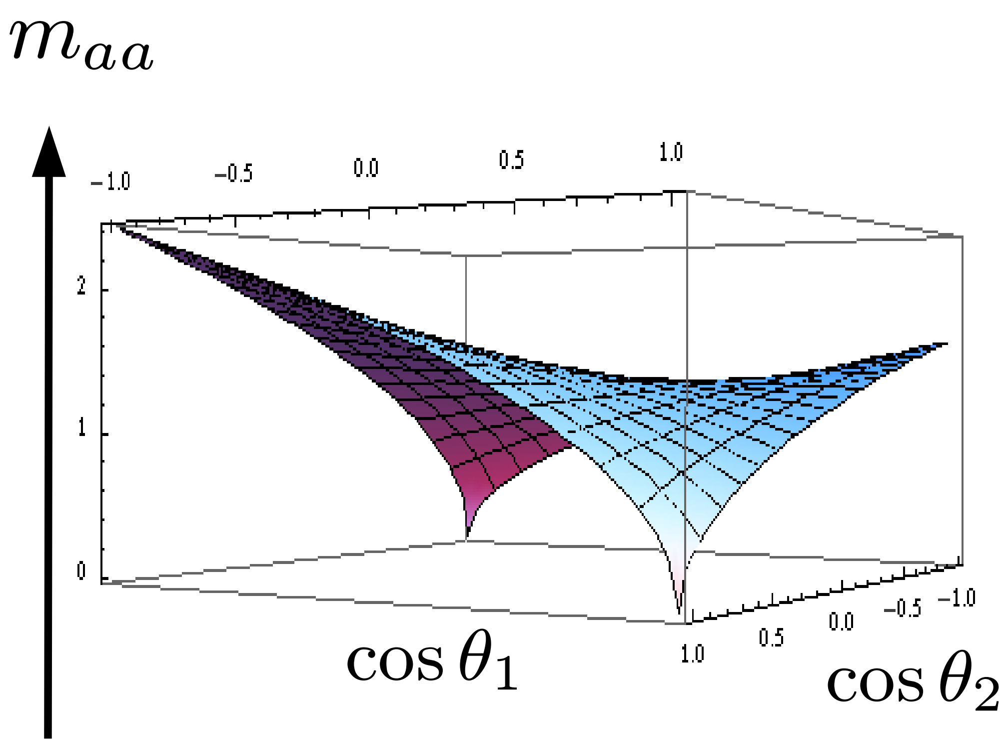
Han, IWK, Song : arXiv:0906.5009

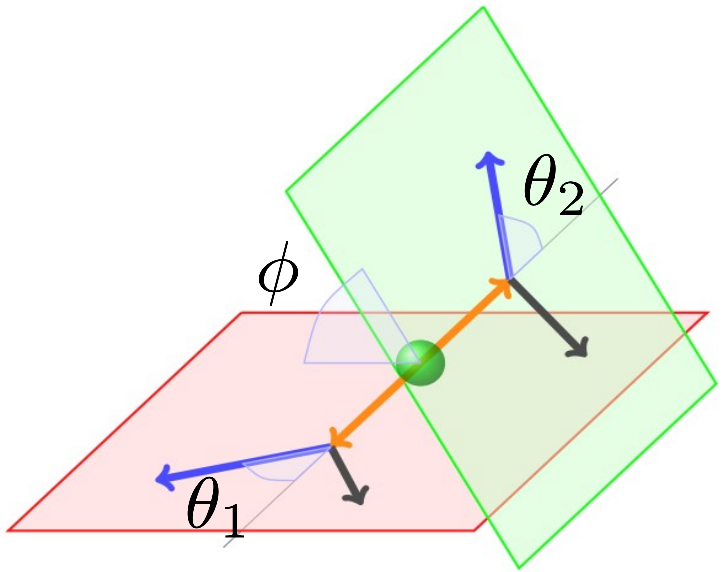
Massless visible particle case:

$$\frac{M_{aa}^{\text{cusp}}}{M_{aa}^{\max}} = \exp(-2\eta) = \frac{m_D^2 - 2m_B^2}{2m_B^2} - \frac{m_D}{m_B}\sqrt{\frac{m_D^2}{4m_B^2} - 1}$$

$$M_{aa}^{\text{cusp}} M_{aa}^{\max} = m_B^2 \left(1 - \frac{m_X^2}{m_B^2}\right)^2$$

Why end point and cusp appear?



