Fourth International Accelerator School for Linear Colliders

Beam Delivery System

Effect of the Inner Detector Solenoid

Questions

1. Transformation matrix for a 'hard edge' solenoid magnet

1.1 We consider the motion of relativistic particles ($v = c$) around a straight reference trajectory $s = z = ct$, in a magnetic field $\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$.
Show that the equations of motion for a particle of charge q, rest mass m_0 and momentum $p_0 = \gamma m_0 c$ can be written as follows:

$$\begin{cases} x'' = \frac{1}{B\rho}\left(y'B_z - B_y\right) \\ y'' = \frac{1}{B\rho}\left(-x'B_z + B_x\right) \end{cases}$$

with $f'(s) = \frac{df}{ds}$, $B\rho = \frac{p_0}{q}$.

1.2. Considering a 'hard edge' solenoid means that the magnet is divided into 3 parts:

- a central part of lenght L_0 where $\vec{B} = B_0\hat{z}$.
- the entrance face ($s = s_i$), where $\frac{\partial B_z}{\partial z} = B_0\,\delta(z - z_i)$,
- and the exit face ($s = s_o$), where $\frac{\partial B_z}{\partial z} = -B_0\,\delta(z - z_o)$,

Show that the transformation matrix R_c of the coordinate vector $X = \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}$ for the solenoid central part is:

$$M_c = \begin{bmatrix} 1 & \frac{\sin(K_sL_0)}{K_s} & 0 & -\frac{1}{K_s}\left[\cos(K_sL_0) - 1\right] \\ 0 & \cos(K_sL_0) & 0 & \sin(K_sL_0) \\ 0 & \frac{1}{K_s}\left[\cos(K_sL_0) - 1\right] & 1 & \frac{\sin(K_sL_0)}{K_s} \\ 0 & -\sin(K_sL_0) & 0 & \cos(K_sL_0) \end{bmatrix}$$

with $K_s = \frac{B_0}{B\rho}$.

1.3. At the faces of a 'hard edge' solenoid, the field is purely radial.
- Calculate B_r as a function of r and B_0, using the definition of divergence in polar coordinates (r, θ, z) : $div\,\vec{B} = \frac{1}{r}\frac{\partial(rB_r)}{\partial r} + \frac{1}{r}\frac{B_\theta}{d\theta} + \frac{\partial B_z}{\partial z}$.
- Express B_x and B_y as a function of x, y, and B_0.

- Calculate the transformation matrices for the edges : R_i and R_o.

1.4. Show that the transformation matrix for the whole 'hard edge' solenoid can be expressed as :

$$R_s = \begin{bmatrix} C^2 & \frac{2}{K_s}SC & SC & \frac{2}{K_s}S^2 \\ -\frac{K_s}{2}SC & C^2 & -\frac{K_s}{2}S^2 & SC \\ -SC & \frac{2}{K_s}S^2 & C^2 & \frac{2}{K_s}SC \\ \frac{K_s}{2}S^2 & -SC & -\frac{K_s}{2}SC & C^2 \end{bmatrix} \quad \text{with} \quad \begin{cases} C = \cos(\frac{K_s L_0}{2}) \\ S = \sin(\frac{K_s L_0}{2}) \end{cases}$$

2. Symplecticity

2.1. Is the transformation given by R_s due to the central part of a 'hard edge' solenoid symplectic? Give an interpretation of this result by analyzing the associated hamiltonian and conjugate variables.

2.2. Repeat the exercise for the transformation through the whole 'hard edge' solenoid.

3. Detector solenoid

3.1. We consider the transport of a beam from the entrance of a solenoid to the Interaction Point (IP) located at the center of the solenoid, that is at a distance L from the entrance. The solenoid is represented by a 'hard edge' solenoid of total length $2L$ and field B_0.

In the case of head-on collisions, beams are going through the solenoid field on axis and no deviation is observed. On the contrary, if we consider a non-zero crossing-angle ($\vartheta_c = 2\,|x_0'|$), the central trajectory is deviated, especially in the vertical plane. This phenomenon is due to the solenoid since the horizontal component of the velocity is not zero.

