Fourth International Accelerator School for Linear Colliders

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Lecture A3

Beam Delivery System and Beam-Beam Effects

Part 1

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Collider Parameters, Physical Constants and Notations

E , energy and p , momentum

$B\rho$, magnetic rigidity ($B\rho = p/e$)

$\mathcal{L}$, luminosity

Q , bunch charge

N , number of particles in the bunch ($N = Q/e$)

n_b , number of bunches in the train

f_{rep} , pulse repetition rate

ε_x and ε_y , horizontal and vertical emittances

σ_x^* and σ_y^*, rms horizontal and vertical beam sizes at the IP

σ_z , bunch length

α, β, γ , Twiss parameters

μ , tune

ψ , phase advance,

$c = 299\,792\,458$ m/s

$e = 1.602\,177\,33\ 10^{-19}$ C

$m_e c^2 = 510\,999\ e\mathrm{V}$

$\mu_0 = 4\pi\ 10^{-7}$ N A^{-2} permeability

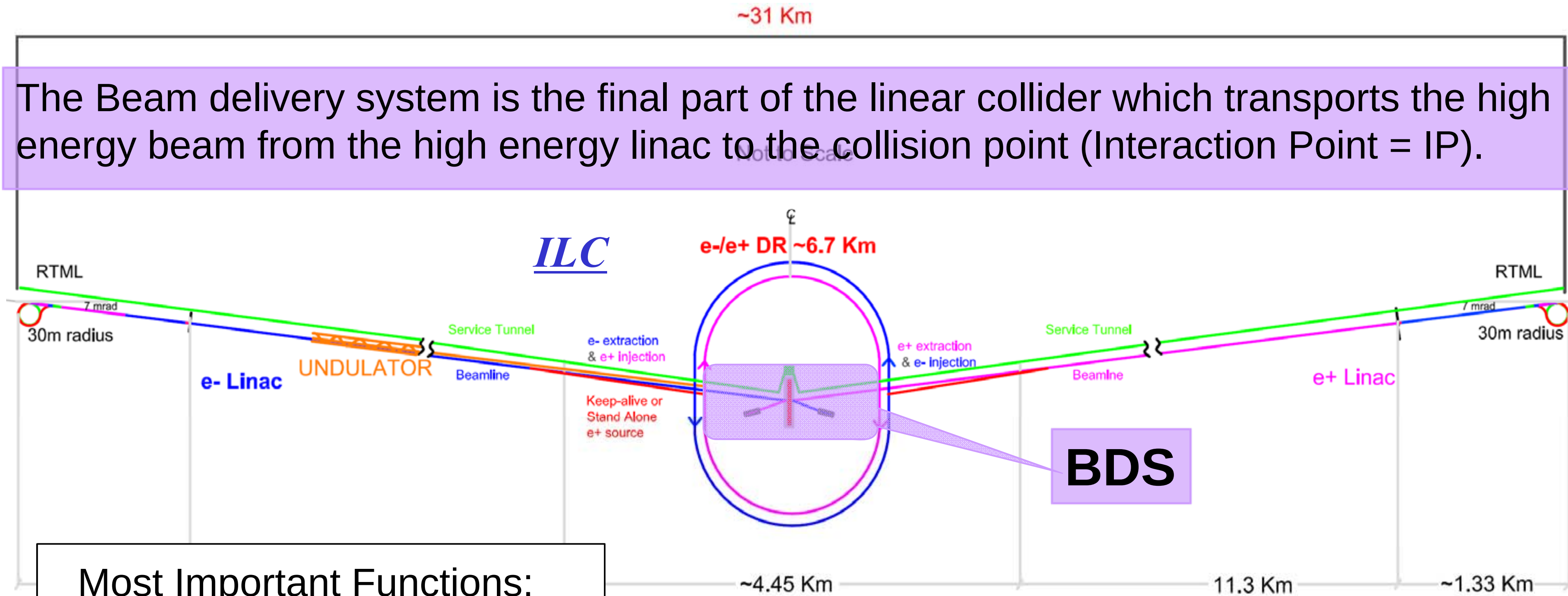
$\varepsilon_0 = 1/\mu_0 c^2$ permittivity

$r_e = 2.818\ 10^{-15}$ m

with $m_e c^2 = \dfrac{e^2}{4\pi\varepsilon_0 r_e}$

Beam Delivery System

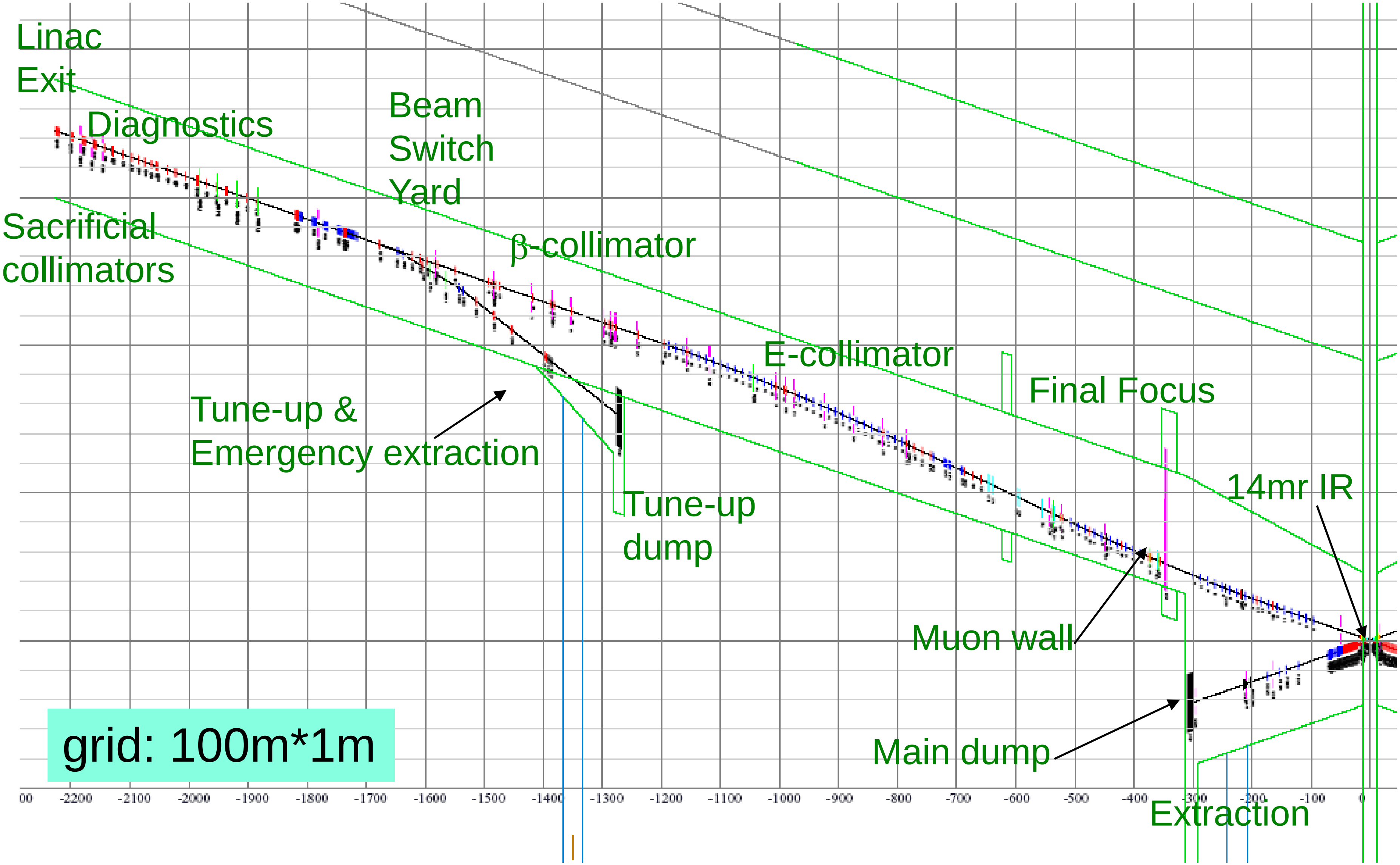
The Beam delivery system is the final part of the linear collider which transports the high energy beam from the high energy linac to the collision point (Interaction Point = IP).



Most Important Functions:

- **Final Focus**: Focus the beams at the interaction point to achieve very small beam sizes.
- **Collimation**: Remove any large amplitude particles (beam halo).
- **Tuning**: Ensure that the small beams collide optimally at the IP.
- **Matching**: Precise beam emittance measurement and coupling correction.
- **Diagnostics**: Measure the key physics parameters such as energy and polarization.
- **Extraction**: Safely extract the beams after collision to the high-power beam dumps.

ILC Beam Delivery System Layout (RDR)



Contents: Four (+ Two) Outstanding Questions ?

Part 1

- Q1: do we need a Final Focus System ?
  - Luminosity, Emittance,
- Q2: do we need High Field Quadrupole Magnets ?
  - Quadrupole Magnets, Multipoles, Superconducting Quadrupoles,

Part 2

- Q4: do we need Flat Beams ?
  - Beam-beam forces, Beamstrahlung, e+e- Pairs, Fast Feedback
- Q3: do we need Corrections Systems ?
  - Beam Optics, Achromat, Emittance Growth

Part 3 (if time permits)

- Q5 : do we need a Crossing Angle ?
  - Beam extraction, Kink Instability, Crab-crossing
- Q6: do we need Collimation ?
  - Synchrotron radiation, SR collimation, Beam collimation

Contents: Fundamentals in Beam Physics

Part 1

- Luminosity
- Emittance
- Magnetism

Part 2

- Beam-Beam Effect
- Beam Optics
- Synchrotron Radiation

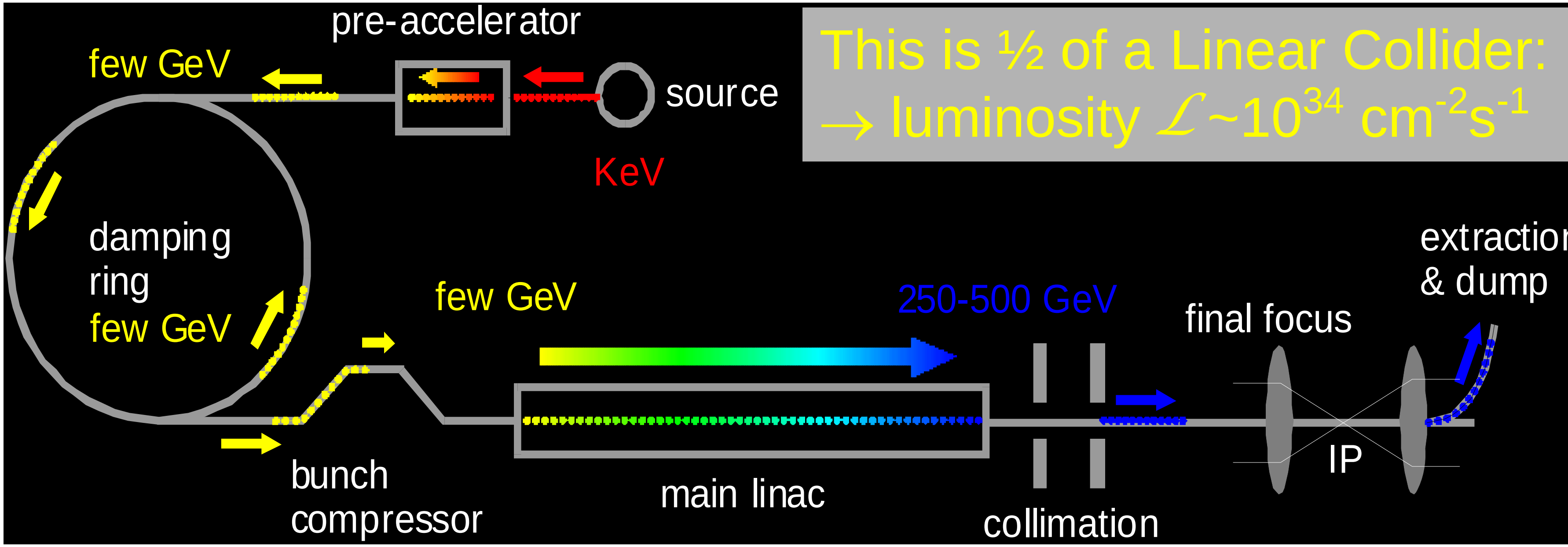
Part 3

- Collimation

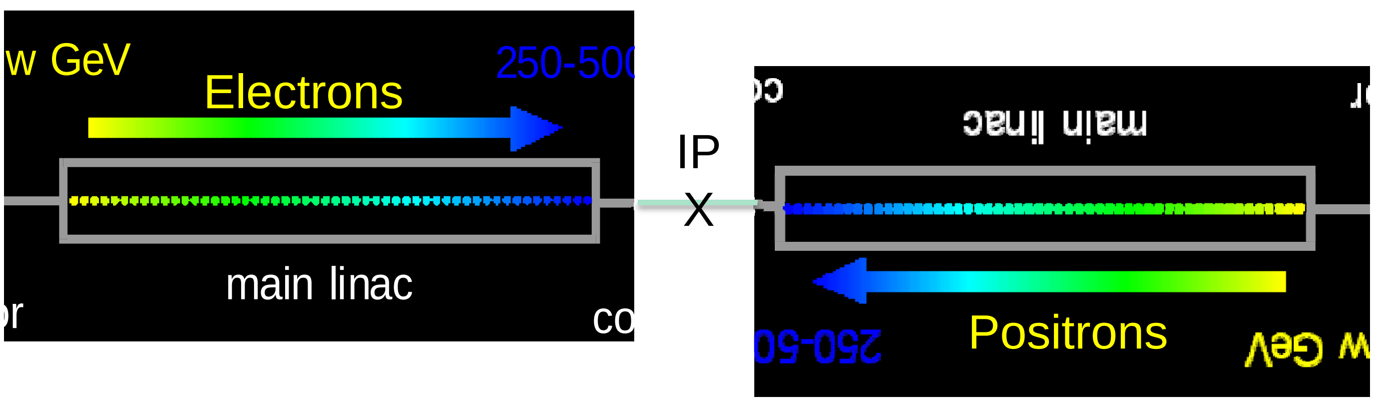
Question n°1:

Do we need a Final Focus System ?