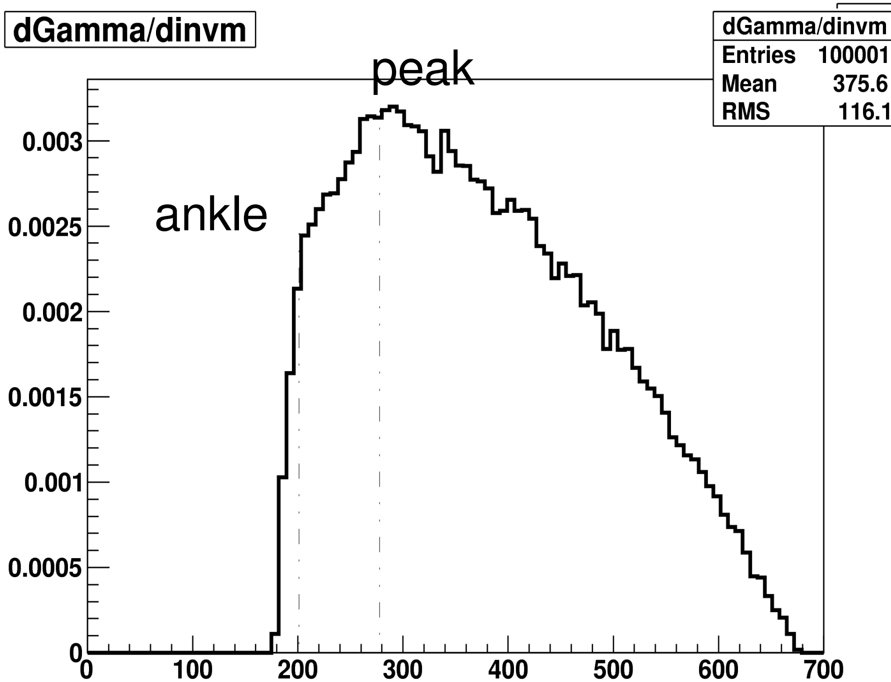


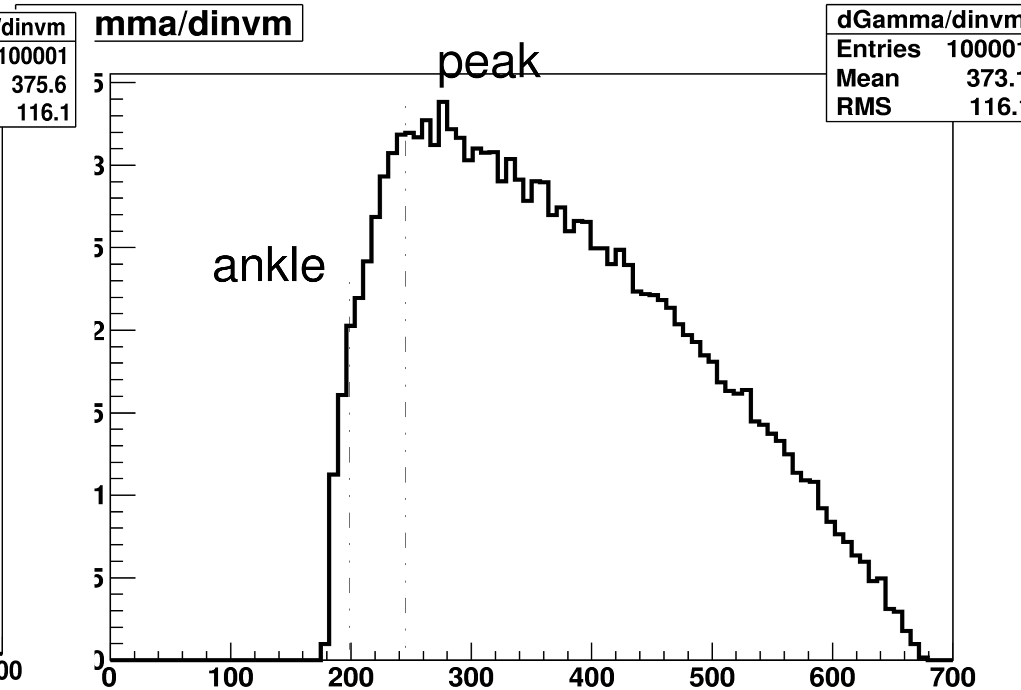
Phase Space is folded for a kinematic variable.

$$
\begin{aligned}
m_{aa}^2 = \quad & \cosh 2\eta + \sinh 2\eta \cos\theta_1 \\
& + \cosh 2\eta \cos\theta_2 + \cosh 2\eta \cos\theta_1 \cos\theta_2 \\
& + \sin\theta_1 \sin\theta_2 \cos\varphi
\end{aligned}
$$

Analytic Formula (Massive)

T.Han, IWK, Song (to appear soon)





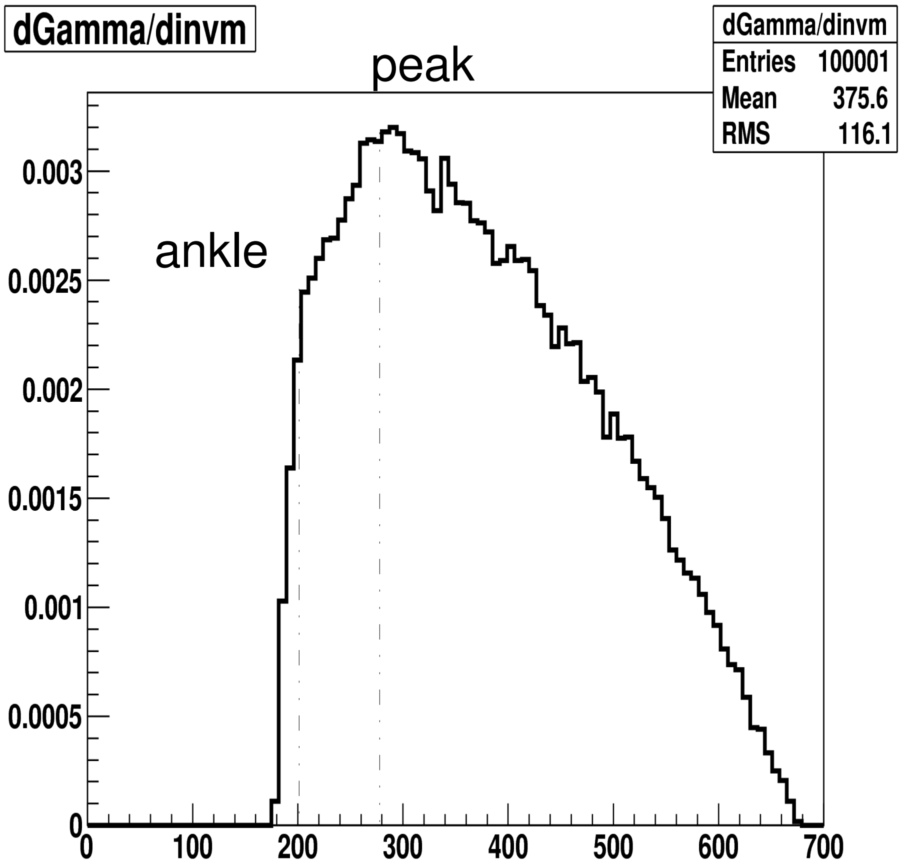
$$\chi \equiv \frac{M_{aa}^2}{(2m_a^2)} - 1$$