Calculate the transfer matrix from the entrance to IP. Given the initial vector is $X_0 = \begin{bmatrix} x_0 \\ x_0' \\ 0 \\ 0 \end{bmatrix}$, calculate the vector X at the IP.

3.2. What is the relation between x_0 and x_0' to have $y_{IP} \equiv y(L) = 0$? Interpret the result in the case where $K_s L << 1$.

4. Vertical orbit at IP

4.1. We make the assumptions that the solenoid is 8m long ($L = 4$m), has a peak field $B_0 = 4$ T, and $\vartheta_c = 14$mrad.

Calculate K_s and fill in the following table.

s (m)	0	0,5	1	1,5	2	2,5	3	3,5	4
y (μm)									

Plot the trajectory (indicate IP position).

4.2. We consider the case where the final quadrupole is located inside of the detector solenoid. Assuming that this quadrupole is a thin lens focusing in the y-plane located at 3m from the IP, of integrated strength $KL = \frac{GL}{B\rho} = -0.3\,\mathrm{m}^{-1}$, give a qualitative explanation (make eventually a diagram) of the observed non zero orbit at IP although the condition 3.2. is verified.
Calculate the value of the y-offset.

5. Calculate the transormation matrix for a magnetic element combining a 'hard edge' solenoid of peak field B_0 and a quadrupole of gradient G.

Solutions

1 Transformation matrix for a 'hard edge' solenoid magnet

1.1 Equation of motion in a magnetic field

Let $\vec{p}$ be the momentum of the particle, $\vec{v}$ its velocity, q its charge and $\vec{B}$ the magnetic field. The Laplace equation is written as follows :

$$\frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B} \tag{1}$$

Introducing the curvilinear abscissa s, it becomes (m_0 being the rest mass):

$$\gamma m_0 \frac{d^2\vec{X}}{ds^2}\,\frac{ds^2}{dt^2} = q\,\frac{d\vec{X}}{ds}\,\frac{ds}{dt} \times \vec{B} \tag{2}$$

Considering $\frac{ds}{dt} = c$:

$$\begin{bmatrix} x'' \\ y'' \\ 0 \end{bmatrix} = \frac{q}{\gamma m_0 c} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \times \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} \tag{3}$$

With $\frac{p_0}{q} = B\rho$, magnetic rigidity of the beam, we get the equations of motion to be solved :

$$\left\{ \begin{array}{l} x''=\frac{1}{B\rho}\,(y'B_z - B_y) \\ y''=\frac{1}{B\rho}\,(-x'B_z + B_x) \end{array} \right. \tag{4}$$

1.2 R_c matrix, Solution for $\vec{B} = B_0\widehat{z}$

According to equations Eq. (4) the coupled system is :

$$\left\{ \begin{array}{l} x''=\frac{B_0}{B\rho}\,y' \\ y''=-\frac{B_0}{B\rho}\,x' \end{array} \right. \tag{5}$$

$$\Rightarrow \left\{ \begin{array}{l} x'=K_s y + x'_0 - K_s y_0 \\ y''=-K_s x' \end{array} \right. \tag{6}$$

$$\Rightarrow \left\{ \begin{array}{l} x'=K_s(y-y_0)+x'_0 \\ y''=-K_s(K_s(y-y_0)+x'_0) \end{array} \right. \tag{7}$$

Solution of the second equation, vertical movement :

$$y = A\cos(K_s s) + B\sin(K_s s) + (y_0 - \frac{x'_0}{K_s}) \tag{8}$$

For $s = 0$, we get :

$$\left\{ \begin{array}{l} y_0 = A + y_0 - \frac{x'_0}{K_s} \Rightarrow A = \frac{x'_0}{K_s} \\ y'_0 = BK_s \Rightarrow B = \frac{y'_0}{K_s} \end{array} \right. \tag{9}$$

Consequently,

$$\left\{ \begin{array}{l} y(s) = x'_0\,[\frac{\cos(K_s s)}{K_s} - \frac{1}{K_s}] + y_0 + y'_0\,\frac{\sin(K_s s)}{K_s} \\ y'(s) = -x'_0\,\sin(K_s s) + y'_0\,\cos(K_s s) \end{array} \right. \tag{10}$$