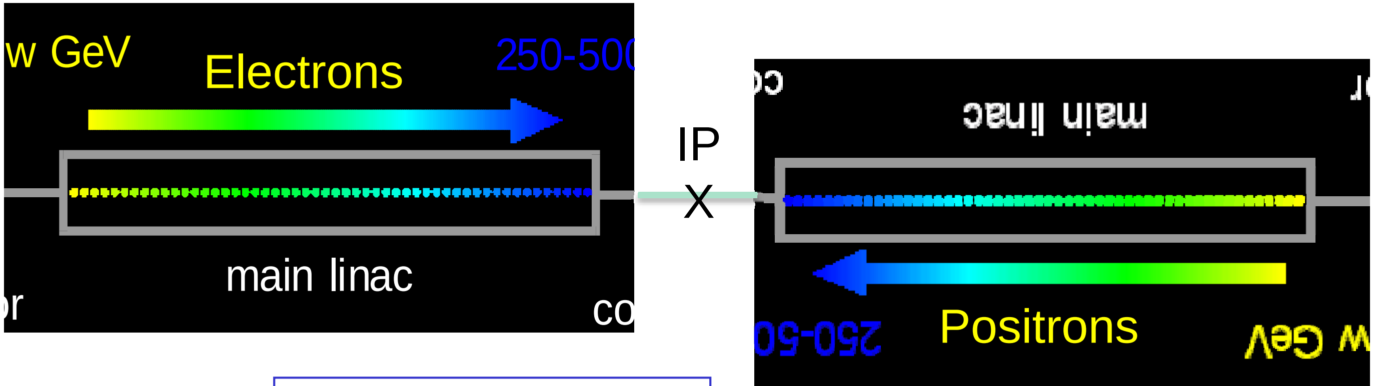
Collider Systematic



Collisions of beams *directly* from RF Linac → luminosity $\mathcal{L} \sim 5\ 10^{28}\ \mathrm{cm}^{-2}\mathrm{s}^{-1}$



Collisions of Linac Beams



Luminosity: $$\mathcal{L} = \frac{n_b N^2 f_{rep}}{4\pi\sigma_x^* \sigma_y^*}$$

Typical electron Linacs are producing:

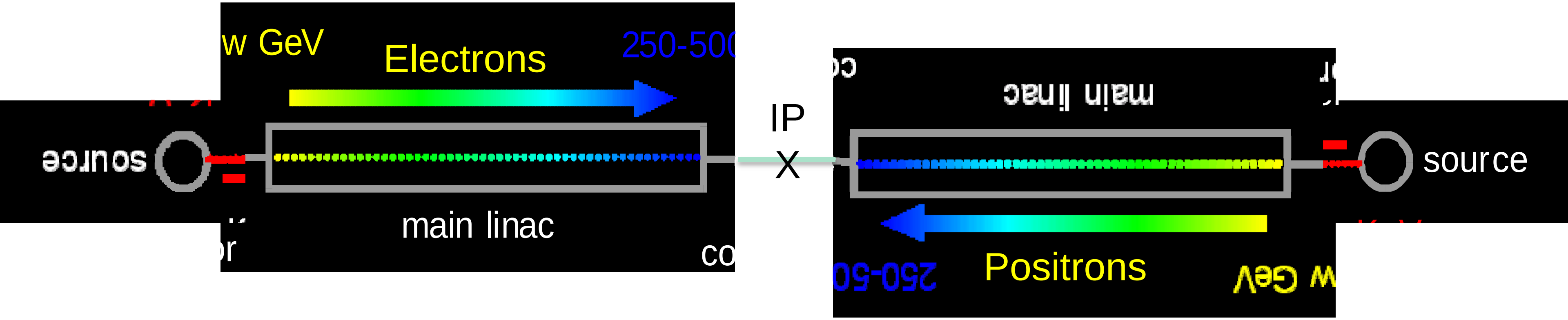
Q = 1 nC bunch charge ($N = 6{\cdot}10^9$) at $\gamma\varepsilon_x{\cdot}\gamma\varepsilon_y$ =1 μm^2 normalized transverse emittances, transported through a FODO lattice with L = 38 m cell length, leading to $\beta_x{\cdot}\beta_y \approx 38^2$ m^2 .

Assuming n_b = 2625 bunches at f_{rep} = 5 Hz and E = 250 GeV per beam

$$\Rightarrow \mathcal{L} = 5\ 10^{28}\ \text{cm}^{-2}\text{s}^{-1}$$ luminosity

generated by $\sigma_x^*{\cdot}\sigma_y^*$ = 80 μm^2 round beam sizes at the IP.

Collisions of Linac Beams from Dedicated Sources



Luminosity: $$\mathcal{L} = \frac{n_b N^2 f_{rep}}{4\pi\sigma_x^*\sigma_y^*}$$

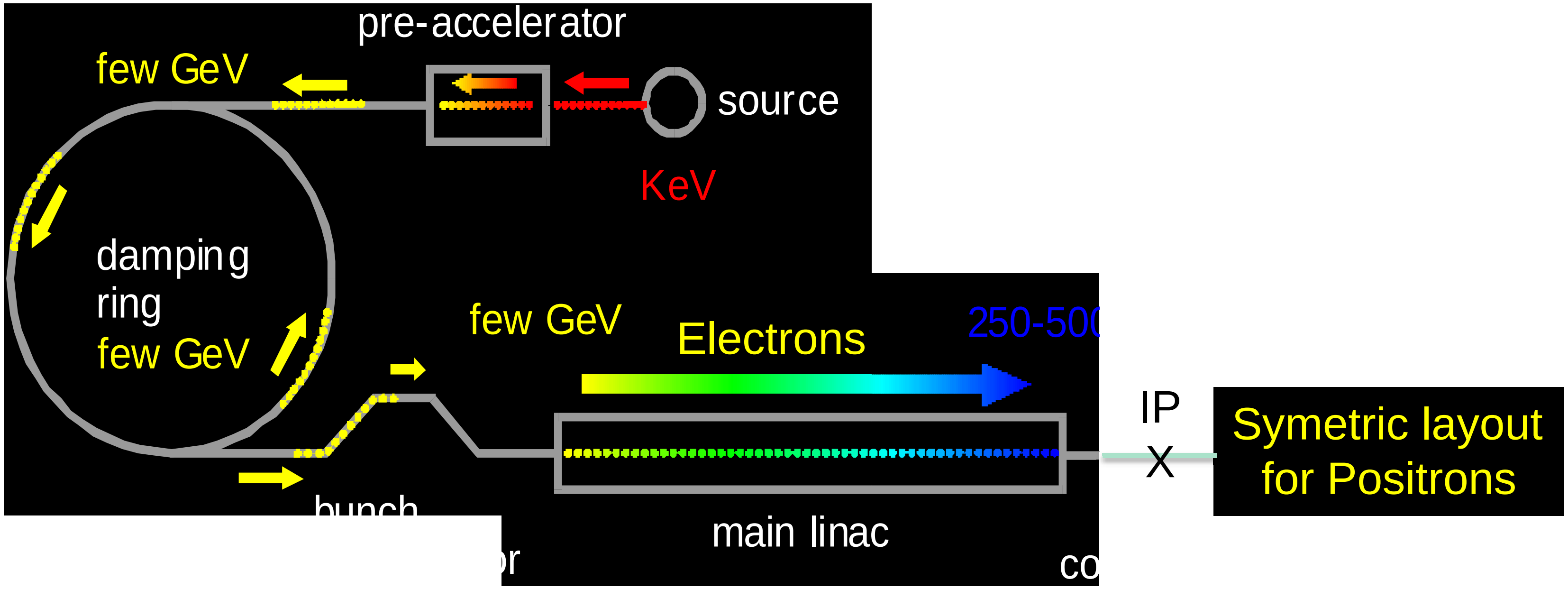
Adding powerful electron and positron sources with

Q = 3.2 nC bunch charge ($N = 2{\cdot}10^{10}$)

$\Rightarrow \mathcal{L} = 5 \cdot 10^{29}$ cm^{-2}s^{-1} luminosity

assuming $\gamma\varepsilon_x \cdot \gamma\varepsilon_y$ =1 µm^2 normalized emittances (*not realistic for* 3.2 nC, *especially for the positron source !!*)

Linac Beams from Dedicated Sources and Damping Rings

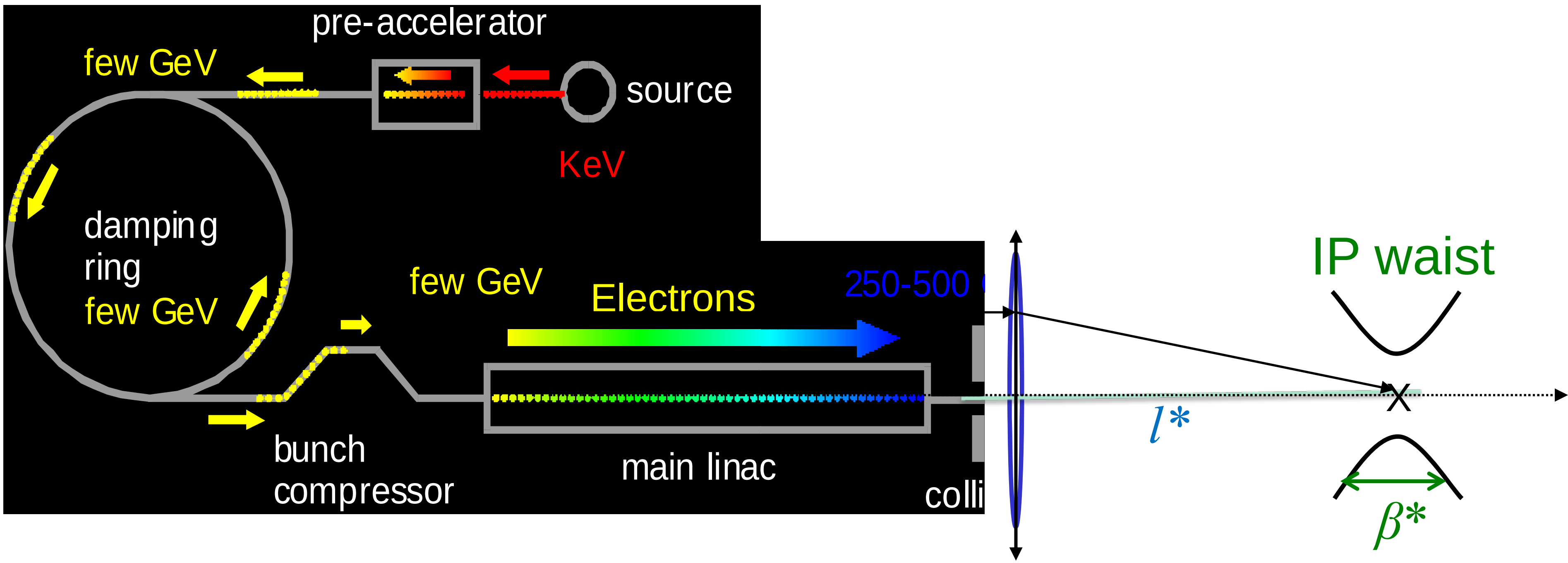


Adding Damping Rings to reduce the normalized transverse emittances of the high charge electron and positron beams, to $\gamma\varepsilon_x \cdot \gamma\varepsilon_y$ =0.4 µm^2

$\Rightarrow \mathcal{L} = 8.5 \cdot 10^{29}$ cm^{-2}s^{-1} luminosity

generated by $\sigma_x{}^* \cdot \sigma_y{}^*$ = 50 µm^2 *flat* beam sizes at the IP

De-magnification from Final Focus Systems

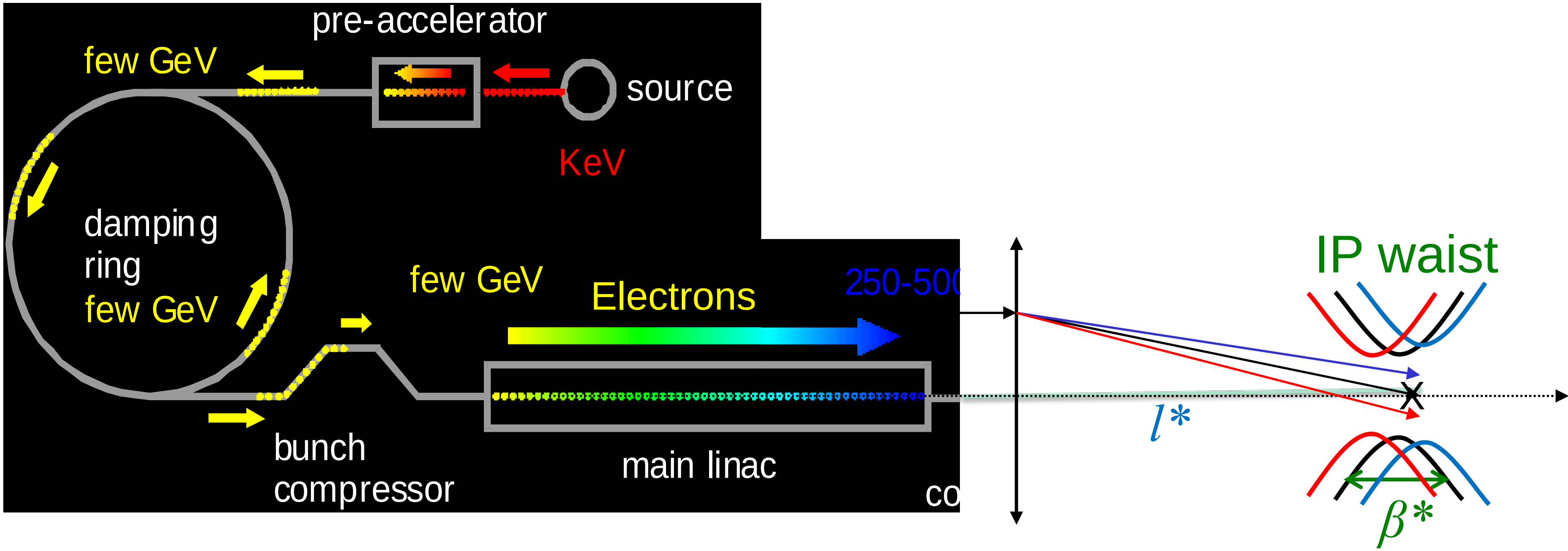


Final Focus Systems are implementing the necessary $\sim 10^4$ demagnification factors to reduce the $\sigma_x^* \cdot \sigma_y^* = 50\ \mu m^2$ *flat* beam sizes to $0.0036\ \mu m^2$

$$\Rightarrow \mathcal{L} \sim 10^{34}\ cm^{-2}s^{-1} \quad \text{luminosity}$$

The final focusing lens is implemented via a doublet (or triplet) of strong Quadrupole Magnets located at distance l^* from the IP, typically 3 to 4 m.

Depth of Focus and Chromaticity



• The Beam Waist is realized over a finite region of longitudinal space called the Depth of Focus (like for a camera), measured by the $\beta*$ parameter. The collision with the opposite beam is optimum only within this region.

• Like all magnets (cf. spectrometer magnets) the final quadrupoles create an energy dependant focusing effect, a 2nd order effect called Chromaticity.

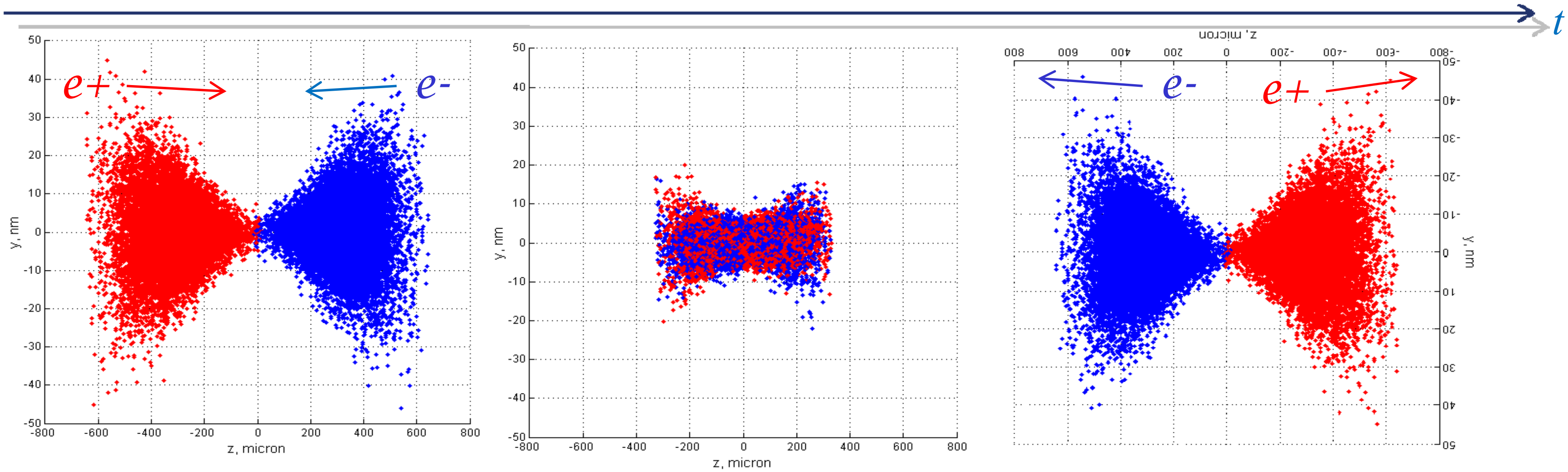
As a consequence, the sizes and positions of the beam waist vary with the energy of the particles : this is called Chromatic Aberrations.