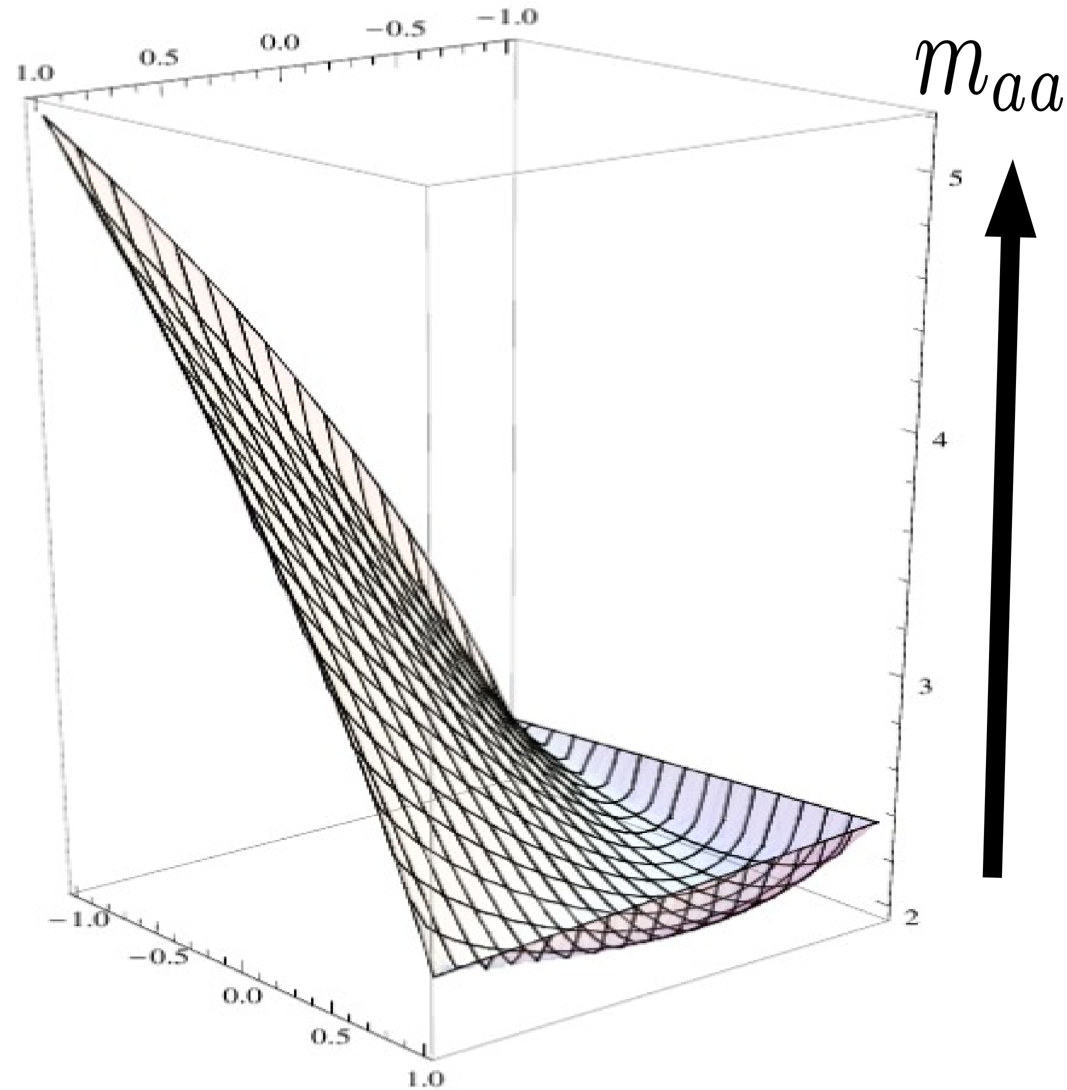
$$\left.\frac{d\Gamma}{d\chi}\right|_{\eta\le\frac{\zeta}{2}} \propto \begin{cases} 2\cosh^{-1}\chi, & \text{if } 1 \le \chi \le c_\eta; \\ 4\eta, & \text{if } c_\eta \le \chi \le c_-; \\ 2\zeta + 2\eta - \cosh^{-1}\chi, & \text{if } c_- \le \chi \le c_+, \end{cases}$$

$$\left.\frac{d\Gamma}{d\chi}\right|_{\frac{\zeta}{2}<\eta<\zeta} \propto \begin{cases} 2\cosh^{-1}\chi, & \text{if } 1 \le \chi \le c_-; \\ 2\zeta - 2\eta + \cosh^{-1}\chi, & \text{if } c_- \le \chi \le c_\eta; \\ 2\zeta + 2\eta - \cosh^{-1}\chi, & \text{if } c_\eta \le \chi \le c_+, \end{cases}$$

$$c_+ = \cosh 2(\eta+\zeta) \qquad c_- = \cosh 2(\eta-\zeta) \qquad c_\eta = \cosh 2\eta$$

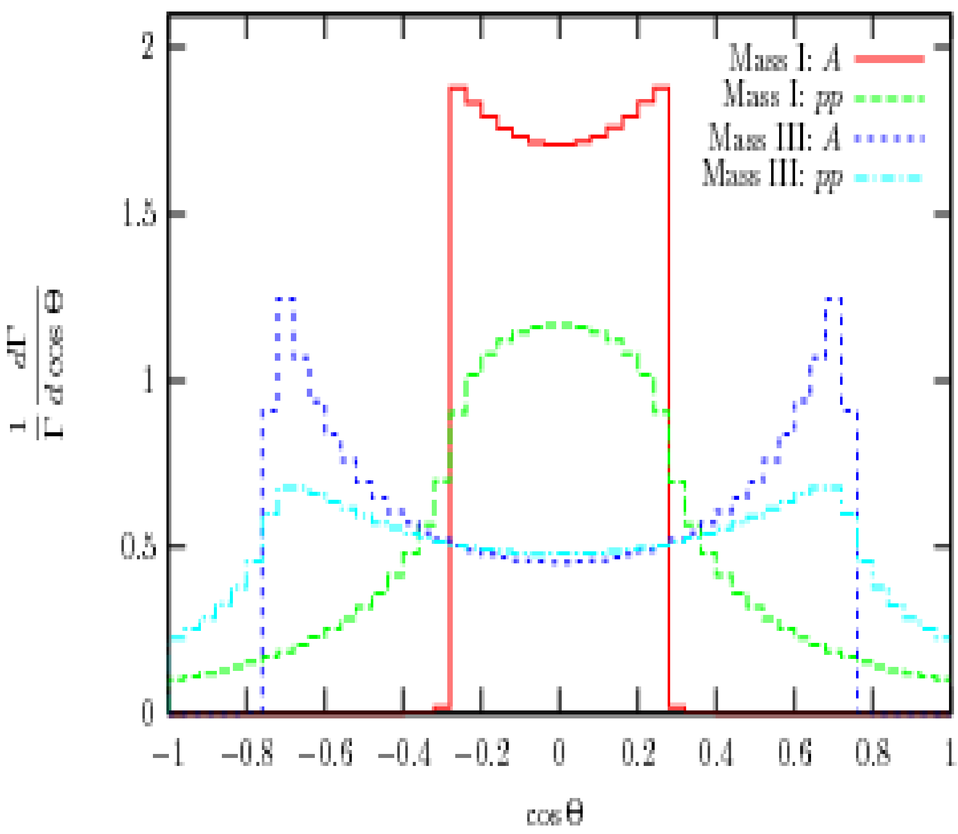
Again, phase space folding gives non-smooth structure in kinematic distribution.