Knowing $y(s)$, we derive the horizontal movement equations thanks to Eq. (6) :

$$\left\{ \begin{array}{l} x(s) = x_0 + x'_0\,\frac{\sin(K_s s)}{K_s} - \frac{y'_0}{K_s}\,[\cos(K_s s) - 1] \\ x'(s) = x'_0\,\cos(K_s s) + y'_0\,\sin(K_s s) \end{array} \right. \tag{11}$$

And the transformation matrix for the central part of a solenoid of length L_0 is :

$$R_c = \begin{bmatrix} 1 & \frac{\sin(K_s L_0)}{K_s} & 0 & -\frac{1}{K_s}\,[\cos(K_s L_0) - 1] \\ 0 & \cos(K_s L_0) & 0 & \sin(K_s L_0) \\ 0 & \frac{1}{K_s}\,[\cos(K_s L_0) - 1] & 1 & \frac{\sin(K_s L_0)}{K_s} \\ 0 & -\sin(K_s L_0) & 0 & \cos(K_s L_0) \end{bmatrix} \tag{12}$$

1.3 R_i and R_o matrices

At the entrance of the magnet, the field is purely radial. According to Maxwell's equations, $div\,\vec{B} = 0$, which means :

$$\frac{1}{r}\frac{\partial r B_r}{\partial r} + \frac{\partial B_z}{\partial z} = 0 \tag{13}$$

$$\Rightarrow B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z} \tag{14}$$

At the transition, $\frac{\partial B_z}{\partial z} = B_0\,\delta(z - z_i)$. Consequently, in the cartesian coordinates system, using the curvilinear abscissa s:

$$\left\{ \begin{array}{l} B_x = -\frac{x}{2}\,B_0\,\delta(s - s_i) \\ B_y = -\frac{y}{2}\,B_0\,\delta(s - s_i) \end{array} \right. \quad \text{since} \quad \left\{ \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right. \tag{15}$$

From Eq. (4), we get the equations of motion :

$$\begin{cases} x''=\frac{B_0}{2\,B\rho}\,[y\,\delta(s-s_i)] \\ y''=-\frac{B_0}{2\,B\rho}\,[x\,\delta(s-s_i)] \end{cases} \tag{16}$$

Integrating the equations, and using the property of Delta Dirac function ($\int_{-\infty}^{\infty} f(x)\;\delta(x-x_0)\;dx = f(x_0)$) the system becomes:

$$\begin{cases} x'-x'_0=\frac{B_0}{2\,B\rho}\,y_0 \\ y'-y'_0=-\frac{B_0}{2\,B\rho}\,x_0 \end{cases} \tag{17}$$

$$\Rightarrow \begin{cases} x=x_0 \\ y=y_0 \end{cases} \tag{18}$$

From Eq. (17) and Eq. (18), we get the transformation matrix R_i for the entrance of a 'hard edge' solenoid :

$$R_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & K_s/2 & 0 \\ 0 & 0 & 1 & 0 \\ -K_s/2 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

The same derivation using $\frac{\partial B_z}{\partial z} = -B_0\,\delta(z-z_o)$ for the exit of the magnet leads to the transformation matrix R_o :

$$R_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -K_s/2 & 0 \\ 0 & 0 & 1 & 0 \\ K_s/2 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

1.4 Transformation matrix R_s for the total 'hard edge' solenoid magnet

R_s is the product of the transformation matrices of the entrance face, the central part and the exit face of the solenoid :

$$R_s = R_o \cdot R_s \cdot R_i$$

$$R_s=\begin{bmatrix} \frac{1+\cos(K_sL_0)}{2} & \frac{\sin(K_sL_0)}{K_s} & \frac{\sin(K_sL_0)}{2} & \frac{1-\cos(K_sL_0)}{K_s} \\ -\frac{K_s}{4}\sin(K_sL_0) & \frac{1+\cos(K_sL_0)}{2} & \frac{K_s}{4}(\cos(K_sL_0)-1) & \frac{\sin(K_sL_0)}{2} \\ -\frac{\sin(K_sL_0)}{2} & \frac{\cos(K_sL_0)-1}{K_s} & \frac{\cos(K_sL_0)+1}{2} & \frac{\sin(K_sL_0)}{K_s} \\ \frac{K_s}{4}(1-\cos(K_sL_0)) & -\frac{\sin(K_sL_0)}{2} & -\frac{K_s}{4}\sin(K_sL_0) & \frac{1+\cos(K_sL_0)}{2} \end{bmatrix} \tag{21}$$