The six important concepts introduced so far

1. Luminosity
2. Beam emittance (β -functions)
3. Final doublet of quadrupoles
4. Beam optics (FODO lattice, 1^{st} order beam optics)
5. Chromatic aberrations (2^{nd} order beam optics)
6. Flat beams

Luminosity

The Collider Luminosity



The Rate $N_{\mathcal{E}}$ of Physics Event $\mathcal{E}$ generated during a beam-beam collision is given by:

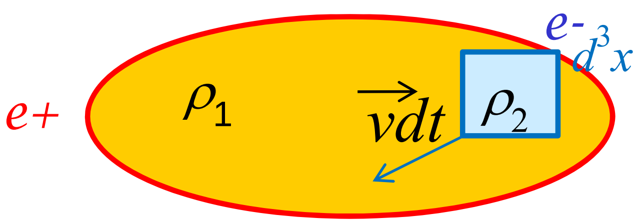
$$N_{\mathcal{E}} = \overline{\mathcal{L}} \cdot \sigma_{\mathcal{E}}$$

where

- $\sigma_{\mathcal{E}}$ is the Cross-Section of the Physics Event $\mathcal{E}$
- $\overline{\mathcal{L}}$ is the Integrated Luminosity over the bunch crossing.

$N_{\mathcal{E}}$, $\sigma_{\mathcal{E}}$, and $\overline{\mathcal{L}}$ are Lorentz invariant quantities !

Luminosity for Fixed Target Collision



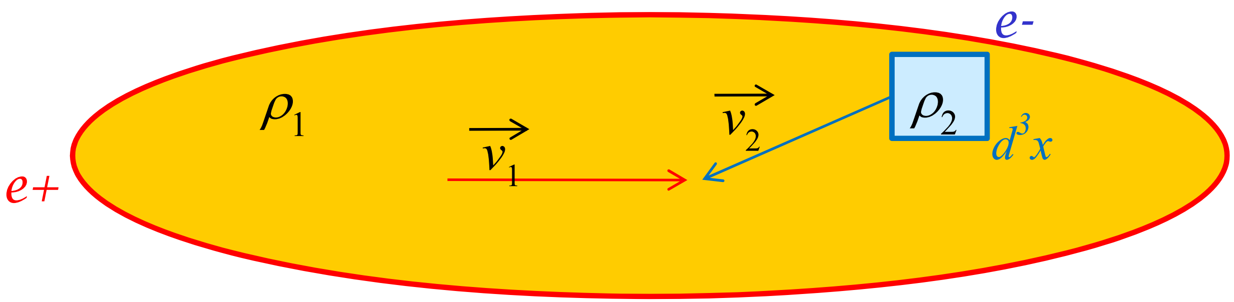
The *integrated luminosity* of an elementary volume element d^3x of particle density ρ_2 moving through a fixed target of density ρ_1 during the time dt , is given by:

$$d\overline{\mathcal{L}} = \rho_1\, \rho_2 d^3x\, vdt$$

The particle densities are defined such that:

$$\int d^3x\, \rho(\vec{x}, t) = N$$

The Luminosity for Mono-kinetic Beams



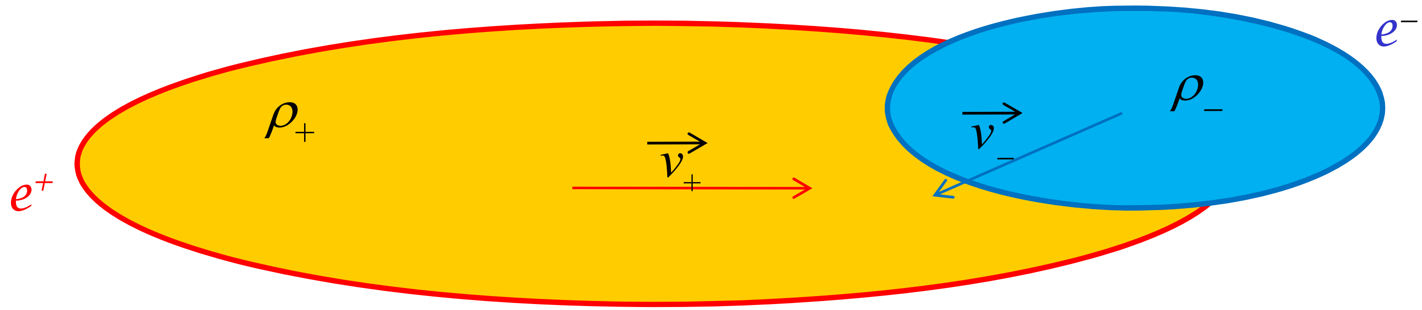
- The integrated luminosity generated by the elementary volume element d^3x of particle density ρ_1 moving through ρ_2 with homogeneous velocities v_1 and v_2 , is:

$$d\overline{\mathcal{L}} = d^3x\, dt\ \rho_1\rho_2 \left[(\vec{v}_1 - \vec{v}_2)^2 - \frac{(\vec{v}_1 \times \vec{v}_2)^2}{c^2} \right]^{1/2}$$

- Introducing the Lorentz current 4-vector $\boldsymbol{J} = (\rho c, \rho\vec{v})$, the integrated luminosity is expressed in an explicitly Lorentz invariant way:

$$d\overline{\mathcal{L}} = \frac{1}{c^2}(cdt\ d^3x)\left((\boldsymbol{J}_1 \cdot \boldsymbol{J}_2)^2 - \boldsymbol{J}_1{}^2\boldsymbol{J}_2{}^2\right)^{1/2}$$

The Luminosity for Mono-kinetic Beams



For relativistic beams $\gamma_1, \gamma_2 \to \infty$, this simplifies to:

$$\overline{\mathcal{L}} = \frac{1}{c^2}\int cdt\ d^3x\, \boldsymbol{J}_1 \cdot \boldsymbol{J}_2 = \frac{1}{c^2}\int cdt\ d^3x\, \rho_1\rho_2(c^2 - \vec{v}_1 \cdot \vec{v}_2)$$

First example: the head-on collision $\vec{v}_+ = -\vec{v}_- = c\hat{z}$ of Gaussian bunches

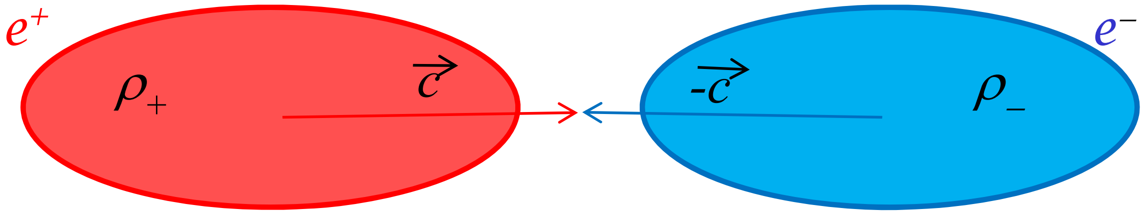
$$\rho_\pm(\vec{x},t) = \frac{N_\pm}{(2\pi)^{3/2}\sqrt{\sigma_x^\pm \sigma_y^\pm \sigma_z^\pm}} \exp -\frac{1}{2}\left(\frac{x^2}{\sigma_x^{\pm 2}} + \frac{y^2}{\sigma_y^{\pm 2}} + \frac{(z \pm ct)^2}{\sigma_z^{\pm 2}}\right)$$

leads to $\overline{\mathcal{L}} = \dfrac{N_+ N_-}{4\pi \Sigma_x \Sigma_y}$ with $\Sigma^2 = \dfrac{1}{2}(\sigma_+^2 + \sigma_-^2)$

The Luminosity for Mono-kinetic Beams

2nd example: head-on collision of mirror Gaussian bunches

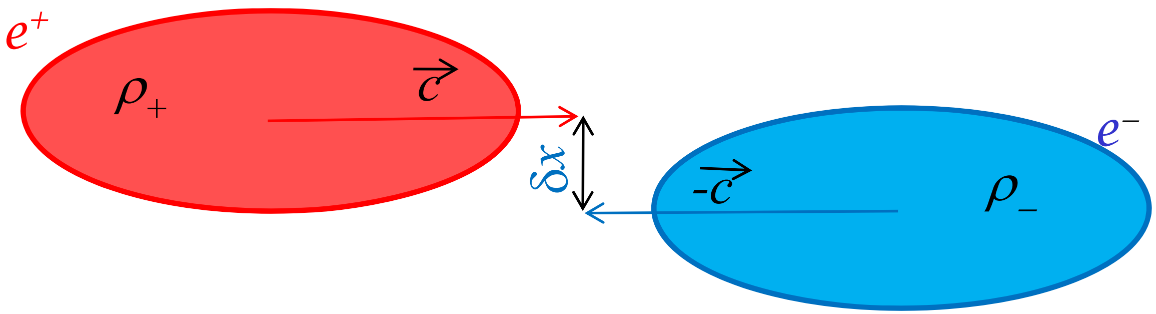
$$\overline{\mathcal{L}}_0 = \frac{N^2}{4\pi\,\sigma_x\sigma_y}$$



$\Rightarrow$ average luminosity over the bunch trains and collider pulse:

$$\boxed{\mathcal{L}_0 = \frac{f_{\text{rep}}\, n_{\text{b}}\, N^2}{4\pi\,\sigma_x\sigma_y}}$$

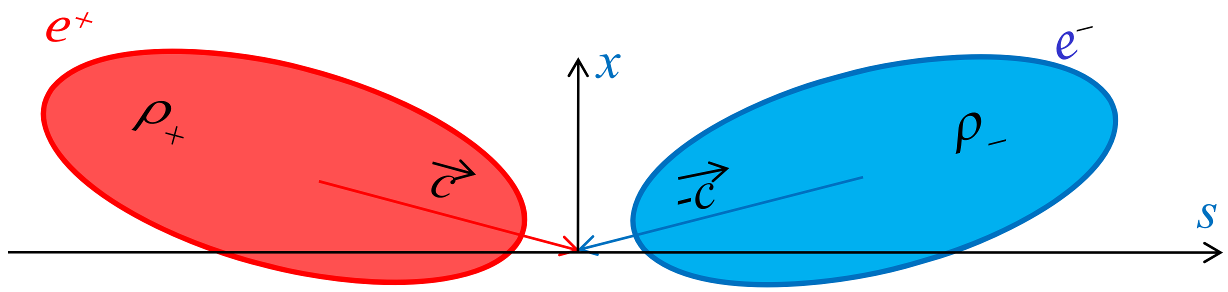
3rd example: idem with transverse offsets δx , δy :



$$\overline{\mathcal{L}} = \overline{\mathcal{L}}_0 \exp -\frac{1}{4}\left(\left(\frac{\delta x}{\sigma_x}\right)^2 + \left(\frac{\delta y}{\sigma_y}\right)^2\right)$$

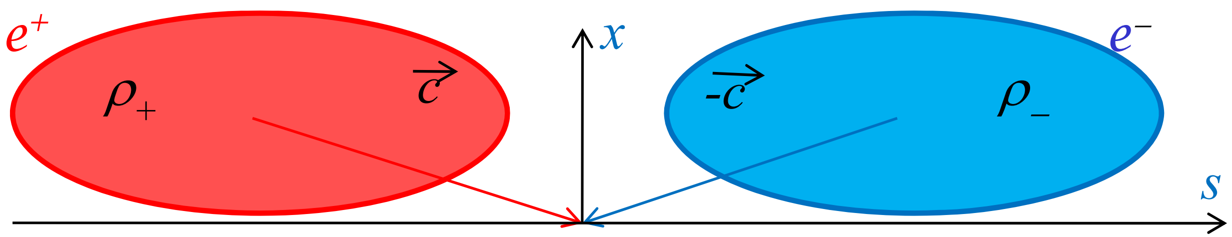
The Luminosity for Mono-kinetic Beams

4th example: case n°3, with a crossing angle $\pm\alpha/2$



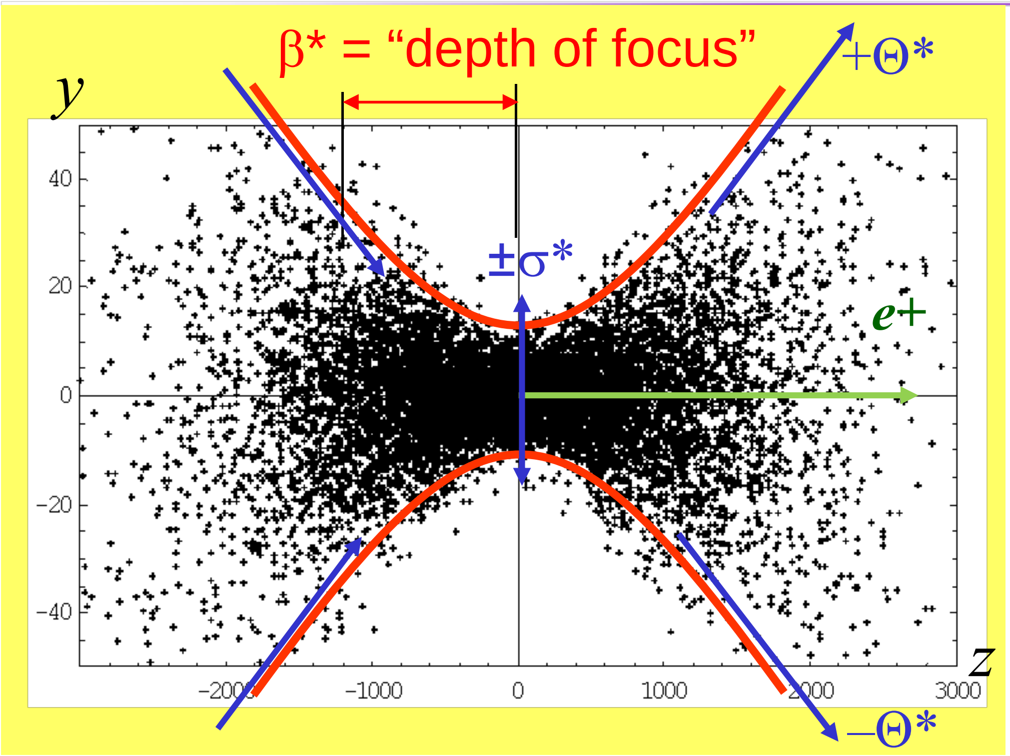
$$\overline{\mathcal{L}} = \overline{\mathcal{L}_0} \Big/ \sqrt{1 + (\tan(\alpha/2)/\Theta_P)^2}$$ with $\Theta_P = \sigma_x/\sigma_z$, 'Piwinski angle' or 'diagonal angle'