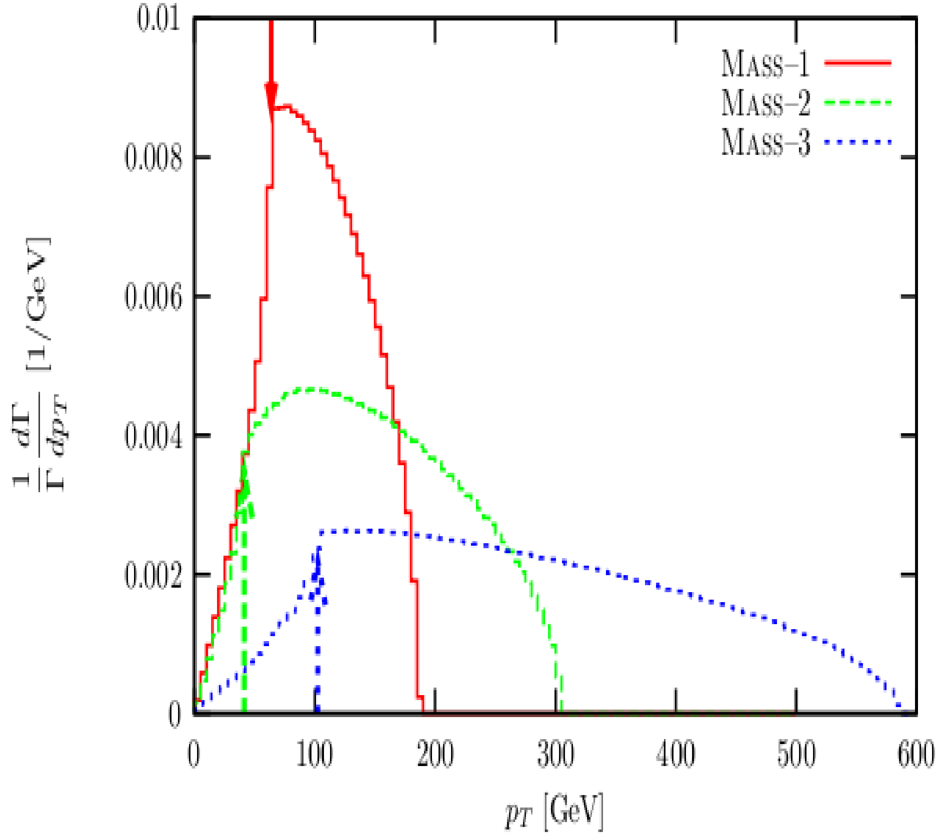


Kinematic Cusps appear in various observables.

Angular distribution



P_T distribution



Other methods usually rely on end-points.

Why End-points and cusps? It's not an accident.

Kinematic analysis : using $\int d\mathrm{PS}$ only

Model independent, using only essential parameters

cannot use event profile since we ignore $|\mathcal{M}|^2$

↓

Reconstructable: check whether each event satisfies mass shell equations

$$(\text{likelihood}) \propto P_{\text{ev1}} \times P_{\text{ev2}} \times \ldots$$

Nonreconstructable: we cannot determine whether each event is on PS or not.

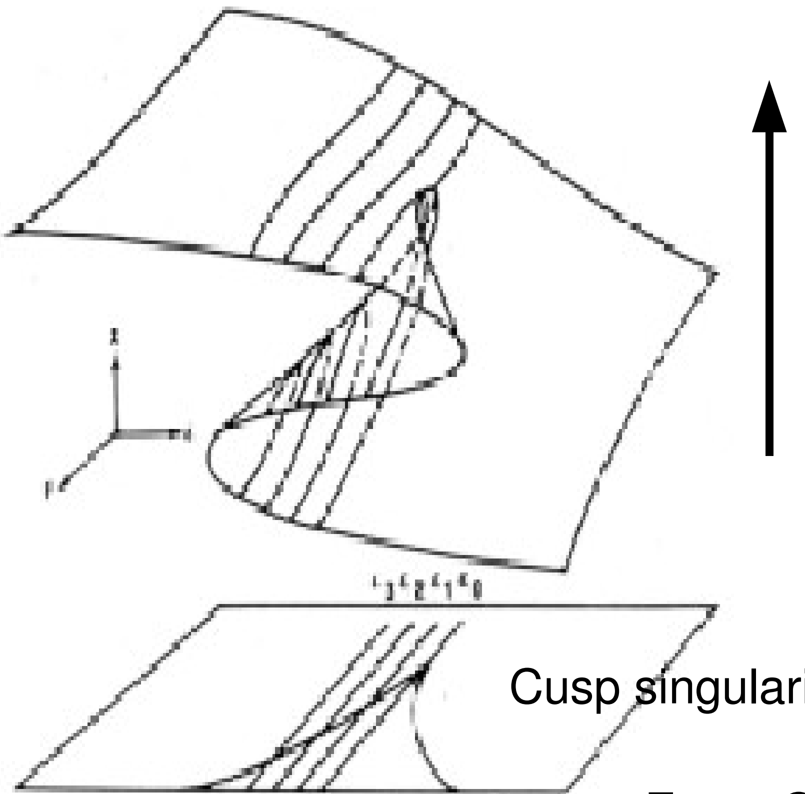
But special points in kinematic distribution can appear

⇒ Kinematic Singularity point

(non-smooth point in a distribution)

Algebraic Singularity Method

IWK, Work in Progress



Event Observable space = projected image of PS.

Multiplicity can change abruptly around a certain point.

$$\int d\mathrm{PS} = \int \ldots \int d^4p \ldots \delta(g_1)\delta(g_2)\ldots$$

Phase space is defined by solution space of coupled polynomial equations.

$$\begin{aligned} g_1 &= 0 \\ g_2 &= 0 \\ g_3 &= 0 \\ &\vdots \end{aligned} \longrightarrow \text{affine variety}$$

x : event unknowns

q : event observables

At a singularity point in event observable space,

Jacobian $\left(\dfrac{\partial g_i}{\partial x_j}\right)$ has a reduced rank

Groebner basis:

With lexicographic ordering $x_1 > x_2 > x_3 > x_4 > \dots$

$$g_1(x_1, x_2, x_3, x_4, \dots) = 0$$
$$g_2(x_2, x_3, x_4, \dots) = 0$$
$$g_3(x_3, x_4, \dots) = 0$$
$$g_4(x_4, \dots) = 0$$

Equations a re sequentially solvable.