This matrix (Eq. (21)) can be further simplified using the two following relations,

$$\left\{ \begin{array}{l} \sin(2a)=2\sin(a)\cos(a) \\ \cos(2a)=\cos(a)^2-\sin(a)^2 \end{array} \right. :$$

$$R_s=\begin{bmatrix} C^2 & \frac{2}{K_s}SC & SC & \frac{2}{K_s}S^2 \\ -\frac{K_s}{2}SC & C^2 & -\frac{K_s}{2}S^2 & SC \\ -SC & \frac{2}{K_s}S^2 & C^2 & \frac{2}{K_s}SC \\ \frac{K_s}{2}S^2 & -SC & -\frac{K_s}{2}SC & C^2 \end{bmatrix} \text{ with } \left\{ \begin{array}{l} C=\cos(\frac{K_sL_0}{2}) \\ S=\sin(\frac{K_sL_0}{2}) \end{array} \right. \tag{22}$$

2 Symplecticity

2.1 Symplecticity of the transformation matrix for the central part of the solenoid

R_c is symplectic if the property ${}^tR_c \cdot J \cdot R_c = J$ is verified,
with $J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$.
But,

$${}^tR_c \cdot J \cdot R_c = \begin{bmatrix} 0 & C^2-S^2 & 0 & 2SC \\ S^2-C^2 & 0 & 2SC & \frac{4}{K_s}(C^4-S^2) \\ 0 & -2SC & 0 & C^2-S^2 \\ -2SC & \frac{4}{K_s}(C^4-S^2) & S^2-C^2 & 0 \end{bmatrix} \neq J \tag{23}$$

The transformation matrix of the vector $X = \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}$ for the central part of the solenoid is not symplectic.

Symplecticity of conjugate variables $(\vec{x}, \vec{p}+q\vec{A})$ transformations is a property of Hamiltonian Mechanics due to energy conservation. Consequently a transfer matrix of the vector X is symplectic if $A_x = A_y = 0$. In the case of a solenoid field, given the symetry of the current distribution, the vector potential axial component is zero and $\vec{A} = A_\theta\widehat{\theta} = A_x\widehat{x} + A_y\widehat{y}$.

Calculation of A_x and A_y :

$$div\vec{B} = 0$$
$$\Rightarrow \vec{B} = r\vec{o}t\vec{B}$$
$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) = B_0$$
$$\Rightarrow A_\theta = \frac{r}{2}B_0 \ (inside\ of\ the\ solenoid)$$
$$\Rightarrow \left\{ \begin{array}{l} A_x=-\frac{y}{2}B_0 \\ A_y=\frac{x}{2}B_0 \end{array} \right. \tag{24}$$

The transfer matrix would be symplectic if a change of frame changing $\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}$ coordinates into $\begin{bmatrix} x \\ p_x + qA_x \\ y \\ p_y + qA_y \end{bmatrix}$ was taken into account.

Change of frame given the potential vector :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & K_s/2 & 0 \\ 0 & 0 & 1 & 0 \\ -Ks/2 & 0 & 0 & 1 \end{bmatrix}. \tag{25}$$

The matrix is identical to the entrance matrix of the solenoid!

2.2 Symplecticity of the transformation matrix for the whole solenoid

Here ${}^tR_s \cdot J \cdot R_s = J$. The transformation is symplectic.

In this case the transverse vector potential components are zero before and after the magnetic element, and the product of three non-symplectic matrices is symplectic.