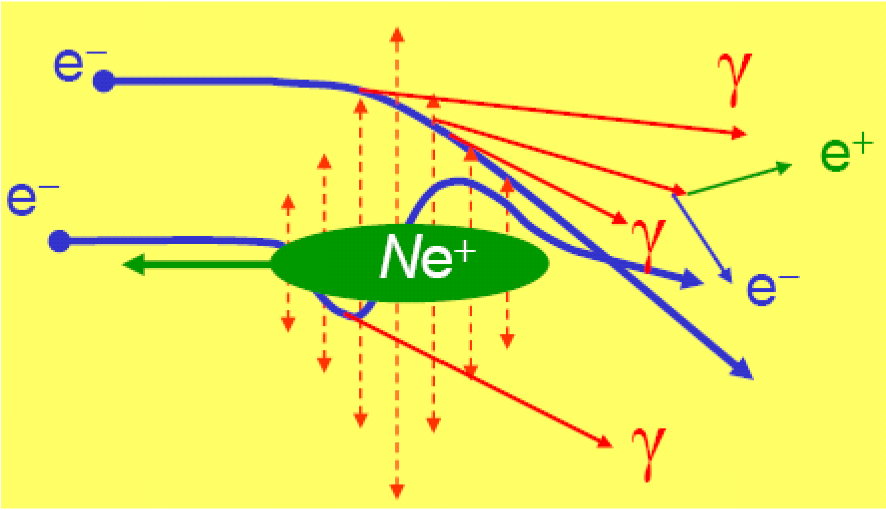
5th example: case n°3, with a crab crossing angle $\pm\alpha/2$



$$\overline{\mathcal{L}} = \overline{\mathcal{L}_0}\cos(\alpha/2)$$

The Luminosity of Realistic Colliding Beams

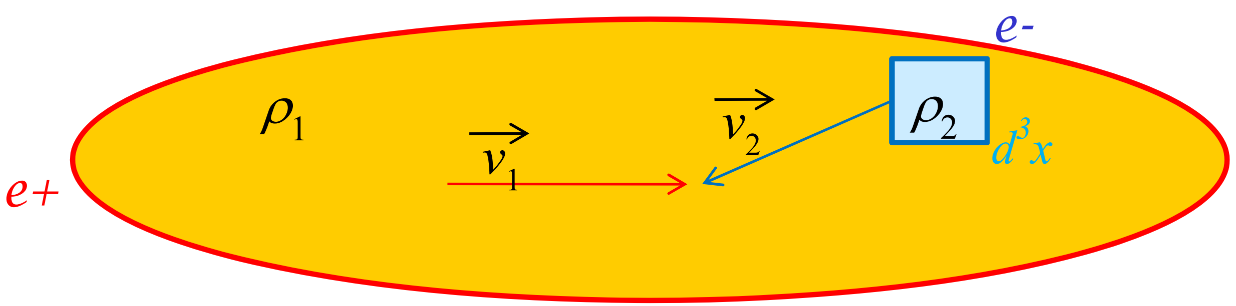




The above "academic" results are modified by the reality of 2 hard factors of Linear Colliders

1. **Real beams are not mono-kinetic**: include angular distribution (*cf. left pict.*)
2. **Real particle trajectories are not straight during the collision**: they are strongly deviated by the electromagnetic field generated by the opposite beam, the so-called "Beam-Beam Forces" (*cf. right pict. and Beam-Beam section*)

The Luminosity for Colliding Beams



Introducing time and velocity dependent distributions $\rho(\vec{x},\vec{v},t)$ such that:

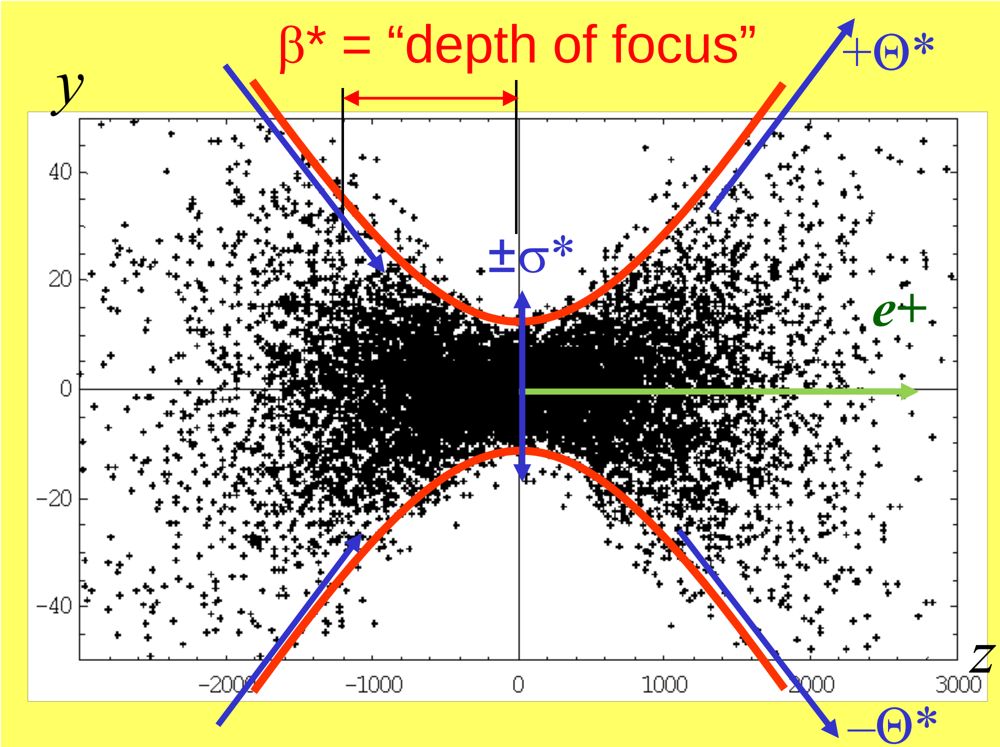
$$\int d^3\vec{x}\, d^3\vec{v}\, \rho(\vec{x},\vec{v},t) = N$$

the integrated luminosity for colliding relativistic beams is given by:

$$\overline{\mathcal{L}} \cong \frac{1}{c^2}\int c\,dt\; d^3\vec{x}\, d^3\vec{v}_1\, d^3\vec{v}_2\, \boldsymbol{J}_1 \cdot \boldsymbol{J}_2$$

$$= \frac{1}{c^2}\int c\,dt\; d^3\vec{x}\, d^3\vec{v}_1\, d^3\vec{v}_2\, \rho_1\rho_2(c^2 - \vec{v}_1\cdot\vec{v}_2)$$

The Hour Glass effect



Beams are naturally converging to their focus point, reaching their minimum sizes σ_x^*, σ_y^*, and diverging from there, with angular spreads : Θ_x, Θ_y .

The focus point is described by a local 'parabola' of finite '*depths of focus*'

$$\beta_x^* = \sigma_x^* / \Theta_x \,,\, \beta_y^* = \sigma_y^* / \Theta_y$$

Introducing the angle variables

$$x' = v_x / c,\ y' = v_y / c$$

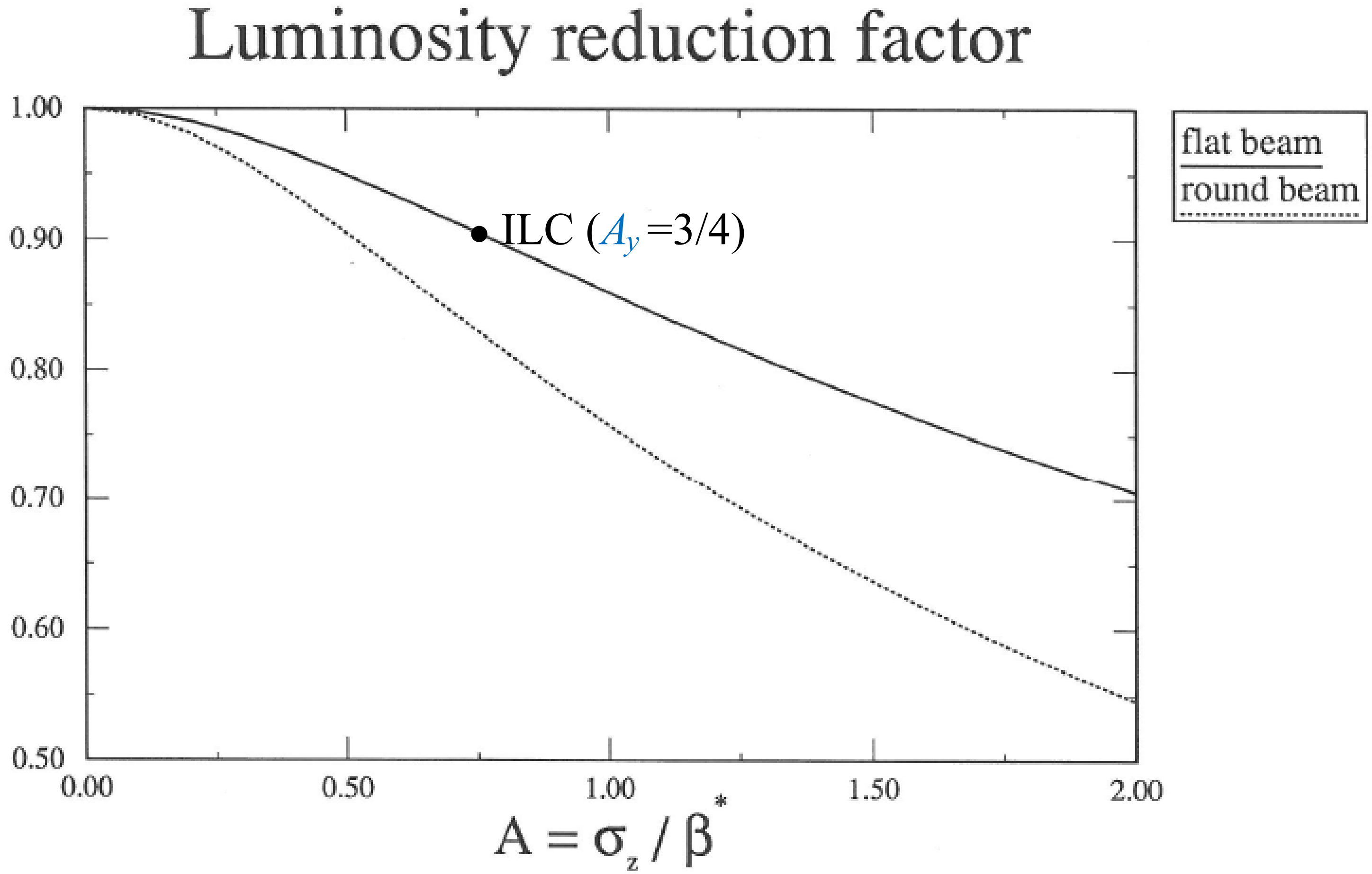
and assuming Gaussian bunches:

$$\rho(\vec{x}, \vec{x}'_{\perp}, t) = \frac{N}{(2\pi)^{5/2}\sqrt{\sigma_x\sigma_y\sigma_z\Theta_x\Theta_y}} \exp -\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{(z \pm ct)^2}{\sigma_z^2} + \frac{x'^2}{\Theta_x^2} + \frac{y'^2}{\Theta_y^2}\right)$$

$$\Rightarrow \overline{\mathcal{L}} = \overline{\mathcal{L}}_0 \frac{2}{\sqrt{\pi}} \int_0^{+\infty} du\ e^{-u^2} \Big/ \sqrt{1 + \left(u\sigma_z / \beta_x^* \right)^2} \sqrt{1 + \left(u\sigma_z / \beta_y^* \right)^2}$$

The Hour Glass Reduction Factor

It is desirable to fulfil the condition $\sigma_z < \beta_x^*, \beta_y^*$ in order to longitudinally contain most of the bunch particles within the depths of focus β_x^* and β_y^*, hence the 1st need for a longitudinal bunch compression system to reduce σ_z.



The Pinch Effect

In e^+e^- colliders, electrons are **focused** by the strong electromagnetic forces generated by the positron beam (f = focal length). This is the well known *beam-beam effect* limiting the luminosity of circular colliders.

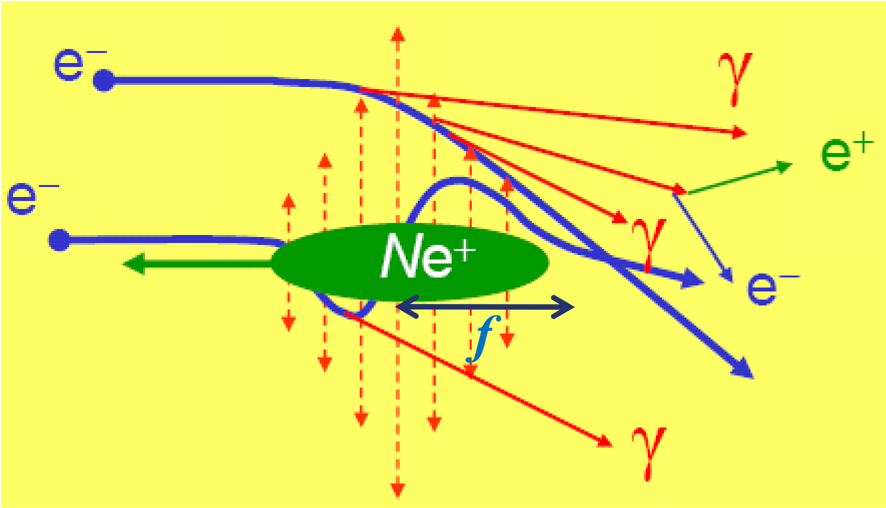
In linear collider (single pass), the beams are 'pinched' and the luminosity is enhanced by this *Pinch Effect*:

$$\mathcal{L} = \frac{f_{\mathrm{rep}} n_{\mathrm{b}} N^2}{4\pi \sigma_x \sigma_y} H_D$$

H_D = disruption enhancement factor (usually includes the hour-glass reduction).

Nota Bene:

H_D > 1 in e^+e^- colliders
H_D < 1 in e^-e^- colliders



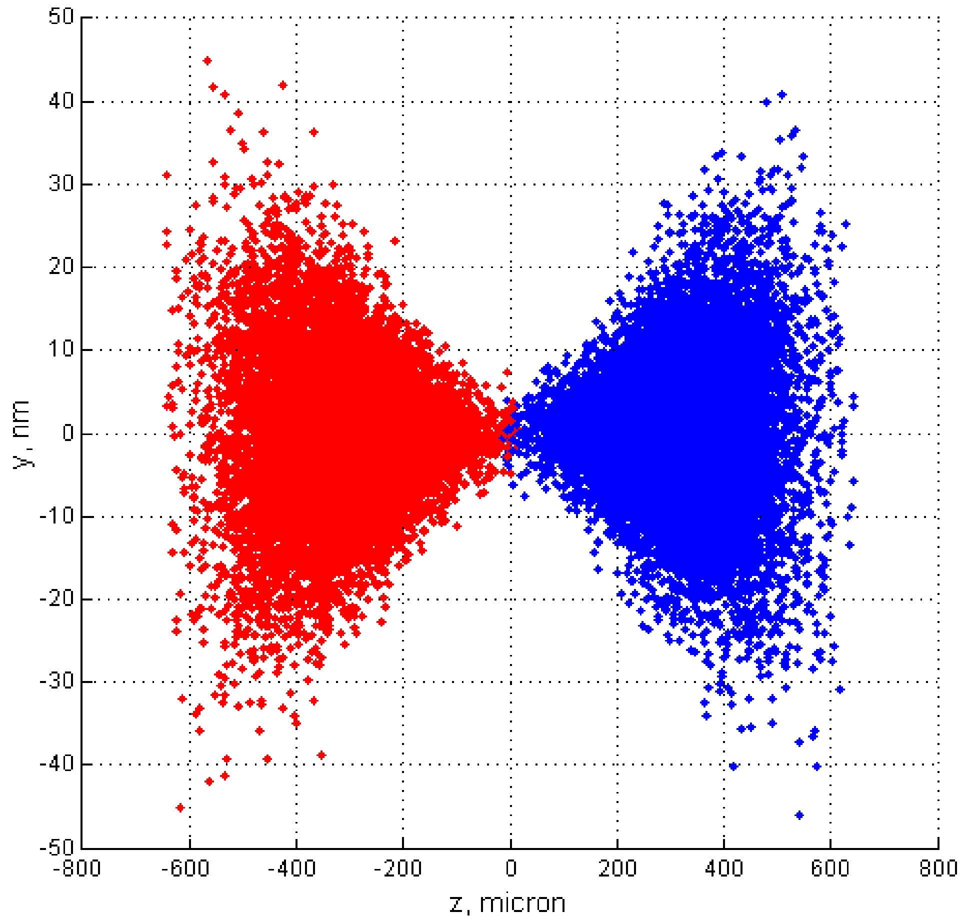
However, for long bunches, several trajectory oscillations can develop and lead to an unstable disruption regime.