Jacobian matrix has upper triangular form.

$$\left(\frac{\partial g_i}{\partial x_j}\right) = \begin{pmatrix} X & \dots & \dots & \\ & X & X & \dots \\ & & X & \dots \end{pmatrix}$$

Vanishing diagonal component is necessary in this basis for a reduced rank of Jacobian.

For the row vectors of Jacobian with vanishing diagonal component, we can directly check whether they are linearly dependent or not.

$$\vec{v}_1 = (a_1, a_2, a_3, \dots)$$

$$\vec{v}_2 = (b_1, b_2, b_3, \dots)$$

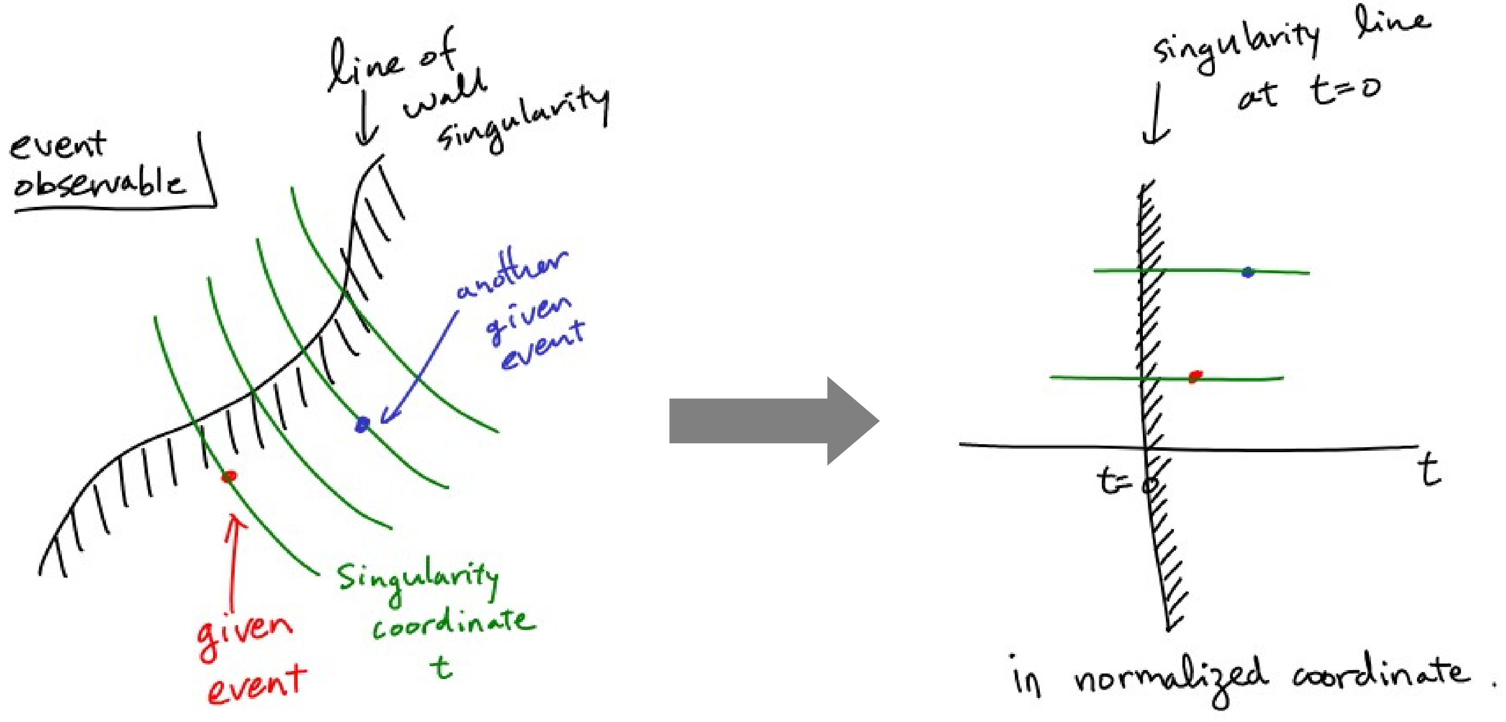
$$\vec{v}_1 \parallel \vec{v}_2 \quad \text{if} \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots$$

By this way, we can classify all the singularities in event observables.