3 Case of the detector solenoid : tracking of a reference particle in half the magnet

Calculation of X at the IP:

$$X = R_c(s \to L) \cdot R_i \cdot X_0 \tag{26}$$

$$X = \begin{bmatrix} 1 & \frac{2SC}{K_s} & 0 & \frac{2S^2}{K_s} \\ 0 & C^2 - S^2 & 0 & 2SC \\ 0 & -\frac{2S^2}{K_s} & 1 & \frac{2SC}{K_s} \\ 0 & -2SC & 0 & C^2 - S^2 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x'_0 \\ 0 \\ -\frac{K_s}{2} x_0 \end{bmatrix} \tag{27}$$

$$X = \begin{bmatrix} x_0 + \frac{2SC}{K_s} x'_0 - S^2 x_0 \\ x'_0(C^2 - S^2) - x_0 K_s SC \\ -x'_0 \frac{2S^2}{K_s} - x_0 SC \\ -x'_0 2SC - x_0 \frac{K_s}{2}(C^2 - S^2) \end{bmatrix} \text{ with } \begin{cases} C = \cos(\frac{K_s L}{2}) \\ S = \sin(\frac{K_s L}{2}) \end{cases} \tag{28}$$

The vertical trajectory crosses the detector solenoid axis on its center if $y(L) = 0$, or, according to Eq. (28), if :

$$-x'_0 \frac{2S^2}{K_s} - x_0 SC = 0 \tag{29}$$

Which leads to the condition on the initial horizontal position and angle :

$$x_0 = -x'_0 \frac{2}{K_s} \tan(\frac{K_s L}{2}) \tag{30}$$

Here $B\rho = 833.91$, we can then consider $\frac{K_s L}{2} << 1$, and the condition becomes :

$$\boxed{x_0 = -x'_0 L} \tag{31}$$

The condition given by Eq. (31) means that the vertical deviation of the trajectory is exactly cancelled at the IP if the initial horizontal trajectory is directed toward the solenoid center, as represented Fig.1.

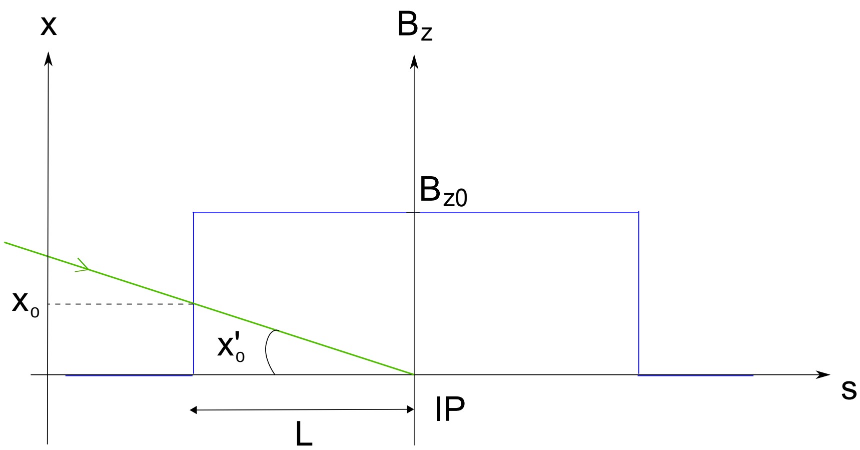

Figure 1: *For the vertical deviation to be cancelled at the IP, the central trajectory must be directed toward the center of the solenoid when the reference particle penetrates the fringe field.*

4 Vertical deviation in the solenoidal field

4.1 Vertical trajectory

Taking into account the given parameters:

$$\begin{cases} L &= 4\text{m} \\ K_s &= \frac{4}{833{,}91} = 4,8.10^{-3}\text{m}^{-1} \\ \vartheta_c &= 14\text{mrad} \end{cases}$$

The tabulated values of $y(s)$ are :

s (m)	0	0,5	1	1,5	2	2,5	3	3,5	4
y (μm)	0	29,3	50,3	62,9	67,1	62,9	50,3	29,3	0

And the resulting trajectory is given Fig. 2 :