The Disruption Parameter is defined as

$$D = \sigma_z / f \propto (\#\ \mathrm{oscillations})^2$$

hence the 2nd need for a short bunches and a longitudinal bunch compression system.

Beam-Beam Simulations



ILC parameters

$D_y \sim 12$

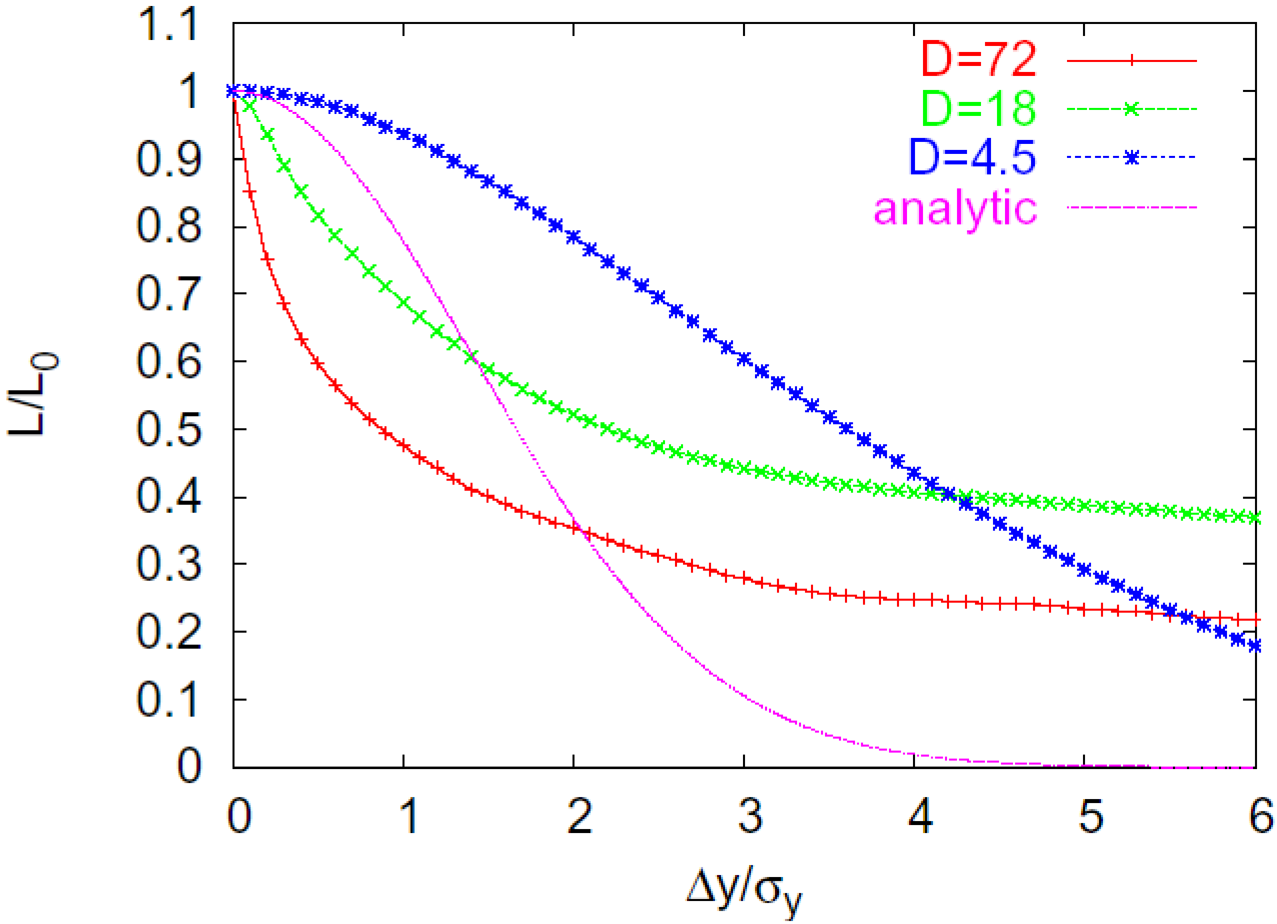
Luminosity enhancement

$H_D \sim 1.4$

Not much of an instability.

Animations produced by A. Seryi using the **GUINEAPIG** beam-beam simulation code (D. Schulte).

Luminosity as a Function of Vertical Beam Offset



• Small disruption is beneficial to the luminosity because the bunches are attracting and focusing each other smoothly.

• Large disruption is detrimental at small offsets because a kink instability develops.

ILC Beam Delivery System Parameters

Max Energy/beam (with more magnets)	GeV	250 (500)
Distance from IP to first quad, L*	m	3.5-(4.5)
Crossing angle at the IP	mrad	14
Nominal beam size at IP, σ^*, x/y	nm	655/5.7
Nominal beam divergence at IP, θ^*, x/y	μrad	31/14
Nominal beta-function at IP, β^*, x/y	mm	21/0.4
Nominal bunch length, σ_z	μm	300
Nominal disruption parameters, x/y		0.162/18.5
Nominal bunch population, N		2×10^{10}
Max beam power at main and tune-up dumps	MW	18

Notice:

- '*Razor blade*' transverse aspect of the beams at collision:

  $R = \sigma_x^* / \sigma_y^* \approx 115$ aspect ratio *(cf. next page)*

- The 14 mrad crossing angle is larger that Piwinski angle $\Theta_P = \sigma_x^* / \sigma_z \approx 2.2\,\mathrm{mrad}$

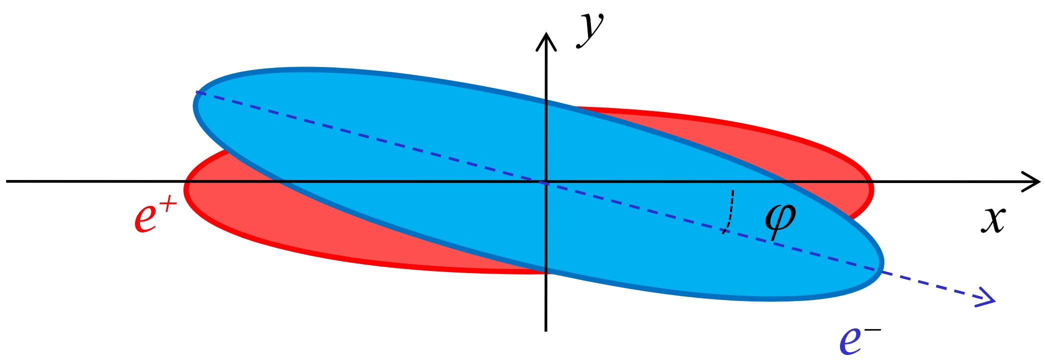
  The reduction of luminosity is too large: $\mathcal{L}(\alpha = 14\,\mathrm{mrad}) = 0.29\,\mathcal{L}_0$, hence crab-crossing is mandatory *(cf. dedicated section on crossing angle)*.

- The vertical disruption is large D_y =18.5 $\Rightarrow$ ~ one oscillation in the vertical plane.

Flat Beams : Luminosity and Horizontality

5th example: error on 'horizontality' of flat beams

$$\overline{\mathcal{L}}_0 = \frac{N^2}{4\pi\,\sigma_x^*\sigma_y^*}$$



The horizontality of flat beams is inherited from their respective Damping Rings. In case of an error, the overlap of their distributions is not perfect and induces a reduction of the luminosity:s

$$\overline{\mathcal{L}} = \overline{\mathcal{L}}_0 \Big/ \sqrt{1+\sin^2(\varphi)(R^2+R^{-2}-2)/4}$$

$$\approx \overline{\mathcal{L}}_0 (1-\varphi^2 R^2/8)$$ for large aspect ratios $R = \sigma_x^* / \sigma_y^*$

Tolerance: $R \approx 100 \quad \Rightarrow \quad \varphi \le 1\,\mathrm{mrad}$

Beam Emittance

Laplace Equation and Hamiltonian Mechanics

• The motion of *individual* trajectories in an external electromagnetic field $(\vec{E},\vec{B})$ derives from the Laplace Equation

$$\frac{d\vec{p}}{dt} = q(\vec{E}+\vec{v}\times\vec{B}) \quad \text{with} \quad \vec{v}=\frac{d\vec{x}}{dt} \quad \text{, and } \vec{p}=\gamma m\vec{v} \quad \text{particle momentum.}$$

• It obeys the laws of Hamiltonian Mechanics:

$$\frac{d\vec{Q}}{dt}=\frac{\partial H}{\partial \vec{P}} \quad , \quad \frac{d\vec{P}}{dt}=-\frac{\partial H}{\partial \vec{Q}} \qquad \textit{(Hamilton-Jacobi equations)}$$

with $\vec{Q}=\vec{x} \quad , \quad \vec{P}=\vec{p}+q\,\vec{A}(\vec{x})$

and $\boxed{H(\vec{Q},\vec{P}) = qV(\vec{Q})+c\sqrt{(\vec{P}-q\vec{A})^2+m^2c^2}}$ is the conserved energy,

where $(V/c\,,\vec{A})$ is the electromagnetic potential 4-vector.

• Hamilton equations derives from a Least Action Principle based on the Action:

$$S=\int_{t_1}^{t_2}(\vec{P}\cdot\dot{\vec{Q}}-H)\,dt$$

Hamiltonian Mechanics and Beam Distribution

Hamiltonian Mechanics is valid as long as

1. There is no Collective Effects (space charge, wake fields)
2. Synchrotron Radiation is ignored
3. There is no beam collimation

These conditions are realized 'most of the time' along the Beam Delivery System.

Hamiltonian Mechanics allows a powerful description of the beam dynamics using *Symplectic Transformations* of the beam particle *6-dimensional Phase Space*

Coordinates $\boldsymbol{X} = \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix}$.

Introducing the 6D Skew Matrix $\boldsymbol{J} = \begin{pmatrix} 0 & \boldsymbol{I}_3 \\ -\boldsymbol{I}_3 & 0 \end{pmatrix}$, with $\boldsymbol{I}_3$ the 3D unit matrix, Hamilton-Jacobi equations read:

$$\boxed{\frac{d\boldsymbol{X}}{dt} = \boldsymbol{J} \cdot \partial_X H}$$

Solving this system of equations amounts to deriving the time evolution of particle trajectories, and therefore the expressions of the transformation which maps the coordinates $\boldsymbol{X}(t_1)$ into $\boldsymbol{X}(t_2)$.

This map is noted $\boldsymbol{X}_2(\boldsymbol{X}_1)$, for convenience.

Some Symplectic Algebra in Even Dimensions D=2,4,6,…

- The skew matrix is anti-symmetric $\boldsymbol{J}^{\mathrm{T}} = -\boldsymbol{J}$ and such that $\boldsymbol{J}^2 = -\boldsymbol{1}_{\mathrm{D}}$.

- The Symplectic Group $\mathcal{Sp}$(D) is composed of the symplectic matrices $\boldsymbol{S}$ such that :

$$\boldsymbol{S}^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{S} = \boldsymbol{J}$$

  In other words (*and by analogy with orthogonal matrices*), they leaves the anti-symmetric quadratic form $\boldsymbol{X}^{\mathrm{T}} \cdot \boldsymbol{J} \cdot \boldsymbol{X}$ invariant. N.B: $\boldsymbol{S}^{\mathrm{T}} = -\boldsymbol{J}\,\boldsymbol{S}^{-1}\boldsymbol{J}$

- The matrix $\boldsymbol{J}$ itself is symplectic: $\boldsymbol{J} \in \mathcal{Sp}(\mathrm{D})$.

- If $\boldsymbol{S} \in \mathcal{Sp}(\mathrm{D})$, then $\boldsymbol{S}^{\mathrm{T}} \in \mathcal{Sp}(\mathrm{D})$

- If $\boldsymbol{S} \in \mathcal{Sp}(\mathrm{D})$, then $\det(\boldsymbol{S}) = 1$; *in other words* $\mathcal{Sp}(\mathrm{D}) \subset \mathcal{Sl}(\mathrm{D})$.

- The group $\mathcal{Sp}$(D) is generated by the matrices $\exp(\boldsymbol{J}\,\boldsymbol{T})$ such that $\boldsymbol{T}$ is a symmetric matrix $\Rightarrow \dim \mathcal{Sp}(\mathrm{D}) = \mathrm{D}(\mathrm{D}+1)/2$.

- In 2 dimensions: $\mathcal{Sp}(2) \equiv \mathcal{Sl}(2)$.
  Indeed, for any regular 2D matrix: $\boldsymbol{M}^{-1} = \boldsymbol{J}^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}}\boldsymbol{J}/\det(\boldsymbol{M})$

Symplectic Maps

1[st] property: Map Symplecticity

The transformation $\boldsymbol{X}(\boldsymbol{X}_0)$ which maps $\boldsymbol{X}_0$ coordinates at t_0 into $\boldsymbol{X}$ coordinates at t, is symplectic in the sense that its Jacobian matrix is a symplectic matrix.

Proof:

Let us note the Jacobian matrix of the map $\boldsymbol{J}_M(t) = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{X}_0}(t)$.