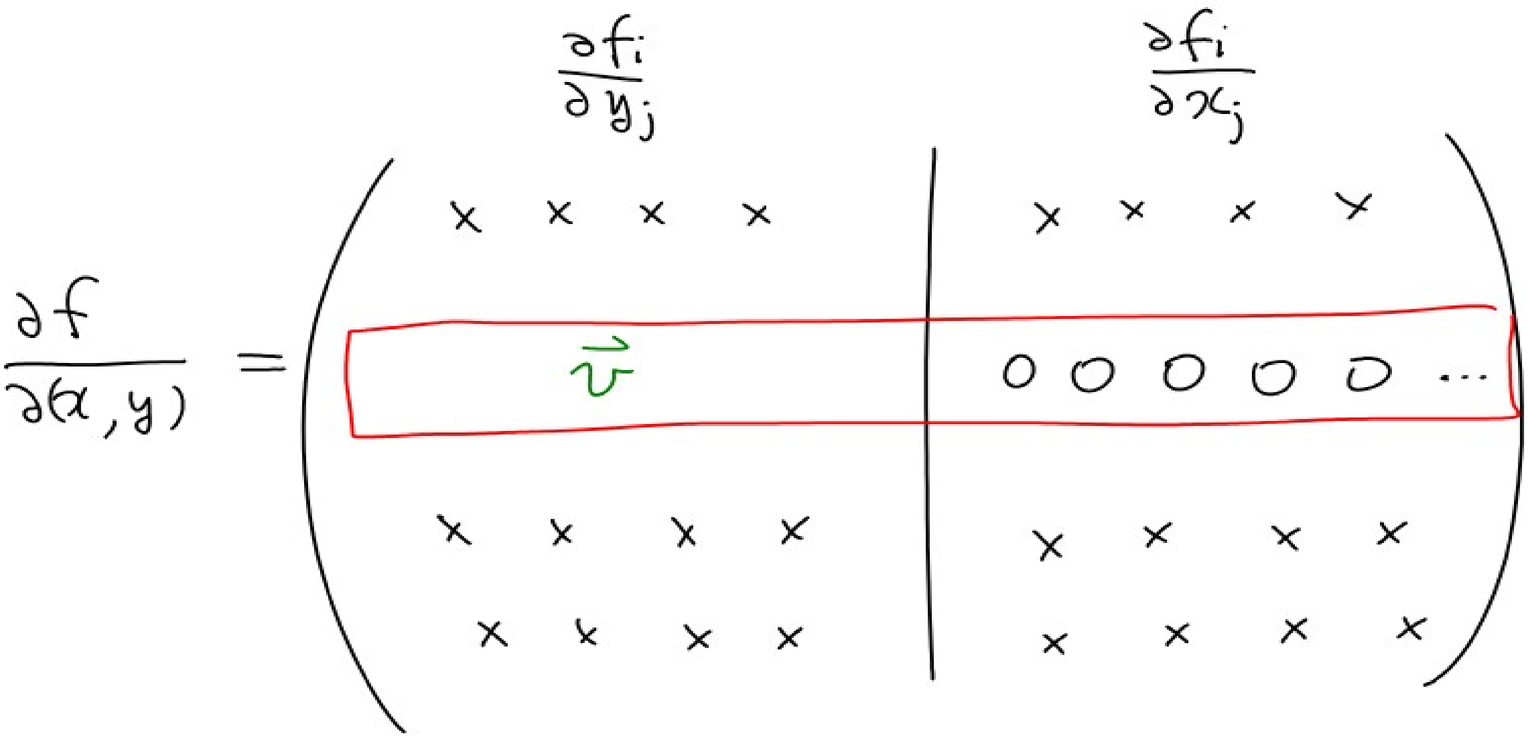
Singularity Coordinate

Once we know where a singularity is located, we need to define a normalized coordinate near the singularity point.

Project all the event points near the singularity on the coordinate.

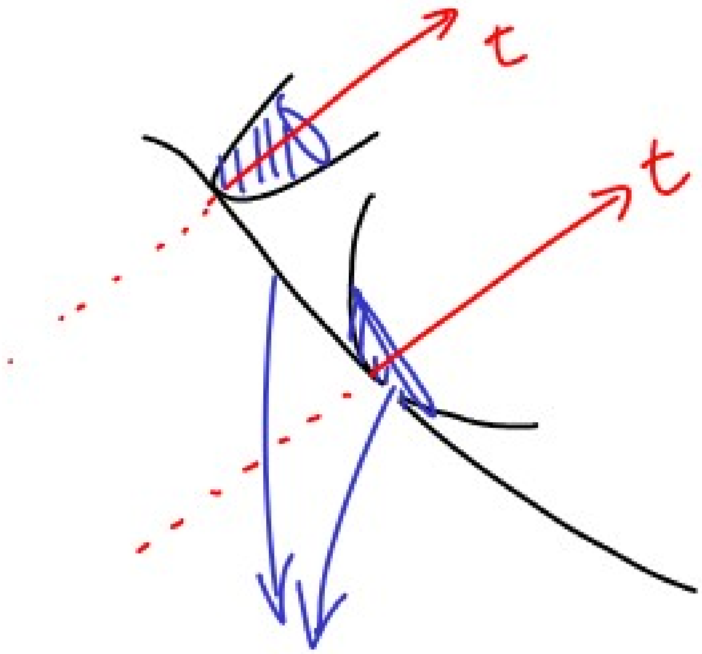


Singularity Coordinate direction is already determined by the reduced rank condition.



$\vec{v}$ is tangent direction for singularity coordinate, which is normal to singularity plane.
I will call it normal direction of singularity.

To determine the scaling of coordinate, we choose equal density normalization.



Local description of phase space is useful for this procedure : second fundamental form

Same event unknown volume give rise to the same singularity coordinate.

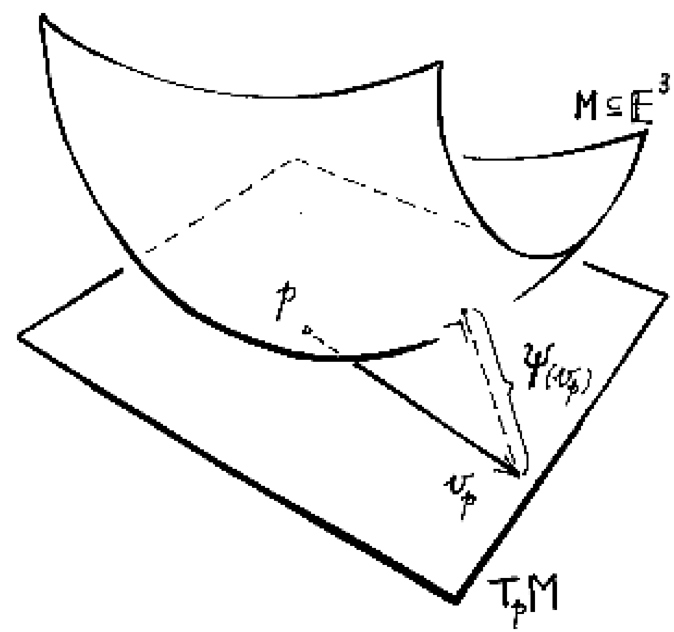
Second Fundamental Form of Algebraic Variety

$$g_i = 0 \quad \Rightarrow \quad g_i + \frac{\partial g_i}{\partial y_j} dy^j + \frac{\partial^2 g_i}{\partial x_j \partial x_k} dx^j dx^k = 0$$