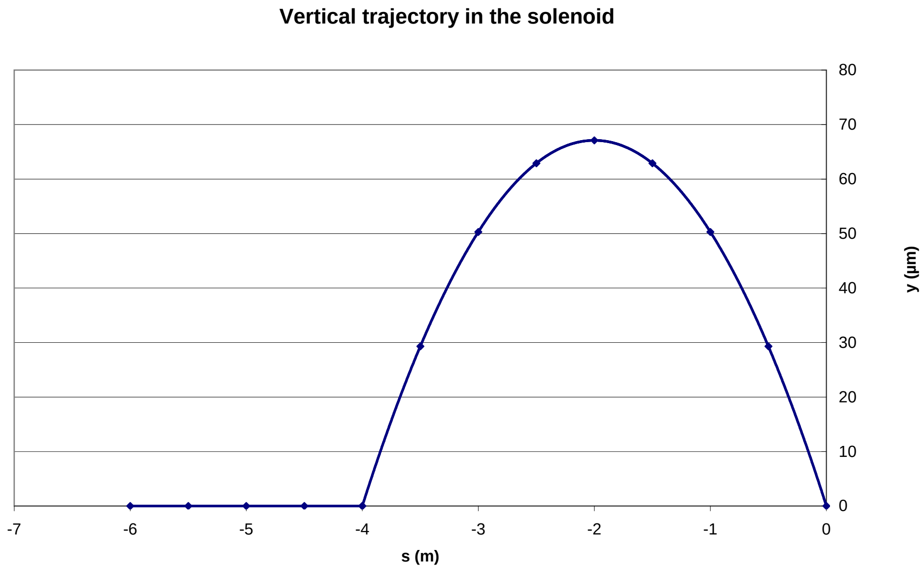

Figure 2: *The IP is at $s = 0$. The vertical trajectory is back on axis at the center of the magnet if $x_0 = -x_0' L$.*

4.2 Insertion of a focusing quadrupole in the solenoid field

The central trajectory being non zero at the focusing lens location, it is deviated. The natural compensation of the effetcs on the orbit of radial and axial components of the solenoid field is broken, as shown Fig. 3. :

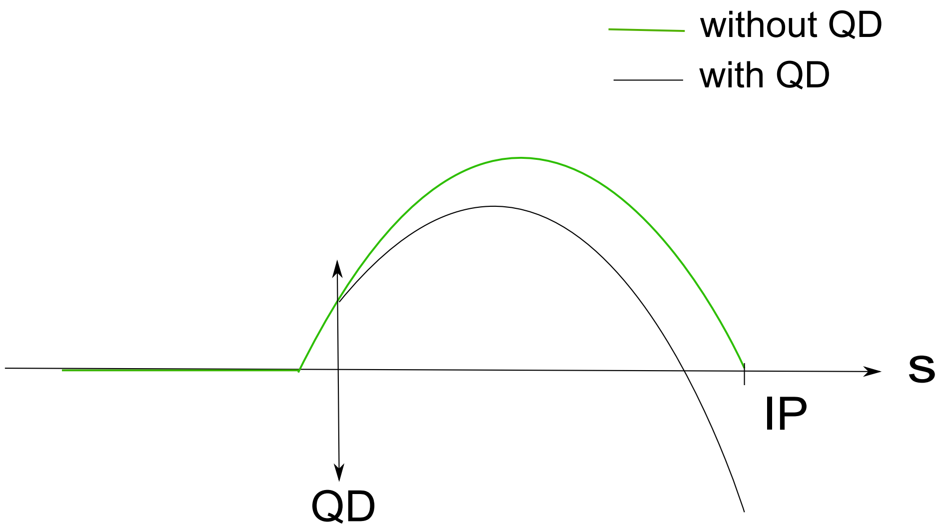

Figure 3: *Effect of the solenoid field overlapping with the final focusing quadrupole.*

Calculation of the offset at the IP when the lens is 3 m far from the IP :

Kick $\delta y'_{QD}$ due to the focusing lens :

$$\delta y'_{QD} = KLy(3m) = -0.3 * 50.3 = -15\mu rad$$

Effect at IP, $f = 3m$:

$$\boxed{y_{IP} = y'_{QD} * 3 = 45\mu m}$$

The offset is so large compared to the beams vertical dimension that it will prevent the beams from colliding. This effect has to be corrected.