Since $\boldsymbol{J}_M(t_0) = \boldsymbol{I}_6$ is a symplectic matrix, we need to show that

$$\frac{d}{dt}(\boldsymbol{J}_M^{\mathrm{T}} \boldsymbol{J}\, \boldsymbol{J}_M) = 0$$

$$\frac{d}{dt}\boldsymbol{J}_M = \frac{\partial}{\partial \boldsymbol{X}_0}\frac{d}{dt}\boldsymbol{X} = \frac{\partial}{\partial \boldsymbol{X}_0}\boldsymbol{J}\,\partial_{\boldsymbol{X}} H = \boldsymbol{J}\,\partial_{\boldsymbol{X}}^2 H\,\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{X}_0} = \boldsymbol{J}\,\partial_{\boldsymbol{X}}^2 H\,\boldsymbol{J}_M$$

$$\frac{d}{dt}(\boldsymbol{J}_M^{\mathrm{T}} \boldsymbol{J}\, \boldsymbol{J}_M) = \boldsymbol{J}_M^{\mathrm{T}}\,\partial_{\boldsymbol{X}}^2 H^{\mathrm{T}}\,\boldsymbol{J}^{\mathrm{T}} \boldsymbol{J}\,\boldsymbol{J}_M + \boldsymbol{J}_M^{\mathrm{T}} \boldsymbol{J}^2 \partial_{\boldsymbol{X}}^2 H\,\boldsymbol{J}_M$$

$$= \boldsymbol{J}_M^{\mathrm{T}}\,\partial_{\boldsymbol{X}}^2 H^{\mathrm{T}}\boldsymbol{J}_M - \boldsymbol{J}_M^{\mathrm{T}}\partial_{\boldsymbol{X}}^2 H\,\boldsymbol{J}_M$$

$$= 0$$

since $\partial_{\boldsymbol{X}}^2 H$ is symmetric

Liouville Theorem

2nd property: Phase Space Volumes are invariant of Motion

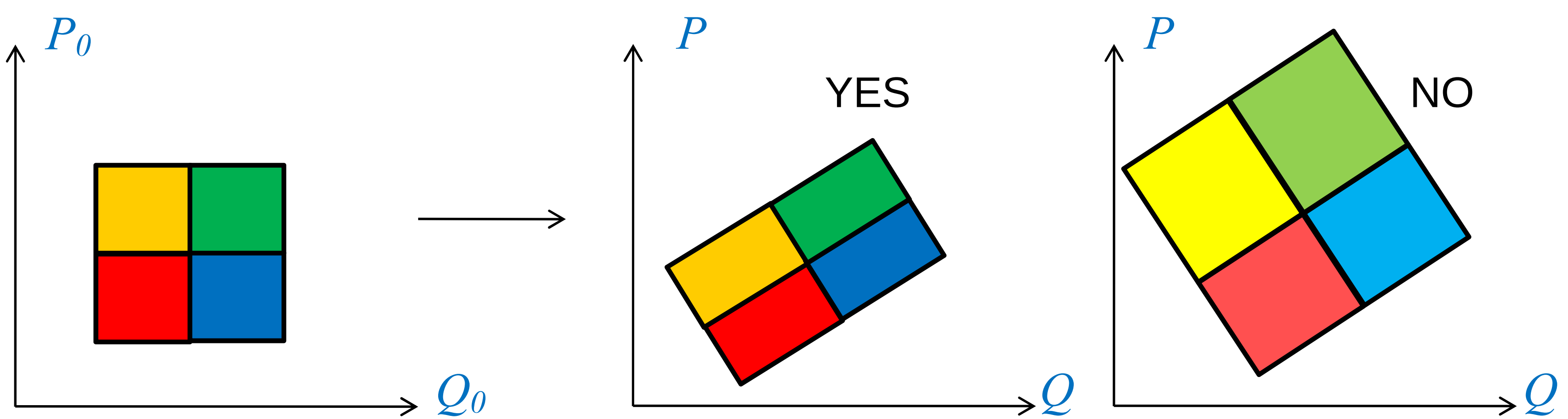
$$\int_{\text{Population}} d^6\boldsymbol{X} = \int_{\text{Population}} \left|\det \boldsymbol{J}_M(t)\right| d^6\boldsymbol{X}_0 = \int_{\text{Population}} d^6\boldsymbol{X}_0$$

3rd property: Particle Densities are invariant of Motion

Given a particle density distribution $\rho(\boldsymbol{X})$ such that $dN = \rho(\boldsymbol{X})d^6\boldsymbol{X}$, then following the motion of the dN particles from t_0 to t:

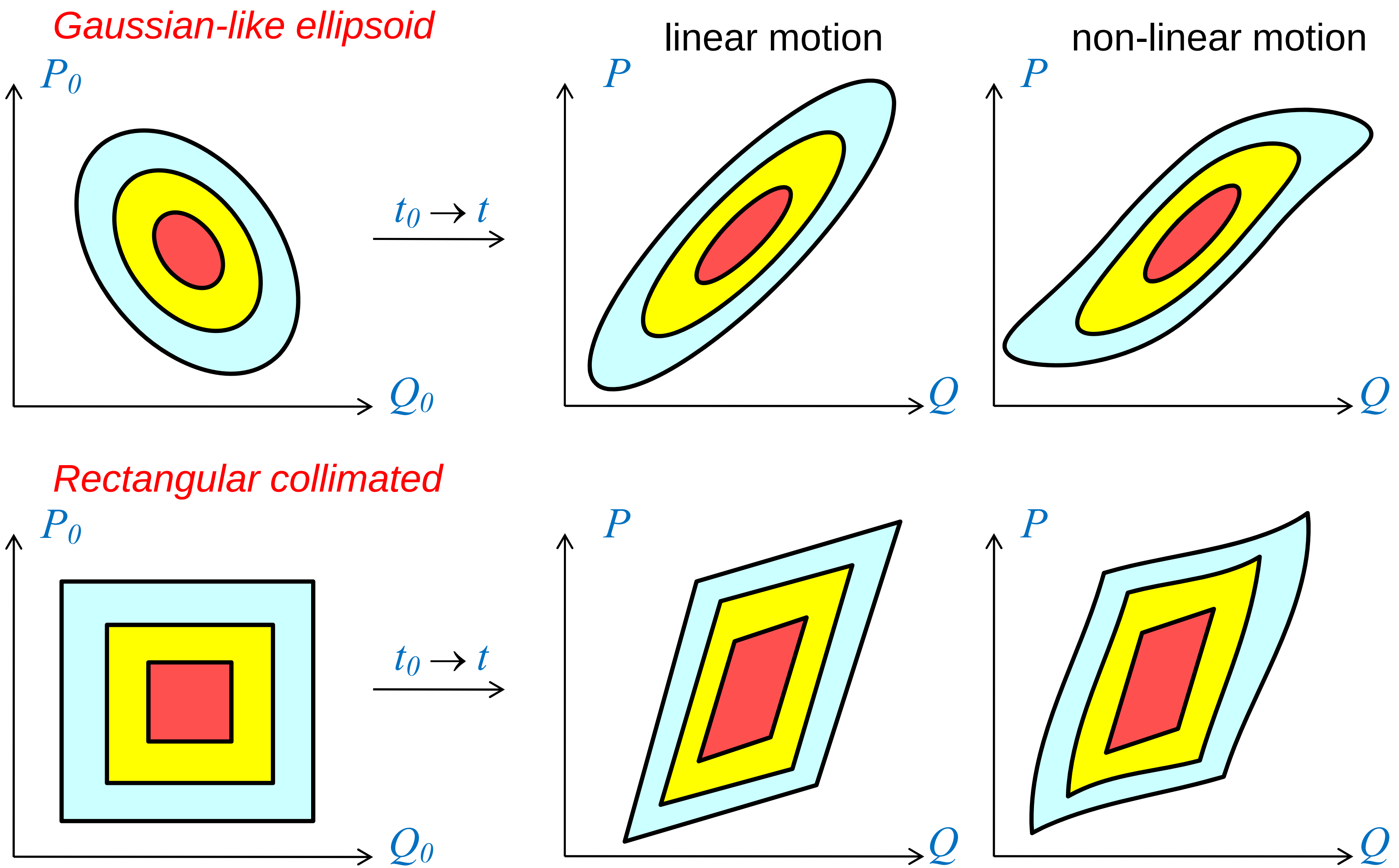
$$dN(t) = dN(t_0) \Rightarrow \rho(\boldsymbol{X})d^6\boldsymbol{X} = \rho(\boldsymbol{X}_0)d^6\boldsymbol{X}_0$$

$$\Rightarrow \rho(\boldsymbol{X}) = \rho(\boldsymbol{X}_0)$$



Liouville Theorem

Global Phase Space Volumes ('areas') are invariant of Motion:



The Particle Density and its Moments

The beam is modelled by a particle density distribution $\rho(\boldsymbol{X})$ which generates a set of characteristic real numbers, its Moments.

The first three moments

0th order moment: beam charge (or population):

$$\int d^6\boldsymbol{X}\ \rho(\boldsymbol{X}) = N$$

1st order moment: centre of mass (or 6-vector beam centroid):

$$\boldsymbol{C} = \frac{1}{N}\int d^6\boldsymbol{X}\ \rho(\boldsymbol{X})\boldsymbol{X} = \langle \boldsymbol{X} \rangle$$

2nd order moment: beam matrix (6x6 matrix):

$$\boldsymbol{\Sigma} \equiv \left\langle (\boldsymbol{X}-\boldsymbol{C})\cdot(\boldsymbol{X}-\boldsymbol{C})^{\mathrm{T}} \right\rangle = \left\langle \boldsymbol{X}\cdot\boldsymbol{X}^{\mathrm{T}} \right\rangle - \boldsymbol{C}\cdot\boldsymbol{C}^{\mathrm{T}}$$

$$= \frac{1}{N}\int d^6\boldsymbol{X}\ \rho(\boldsymbol{X})(\boldsymbol{X}-\boldsymbol{C})\cdot(\boldsymbol{X}-\boldsymbol{C})^{\mathrm{T}}$$

are widely used to characterize the beam transport.

The Gaussian Particle Density

• The Gaussian density distribution, in even D dimensions

$$\rho_{\mathrm{G}}(\boldsymbol{X}) = \frac{N}{\sqrt{(2\pi)^{\mathrm{D}} \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2} (\boldsymbol{X} - \boldsymbol{C})^{\mathrm{T}} \cdot \boldsymbol{\Sigma}^{-1} \cdot (\boldsymbol{X} - \boldsymbol{C}) \right)$$

is fully characterized by its 3 first moments $N, \boldsymbol{C}, \boldsymbol{\Sigma}$.

• It simplifies to the '*usual*' Gaussian distribution functions, assuming for instance

$$\boldsymbol{C} = 0 \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_Q^2 & 0 \\ 0 & \sigma_P^2 \end{pmatrix}$$ in D=2 dimensions

then

$$\rho_{\mathrm{G}}(Q, P) = \frac{N}{2\pi\sigma_Q \sigma_P} \exp\left(-\frac{Q^2}{2\sigma_Q^2} - \frac{P^2}{2\sigma_Q^2} \right)$$

• It is one of the few typical beam distribution used in the Beam Tracking simulation codes which probe the beam transport properties.

Other distributions widely used: Uniform ('Water-bag'), Ellipsoid surface (K-V), …

Some Algebra about the Beam Matrix

1st property: $\boldsymbol{\Sigma}$ is a symmetric definite-positive matrix

$$\boldsymbol{\Sigma} \equiv \left\langle (\boldsymbol{X}-\boldsymbol{C})\cdot(\boldsymbol{X}-\boldsymbol{C})^{\mathrm{T}} \right\rangle$$

$$\Rightarrow \boldsymbol{\Sigma}^{\mathrm{T}} = \boldsymbol{\Sigma} \quad \text{and} \quad \boldsymbol{X}_0^{\mathrm{T}}\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{X}_0 = \left\langle \left(\boldsymbol{X}_0^{\mathrm{T}}\cdot(\boldsymbol{X}-\boldsymbol{C})\right)^2 \right\rangle \geq 0$$

2nd property: $\boldsymbol{\Sigma}$ is strictly positive (except for the point-like distribution) and can be inverted.

3rd property: Normal form of $\boldsymbol{\Sigma}$:

$$\boldsymbol{\Sigma} = \boldsymbol{S}\,\boldsymbol{E}\,\boldsymbol{S}^{\mathrm{T}} \quad \text{with} \quad \boldsymbol{S} \text{ symplectic} \quad ,$$

$$\boldsymbol{E} = \begin{pmatrix} \boldsymbol{\varepsilon} & \boldsymbol{0}_3 \\ \boldsymbol{0}_3 & \boldsymbol{\varepsilon} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}, \quad \varepsilon_i > 0$$

Some Algebra about the Beam Matrix

3rd property: Normal form of $\boldsymbol{\Sigma}$

Proof:

Introducing the Square Root Matrix $\boldsymbol{\Sigma}^{1/2}$ such that $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{1/2}$, and the anti-symmetric matrix $\boldsymbol{A} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{J}\,\boldsymbol{\Sigma}^{1/2}$, one can find an orthogonal matrix O such that

$$\boldsymbol{A} = \boldsymbol{O}^{\mathrm{T}}\widetilde{\boldsymbol{A}}\,\boldsymbol{O} \quad \text{with} \quad \widetilde{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{0}_3 & \boldsymbol{\varepsilon} \\ -\boldsymbol{\varepsilon} & \boldsymbol{0}_3 \end{pmatrix}$$

(*kind of Jordan normal form for anti-symmetric matrices*)

Then, introducing $\boldsymbol{S} = \boldsymbol{E}^{1/2}\boldsymbol{O}\,\boldsymbol{\Sigma}^{-1/2}$ with $\boldsymbol{E} = \begin{pmatrix} \boldsymbol{\varepsilon} & \boldsymbol{0}_3 \\ \boldsymbol{0}_3 & \boldsymbol{\varepsilon} \end{pmatrix}$

one can show that

1) $\boldsymbol{S}\,\boldsymbol{\Sigma}\,\boldsymbol{S}^{\mathrm{T}} = \boldsymbol{E}^{1/2}\boldsymbol{O}\,\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}\,\boldsymbol{\Sigma}^{-1/2}\boldsymbol{O}^{\mathrm{T}}\boldsymbol{E}^{1/2} = \boldsymbol{E}$

2) $\boldsymbol{S}\,\boldsymbol{J}\,\boldsymbol{S}^{\mathrm{T}} = \boldsymbol{E}^{1/2}\boldsymbol{O}\,\boldsymbol{\Sigma}^{-1/2}\boldsymbol{J}\,\boldsymbol{\Sigma}^{-1/2}\boldsymbol{O}^{\mathrm{T}}\boldsymbol{E}^{1/2}$

$$= -\boldsymbol{E}^{1/2}\boldsymbol{O}\,\boldsymbol{A}^{-1}\boldsymbol{O}^{\mathrm{T}}\boldsymbol{E}^{1/2}$$

$$= -\boldsymbol{E}^{1/2}\widetilde{\boldsymbol{A}}^{-1}\boldsymbol{E}^{1/2} = \boldsymbol{J}$$

6D generalized
Twiss parameters
Phase Advance
Emittance

Linearized Motion

- For small deviations about the accelerator reference trajectory $\boldsymbol{X}_{\text{ref}}(t)$, the transformation map $\boldsymbol{X}(\boldsymbol{X}_0)$ can be linearized into:

$$\boldsymbol{X} = \boldsymbol{X}_{\text{ref}}(t) + \boldsymbol{R}(t,t_0)\cdot(\boldsymbol{X}_0 - \boldsymbol{X}_{\text{ref}}(t_0)) + O((\boldsymbol{X}_0 - \boldsymbol{X}_{\text{ref}}(t_0))^2)$$

with

$$\boldsymbol{R}(t,t_0) = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{X}_0}(\boldsymbol{X}_{\text{ref}}(t_0))$$ a symplectic matrix

- At the first order, the beam centre and beam matrix transform as:

$$\boldsymbol{C} = \boldsymbol{X}_{\text{ref}}(t) + \boldsymbol{R}(t,t_0)\cdot(\boldsymbol{C}_0 - \boldsymbol{X}_{\text{ref}}(t_0))$$

$$\boldsymbol{\Sigma} = \boldsymbol{R}(t,t_0)\,\boldsymbol{\Sigma}_0\,\boldsymbol{R}^{\mathrm{T}}(t,t_0)$$

Intrinsic Emittances

- The 3 quantities $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are called the *Intrinsic Emittances* : they are invariant of the motion, in the linearized motion approximation.

- To calculate these intrinsic emittances, one uses the 2 properties:

$$\det(\boldsymbol{\Sigma}) = (\varepsilon_1 \varepsilon_2 \varepsilon_3)^2$$

$$\boldsymbol{\Sigma}\,\boldsymbol{J} = \boldsymbol{S}(\boldsymbol{E}\,\boldsymbol{J})\boldsymbol{S}^{-1} \Rightarrow \mathrm{tr}(\boldsymbol{\Sigma}\,\boldsymbol{J})^{2n} = \mathrm{tr}(\boldsymbol{E}\,\boldsymbol{J})^{2n} = 2(-1)^n\,\mathrm{tr}(\boldsymbol{\varepsilon}^{2n})$$

D=6 dimensions: ???