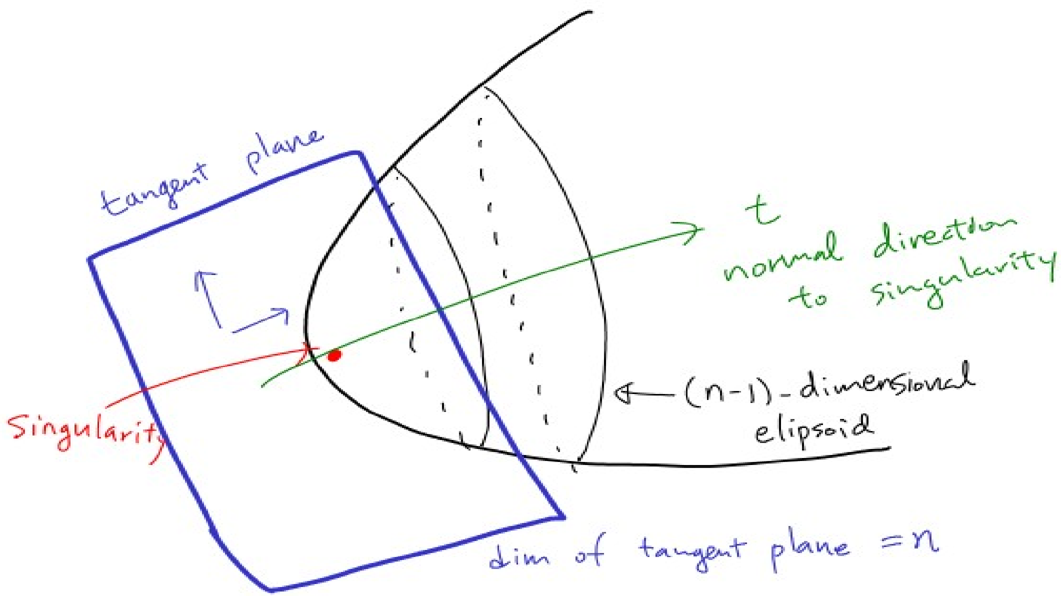
local variation

dy is confined to the normal space and dx is confined to the tangential space.

$$\frac{\partial g_i}{\partial y_j} dy^j = -\frac{\partial^2 g_i}{\partial x_j \partial x_k} dx^j dx^k$$



defines a map : tangent space to normal space



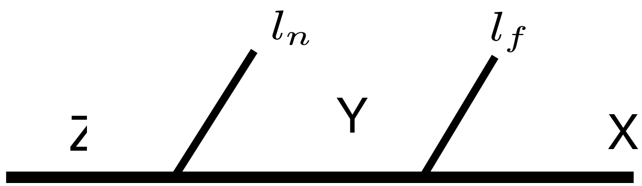
Locally,

$$t = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2}$$

$$(\mathrm{vol}) \propto r^n \sim S(a_1, a_2, \ldots, a_n) t^{n/2}$$

$$\frac{d\Gamma}{dt} \propto S(a_1, \ldots, a_n) t^{\frac{n}{2}-1} \frac{d\Gamma}{d(\mathrm{vol})}$$

Reanalysis of Invariant mass in cascade decay



Kinematic constraint equations:

$$X^2 = m_X^2$$
$$(X + l_f)^2 = m_Y^2$$
$$(X + l_f + l_n)^2 = m_Z^2$$

Event unknowns :

$$X^\mu = (X_0, X_1, X_2, X_3)$$

Event Observable : $l_n^\mu \quad l_f^\mu$

Coordinate in C.M. frame of l_n and l_f

$$l_n'^{\mu} = \left(\frac{E_{\rm cm}}{2}, 0, 0, \frac{E_{\rm cm}}{2}\right) \qquad l_f'^{\mu} = \left(\frac{E_{\rm cm}}{2}, 0, 0, -\frac{E_{\rm cm}}{2}\right)$$

Groebner basis : $X_0' > X_3' > X_1' > X_2'$

$$g_1 = 2E_{\rm cm}X_0' + (E_{\rm cm}^2 + m_X^2 - m_Z^2) = 0$$

$$g_2 = 2E_{\rm cm}X_3' + E_{\rm cm}^2 + 2m_Y^2 - m_X^2 - m_Z^2 = 0$$

$$g_3 = E_{\rm cm}^2 X_1'^2 + E_{\rm cm}^2 X_2'^2 + E_{\rm cm}^2 m_Y^2 - (m_Z^2 - m_Y^2)(m_Y^2 - m_X^2) = 0$$

Jacobian matrix for Groebner basis:

$$\left(\frac{\partial g_i}{\partial X_j'}\right) = \begin{pmatrix} 2E_{\rm cm} & 0 & 0 & 0 \\ & 2E_{\rm cm} & 0 & 0 \\ & & E_{\rm cm}^2 X_1' & E_{\rm cm}^2 X_2' \end{pmatrix}$$

Nontrivial reduced rank Jacobian can arise when

$$X_1' = X_2' = 0$$

Then, $E_{\rm cm}^2 = \frac{(m_Z^2 - m_Y^2)(m_Y^2 - m_X^2)}{m_Y^2} \equiv E_{\rm cm0}^2$ by $g_3 = 0$

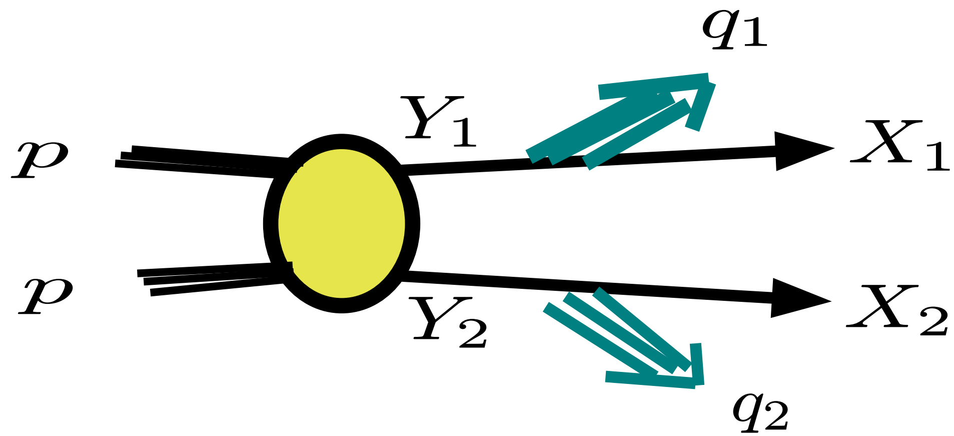
Perpendicular direction to tangent space at the singularity: rather trivial

$$\frac{\partial g_3}{\partial E_{\rm cm}} = 2m_Y^2 E_{\rm cm}$$

Normalized coordinate:

$$t \equiv 4\pi m_Y^2 \left(\frac{E_{\rm cm0}^2 - E_{\rm cm}^2}{E_{\rm cm0}^2} \right)$$

Double Missing Particle Chain Topology:



Kinematic Constraint Equations :

$$X_1^2 = m_X^2 \qquad X_2^2 = m_X^2$$

$$(X_1 + q_1)^2 = m_Y^2 \qquad (X_2 + q_2)^2 = m_Y^2$$

$$\vec{X}_{1T} + \vec{X}_{2T} = \vec{P}_T^{\text{miss}}$$

Event unknowns:

$$X_1^\mu = (X_{10}, X_{11}, X_{12}, X_{13}) \qquad X_2^\mu = (X_{20}, X_{21}, X_{22}, X_{23})$$

Event observables:

$$q_1^\mu = (q_{10}, q_{11}, q_{12}, q_{13}) \quad q_2^\mu = (q_{20}, q_{21}, q_{22}, q_{23}) \quad \vec{P}_T^{\text{miss}} = (P_{T1}, P_{T2})$$

Groebner basis : $X_{10} > X_{13} > X_{20} > X_{21} > X_{22} > X_{23} > X_{11} > X_{12}$

$$
\begin{aligned}
g_1 = & \left(q_{20}^2 - q_{23}^2\right)\chi_{23}^2 + 2q_{21}q_{23}\chi_{23}\chi_{11} + 2q_{22}q_{23}\chi_{23}\chi_{12} - 2C_2 q_{23}\chi_{23} \\
& + \left(q_{20}^2 - q_{21}^2\right)\chi_{11}^2 - 2q_{21}q_{22}\chi_{11}\chi_{12} + \left(q_{20}^2 - q_{22}^2\right)\chi_{12}^2 \\
& + \left(-2P_{T1}q_{20}^2 + 2C_2 q_{21}\right)\chi_{11} + \left(-2P_{T2}q_{20}^2 + 2C_2 q_{22}\right)\chi_{12} \\
& + \left(\vec{P}_T^2 + m_\chi^2\right) q_{20}^2 - C_2^2, \\
g_2 = & \chi_{22} + \chi_{12} - P_{T2}, \\
g_3 = & \chi_{21} + \chi_{11} - P_{T1}, \\
g_4 = & q_{20}\chi_{20} - q_{23}\chi_{23} + q_{21}\chi_{11} \\
& + q_{22}\chi_{12} - C_2, \\
g_5 = & \left(q_{10}^2 - q_{13}^2\right)\chi_{13}^2 - 2q_{11}q_{13}\chi_{13}\chi_{11} - 2q_{12}q_{13}\chi_{13}\chi_{12} - 2C_1 q_{13}\chi_{13} \\
& + \left(q_{10}^2 - q_{11}^2\right)\chi_{11}^2 - 2q_{11}q_{12}\chi_{11}\chi_{12} \\
& + \left(q_{10}^2 - q_{12}^2\right)\chi_{12}^2 - 2C_1 q_{11}\chi_{11} - 2C_1 q_{12}\chi_{12} \\
& + \left(m_\chi^2 q_{10}^2 - C_1^2\right), \\
g_6 = & q_{10}\chi_{10} - q_{13}\chi_{13} - q_{11}\chi_{11} - q_{12}\chi_{12} - C_1,
\end{aligned}
$$

Solution as a function of χ_{11} and χ_{12}

$$\chi_{23}^{\text{soln}} = \frac{A_2 q_{23} \pm q_{20}\sqrt{A_2^2 - \left(q_{20}^2 - q_{23}^2\right)\left(m_\chi^2 + \vec{k}_{2T}^2\right)}}{q_{20}^2 - q_{23}^2},$$

$$\chi_{22}^{\text{soln}} = P_{T2}^{\text{miss}} - \chi_{12},$$

$$\chi_{21}^{\text{soln}} = P_{T1}^{\text{miss}} - \chi_{11},$$

$$\chi_{20}^{\text{soln}} = \frac{A_2 + q_{23}\chi_{23}}{q_{20}},$$

$$\chi_{13}^{\text{soln}} = \frac{A_1 q_{13} \pm q_{10}\sqrt{A_1^2 - \left(q_{10}^2 - q_{13}^2\right)\left(m_\chi^2 + \vec{k}_{1T}^2\right)}}{q_{10}^2 - q_{13}^2},$$

$$\chi_{10}^{\text{soln}} = \frac{A_1 + q_{13}\chi_{13}}{q_{10}},$$

$$A_1 = C_1 + \vec{q}_{1T} \cdot \vec{\chi}_{1T},$$

$$A_2 = C_2 - \vec{q}_{2T} \cdot \vec{\chi}_{1T}.$$

Jacobian Matrix :

$$
\begin{matrix} g_6 \\ g_5 \\ g_4 \\ g_3 \\ g_2 \\ g_1 \end{matrix}
\begin{pmatrix}
X & X & & & & & X & X \\
 & X & & & & & X & X \\
 & & X & & & X & X & X \\
 & & & 1 & & & 1 & \\
 & & & & 1 & & & 1 \\
 & & & & & X & X & X
\end{pmatrix}
$$

reduced rank condition :

$$
\left.\begin{aligned}
2(q_{10}^2 - q_{13}^2)X_{13} - 2A_1 q_{13} = 0 \\
2(q_{20}^2 - q_{23}^2)X_{23} - 2A_2 q_{23} = 0
\end{aligned}\right\} \rightarrow \text{MAOS momentum}
$$

$$
\det \begin{pmatrix} \frac{\partial g_5}{\partial X_{11}} & \frac{\partial g_5}{\partial X_{12}} \\ \frac{\partial g_1}{\partial X_{11}} & \frac{\partial g_1}{\partial X_{12}} \end{pmatrix} = 0 \rightarrow \text{Maximum MT2}
$$

These conditions give rise to the maximum MT2 condition and MAOS momentum.

Numerical analysis with singularity coordinate is now in progress. ⇨ Wait for our paper!

Conclusion

Kinematic Cusps in Resonant Decay Channel can be very helpful for mass measurement. This is very adequate for the ILC physics.

General description of mass measurement method leads us to investigate singularity structure in phase space.

We develop a new algebraic geometrical method for seeking for singularities in PS and find out tailored implicit variable for singularity.

This method naturally includes previous methods for nonreconstructable event topology such as end point and cusp in invariant mass and mT2. We can enhance such cases with this generalized method and generalize to different event topologies.