D=4 dimensions:

$$\det(\boldsymbol{\Sigma}) = (\varepsilon_1 \varepsilon_2)^2$$

$$\mathrm{tr}(\boldsymbol{\Sigma}\,\boldsymbol{J})^2 = -2(\varepsilon_1^2 + \varepsilon_2^2)$$

$$\Rightarrow \begin{cases} \varepsilon_1 = \dfrac{1}{2}\sqrt{-\mathrm{tr}(\boldsymbol{\Sigma}\,\boldsymbol{J})^2 + \sqrt{\mathrm{tr}^2(\boldsymbol{\Sigma}\,\boldsymbol{J})^2 - 16\det(\boldsymbol{\Sigma})}} \\ \varepsilon_2 = \dfrac{1}{2}\sqrt{-\mathrm{tr}(\boldsymbol{\Sigma}\,\boldsymbol{J})^2 - \sqrt{\mathrm{tr}^2(\boldsymbol{\Sigma}\,\boldsymbol{J})^2 - 16\det(\boldsymbol{\Sigma})}} \end{cases}$$

D=2 dimensions:

$$\boxed{\varepsilon = \sqrt{\det(\boldsymbol{\Sigma})} = \frac{A}{\pi}}$$

where A is the area of the 2D ellipse $\boldsymbol{X}^{\mathrm{T}} \cdot \boldsymbol{\Sigma}^{-1} \cdot \boldsymbol{X} = 1$

Switching Basis

We (will) switch to a new coordinates basis

$$\boldsymbol{X} = \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix} \rightarrow \boldsymbol{X} = \begin{pmatrix} Q_1 \\ P_1 \\ Q_2 \\ P_2 \\ Q_3 \\ P_3 \end{pmatrix}$$

in which $\boldsymbol{J} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$ is the skew matrix,

and the normal form of the beam matrix is $\boldsymbol{E} = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_3 \end{pmatrix}$

Projected Emittances

- **Case of Uncoupled Beam Matrix**

In some ideal locations along perfect beam lines, the beam matrix is uncoupled in the 3 physical dimensions $(x,y\ z)$:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_x & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}_y & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\Sigma}_z \end{pmatrix}$$

Then, the 3 emittances are given by

$$\varepsilon_x = \sqrt{\det(\boldsymbol{\Sigma}_x)},\ \varepsilon_y = \sqrt{\det(\boldsymbol{\Sigma}_y)},\ \varepsilon_z = \sqrt{\det(\boldsymbol{\Sigma}_z)}$$

- **Case of 4D Transverse Coupled Matrix**

In reality, either by concept or by the effect of misalignment errors, the beam matrix is x-y coupled

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_x & \boldsymbol{\Sigma}_{xy} \\ \boldsymbol{\Sigma}_{xy}^{\mathrm{T}} & \boldsymbol{\Sigma}_y \end{pmatrix} .$$

Accelerator instrumentation, usually built in horizontal and vertical frames, allows to measure the projected emittances

$$\tilde{\varepsilon}_x = \sqrt{\det(\boldsymbol{\Sigma}_x)},\ \tilde{\varepsilon}_y = \sqrt{\det(\boldsymbol{\Sigma}_y)}$$

which are not invariant of motion, and such that $\tilde{\varepsilon}_{x,y} > \varepsilon_{x,y}$

Beam Line Coordinate System

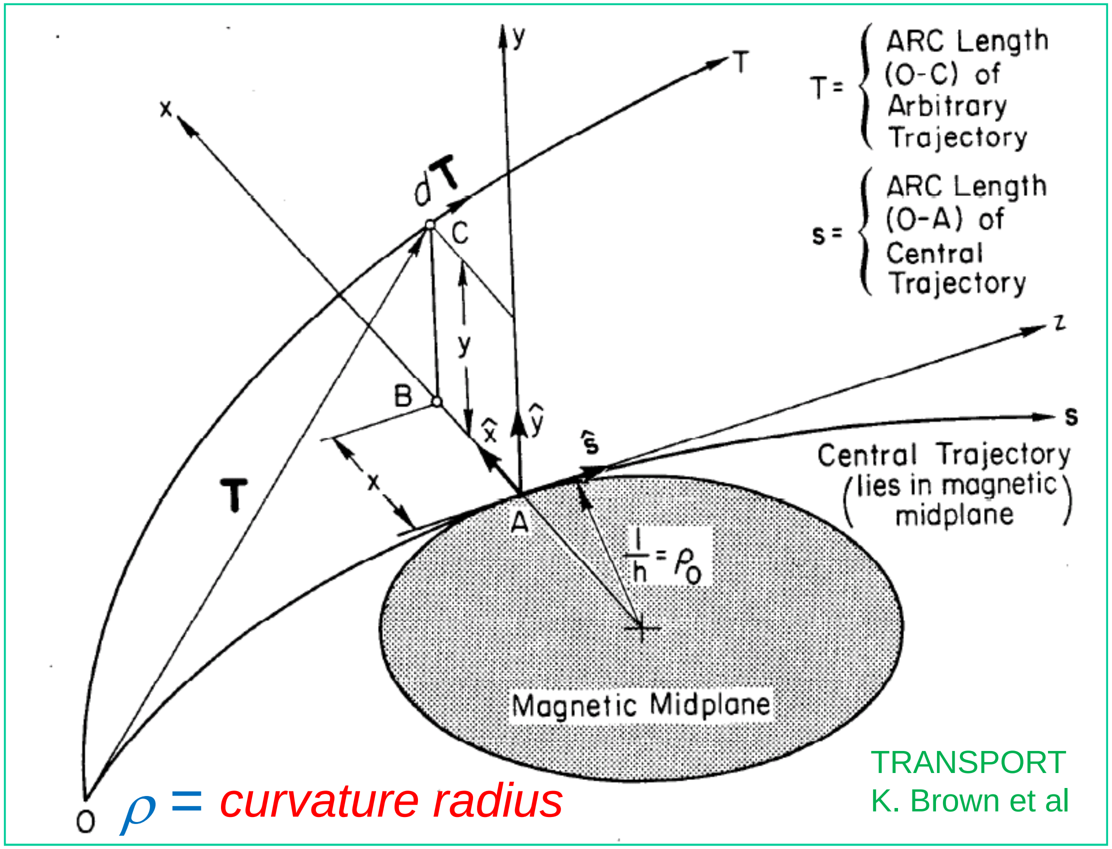
Particle trajectories are parameterized with respect to a Reference Trajectory $\vec{R}_{\text{ref}}(t)$ associated to a Reference Momentum $\vec{P}_{\text{ref}}(t) = m\gamma\,\dot{\vec{R}}_{\text{ref}}(t)$.

One introduces the curvilinear coordinate system defined by the Moving Frame $(\hat{x}, \hat{y}, \hat{s})$ of *(planar)* Reference Trajectory, such that :

$$ds \equiv \left|\dot{\vec{R}}_{\text{ref}}\right| dt$$ *arc length*

$$\hat{s} = \frac{\dot{\vec{R}}_{\text{ref}}}{\left|\dot{\vec{R}}_{\text{ref}}\right|}$$ *tangent vector*

$$\Rightarrow \begin{cases} \dfrac{d\hat{s}}{ds} = -\dfrac{\hat{x}}{\rho} \\ \dfrac{d\hat{x}}{ds} = \dfrac{\hat{s}}{\rho} \end{cases}$$



Particle Motion Coordinates

Then, the particle motion is parameterized by $(x, y, \tau)(s)$ through

$$\begin{cases} \vec{R}(s) = \vec{R}_{\text{ref}}(s) + x(s)\,\hat{x} + y(s)\,\hat{y} \\ t(s) = (t_{\text{ref}}(s) - \tau(s)) \end{cases}$$

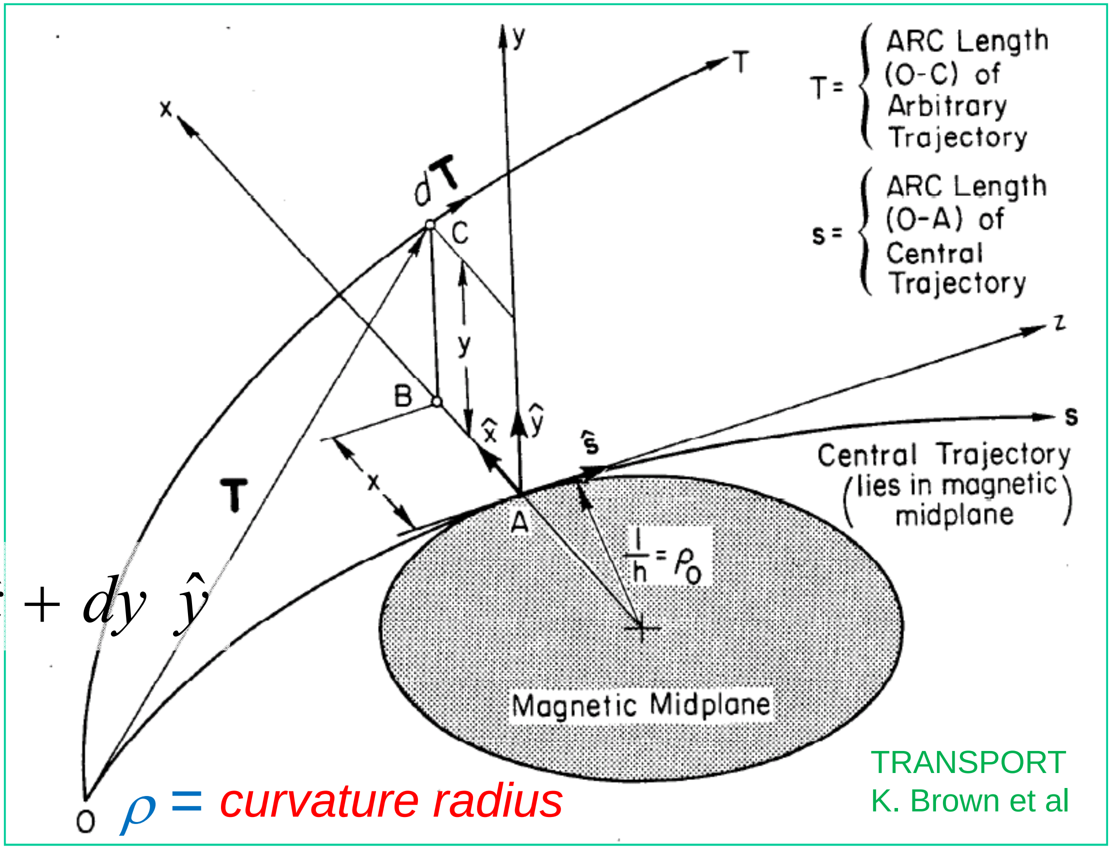
where $t(s)$, $\vec{R}(s)$ are the time and position of the particle when crossing the plane normal to the reference trajectory at $\vec{R}_{ref}(s)$.

Note 1: $\tau > 0$ for particles ahead of the reference particle.

Note 2: for a planar reference trajectory:

$$d\vec{R} = (1 + xh)ds\ \hat{s} + dx\ \hat{x} + dy\ \hat{y}$$

with $h \equiv \dfrac{1}{\rho}$



ρ = *curvature radius*

TRANSPORT
K. Brown et al

The Resulting Hamiltonian

To study the evolution of Particle Motion, Beam Physicists like to trade the Time variable t with the Arc Length variable s defined by :

$$s = \int_{t_0}^{t} \left| \dot{\vec{R}}_{\text{ref}}(u) \right| du$$

After some *Canonical* manipulations, this leads to a new Hamiltonian $G(\vec{Q}, \vec{P})$

$$G(\vec{Q},\vec{P}) = -(1 + hQ_x)(qA_s + \sqrt{(P_t - qV)^2 / c^2 - m^2c^2 - (\vec{P}_\perp - q\vec{A}_\perp)^2})$$

in terms of the conjugate variables:

$$\vec{Q} = \begin{pmatrix} x \\ y \\ \tau \end{pmatrix} \quad , \quad \vec{P} = \begin{pmatrix} p_x + qA_x \\ p_y + qA_y \\ H \end{pmatrix}$$

Hamiltonian mechanics and Symplecticity provide powerful tools to study beam dynamics over very long times, like the beam orbits in circular colliders.
They are of lesser importance for the design of linacs and transfer beam lines.

TRANSPORT Coordinate System

• For transfer lines, like BDS and linear accelerators, we turn to more familiar, but not symplectic conjugate variables, the TRANSPORT coordinates:

$$\boldsymbol{X} = \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{pmatrix} \quad \text{with} \quad \begin{cases} z = c\tau \\ x' = dx/ds = (1+xh)v_x/v_s \\ y' = dy/ds = v_y/v_s \\ \delta = (p - p_{\text{ref}})/p_{\text{ref}} \end{cases}$$

- $z = c\tau$ ← a time variable !
- x', y' ← transverse 'angles'
- δ ← relative momentum deviation

• These coordinates are to the first order proportional to symplectic conjugate variables:

$$\begin{cases} x' = v_x/v(1+o(x,y^2)) \approx p_x/\beta\gamma mc \\ y' = v_y/v(1+o(x^2,y^2)) \approx p_y/\beta\gamma mc \end{cases} \Rightarrow \begin{cases} \beta\gamma\, x' \propto p_x \\ \beta\gamma\, y' \propto p_y \end{cases}$$

Geometric and Normalized Emittances

- **Geometric emittances of uncoupled motion**

In TRANSPORT coordinates, the ‘geometric’ emittances calculated from the beam matrix

$$\varepsilon_x = \sqrt{\det(\boldsymbol{\Sigma}_x)},\ \varepsilon_y = \sqrt{\det(\boldsymbol{\Sigma}_y)} \quad \text{with} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_x & 0 \\ 0 & \boldsymbol{\Sigma}_y \end{pmatrix}$$

is not invariant under acceleration.

- **Normalized emittances**

Normalized emittance, invariant under linearized motion, are given by

$$\varepsilon_{N,x} = \beta\gamma\sqrt{\det(\boldsymbol{\Sigma}_x)},\ \varepsilon_{N,y} = \beta\gamma\sqrt{\det(\boldsymbol{\Sigma}_y)}$$

For high energy beam lines $\beta = 1$ and

$$\boxed{\varepsilon_{N,x} = \gamma\sqrt{\det(\boldsymbol{\Sigma}_x)} = \gamma\varepsilon_x,\ \varepsilon_{N,y} = \gamma\sqrt{\det(\boldsymbol{\Sigma}_y)} = \gamma\varepsilon_y}$$

The Beta Functions of the Beam (D=2)

- The 2D beam matrix is parameterized as follows:

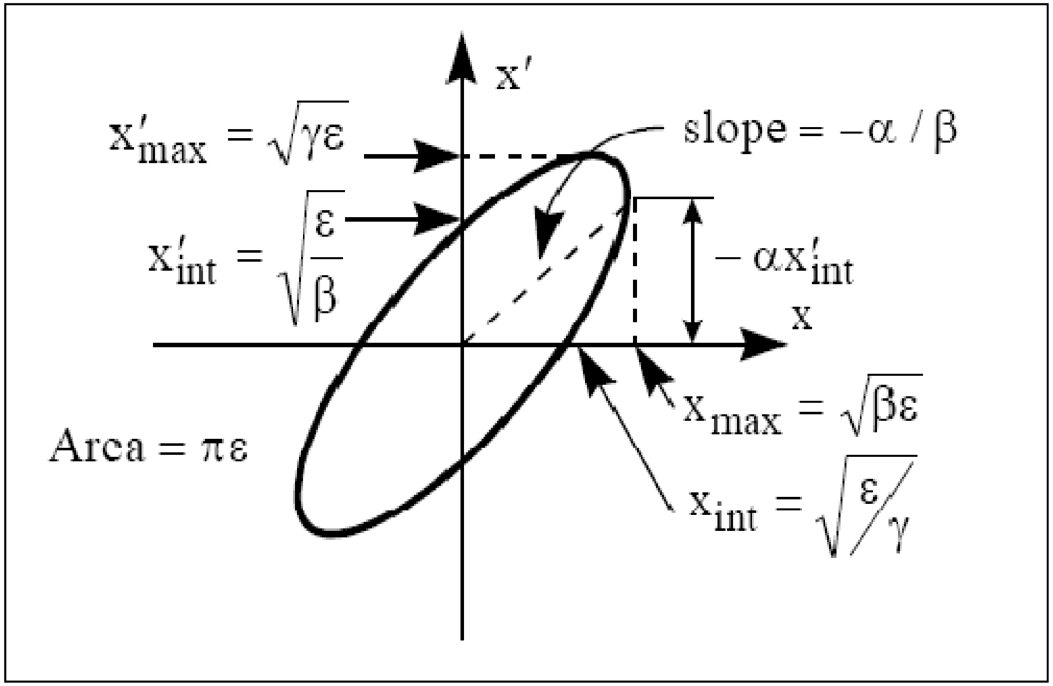
$$\boldsymbol{\Sigma}_x = \varepsilon_x \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} \quad \text{with} \quad \det\begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} = \beta_x\gamma_x - \alpha_x^2 = 1$$

By definition $\alpha_x = -\frac{1}{\varepsilon_x}\left(\langle xx'\rangle - \langle x\rangle\langle x'\rangle\right) = -\frac{1}{2\varepsilon_x}\frac{d}{ds}\left(\langle x^2\rangle - \langle x\rangle^2\right) = -\frac{\beta_x{}'}{2}$.

- The 2D beam ellipse $\boldsymbol{X}^{\mathrm{T}} \cdot \boldsymbol{\Sigma}_x^{-1} \cdot \boldsymbol{X} = 1$ can then be parametrized by

$$\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2 = \varepsilon_x$$

Courant-Snyder Invariant



- The remarkable points can be recovered from the above equation, and also from the search for extrema:

$$(\gamma_x x + \alpha_x x')dx + (\alpha_x x + \beta_x x')dx' = 0$$

e.g. $dx = 0 \Rightarrow (\alpha_x x_{\max} + \beta_x x') = 0$

$$\Rightarrow x_{\max}^2 = \beta_x \varepsilon_x$$

Normalized Variables (D=2)

• The 2D beam matrix takes the normal form:

$$\boldsymbol{\Sigma}_x = \boldsymbol{S}\begin{pmatrix} \varepsilon_x & 0 \\ 0 & \varepsilon_x \end{pmatrix}\boldsymbol{S}^{\mathrm{T}} = \varepsilon_x \boldsymbol{S}\,\boldsymbol{S}^{\mathrm{T}} \quad \text{with} \quad \boldsymbol{S} = \frac{1}{\sqrt{\beta_x}}\begin{pmatrix} \beta_x & 0 \\ -\alpha_x & 1 \end{pmatrix} \qquad (\det(S)=1)$$

This suggests introducing the Normalized Variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = \boldsymbol{S}^{-1}\cdot\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}$$

such that

$$\boldsymbol{\Sigma}_u = \left\langle \begin{pmatrix} u \\ v \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}^{\mathrm{T}} \right\rangle - \left\langle \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle^{\mathrm{T}} = \boldsymbol{S}^{-1}\boldsymbol{\Sigma}_x \boldsymbol{S}^{-1\,\mathrm{T}} = \varepsilon_x \boldsymbol{I}_2$$

• In these variables, the particle motion is described by simple rotations around a circle in (u,v) ‘phase space’ (*cf. Beam Optics section*).
But most of the information is retained in the β - functions.

The General Form of the Beam Matrix (D=6)

In Final Focus Systems, several simplifications are in force :

1. the longitudinal motion is frozen: the arc length differences between trajectories are negligible and the bunch length is constant.
2. the transverse (x,y) motions are not coupled.

The resulting 6D beam matrix is parametrized as follows:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \varepsilon_x\beta_x & -\varepsilon_x\alpha_x & 0 & 0 & 0 & \eta_x\sigma_\delta^2 \\ -\varepsilon_x\alpha_x & \varepsilon_x\gamma_x & 0 & 0 & 0 & \eta_x'\sigma_\delta^2 \\ 0 & 0 & \varepsilon_y\beta_y & -\varepsilon_y\alpha_y & 0 & \eta_y\sigma_\delta^2 \\ 0 & 0 & -\varepsilon_y\alpha_y & \varepsilon_y\gamma_y & 0 & \eta_y'\sigma_\delta^2 \\ 0 & 0 & 0 & 0 & \sigma_z^2 & 0 \\ \eta_x\sigma_\delta^2 & \eta_x'\sigma_\delta^2 & \eta_y\sigma_\delta^2 & \eta_y'\sigma_\delta^2 & 0 & \sigma_\delta^2 \end{pmatrix}$$

with the dispersion coefficients $(\eta_x, \eta_x', \eta_y, \eta_y')$ measure the correlations between transverse position-angle and energy, e.g.

$$r_{16} \equiv \frac{\langle x\delta \rangle}{\sqrt{\sigma_x^2\sigma_\delta^2}} = \eta_x \frac{\sigma_\delta}{\sigma_x} \;, \quad r_{26} \equiv \frac{\langle x'\delta \rangle}{\sqrt{\sigma_{x'}^2\sigma_\delta^2}} = \eta_x' \frac{\sigma_\delta}{\sigma_{x'}} \quad .$$

Sources of Emittance Growth

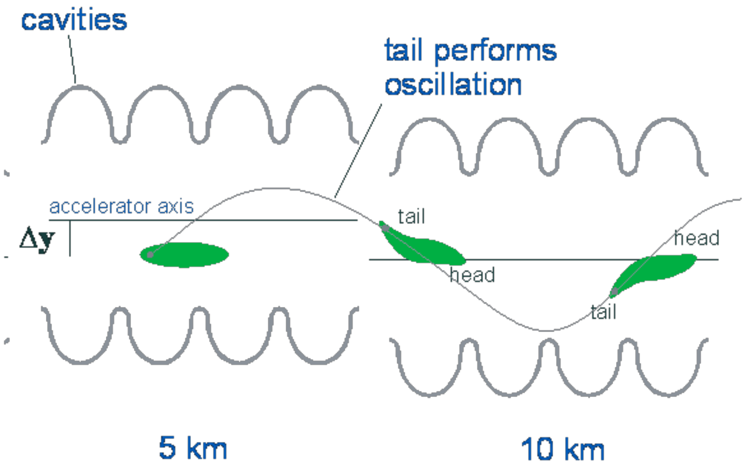
Linear Collider beams are experiencing many sources of emittance growth (or emittance degradation), in particular through the Beam Delivery Systems :

- Non Hamiltonian Mechanics :
  - ➢ Synchrotron Radiation (SR) : an interesting case because it works both sides:
    1. at **low energy**, big injected emittances (transverse and longitudinal) are damped (i.e. reduced) down to the equilibrium emittances of Damping Rings;
    2. at **high energy**, the longitudinal emittance growth dominates due to the strong energy dependence of SR, and couples to the transverse motion through magnet chromaticity.
  - ➢ Collective Effects, e.g. beam-beam interactions

- Non-linear motion
  - ✓ a source of the Linear Emittance growth
  - ✓ but does not violate Liouville Theorem

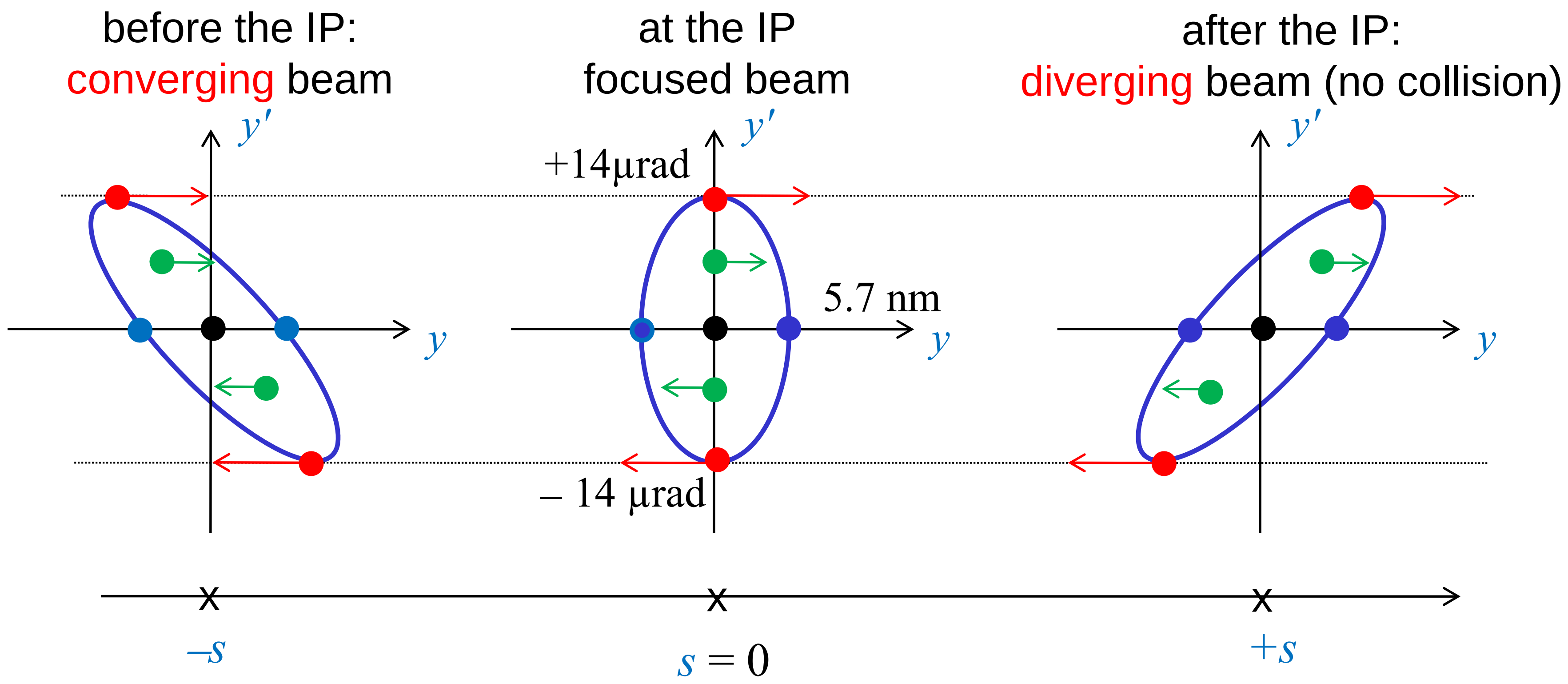
- Coupling
  - ✓ a source of 4D projected emittance growth
  - ✓ but not of 4D intrinsic emittance growth

- Transverse Wakefields
  - ✓ a source of 4D projected emittance growth
  - ✓ but not of 4D intrinsic emittance growth.



Kinematics at the Interaction Point

We assume a perfect beam at the IP, without any coupling, and we investigate the kinematics of the beam in the vertical plane at the 5.7 nm focus.

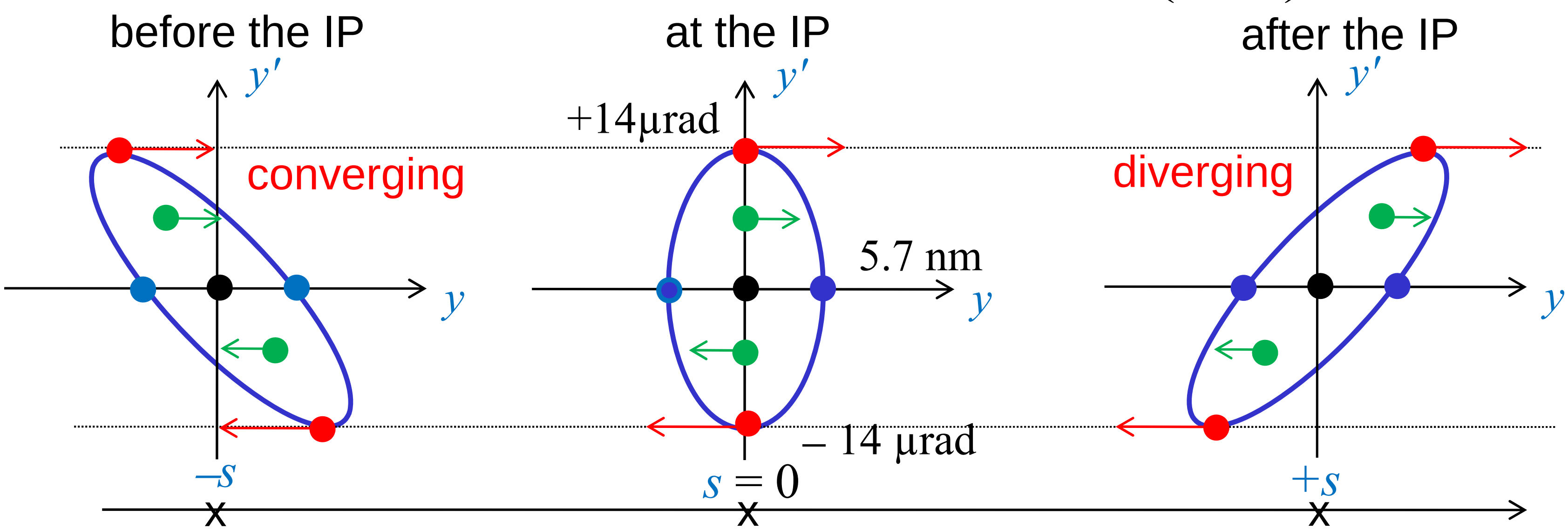


Linear motion
In a Drift space:

$$\begin{cases} y(s) = y_0 + y'_0\, s \\ y'(s) = y'_0 \end{cases} \Leftrightarrow \begin{pmatrix} y \\ y' \end{pmatrix}(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}(s=0)$$

Beta Functions at the Interaction Point

Introducing the transfer matrix of the Drift Space $D(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$



we can calculate the Beam Matrix at and after the IP, using $\varepsilon_y = \sigma_y \sigma_{y'} = 82\ \text{fm}$

$$\Sigma^* \equiv \Sigma(s=0) = \begin{pmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \beta_y^* = \sigma_y^2 / \varepsilon_y = 400\,\mu\text{m} \\ \alpha_y^* = 0 \ ; \ \gamma_y^* = 1/\beta_y^* \end{cases}$$

$$\Sigma(s) = D(s) \begin{pmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_{y'}^2 \end{pmatrix} D(s)^{\mathrm{T}} = \begin{pmatrix} \sigma_y^2 + s^2 \sigma_{y'}^2 & s\sigma_{y'}^2 \\ s\sigma_{y'}^2 & \sigma_{y'}^2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \beta_y(s) = \beta_y^* + s^2 / \beta_y^* \approx s^2 / \beta_y^* \\ \alpha_y(s) = -s / \beta_y^* \ ; \ \gamma_y(s) = \gamma_y^* \end{cases}$$

Question n°2:

Do we need High Field Quadrupole Magnets ?

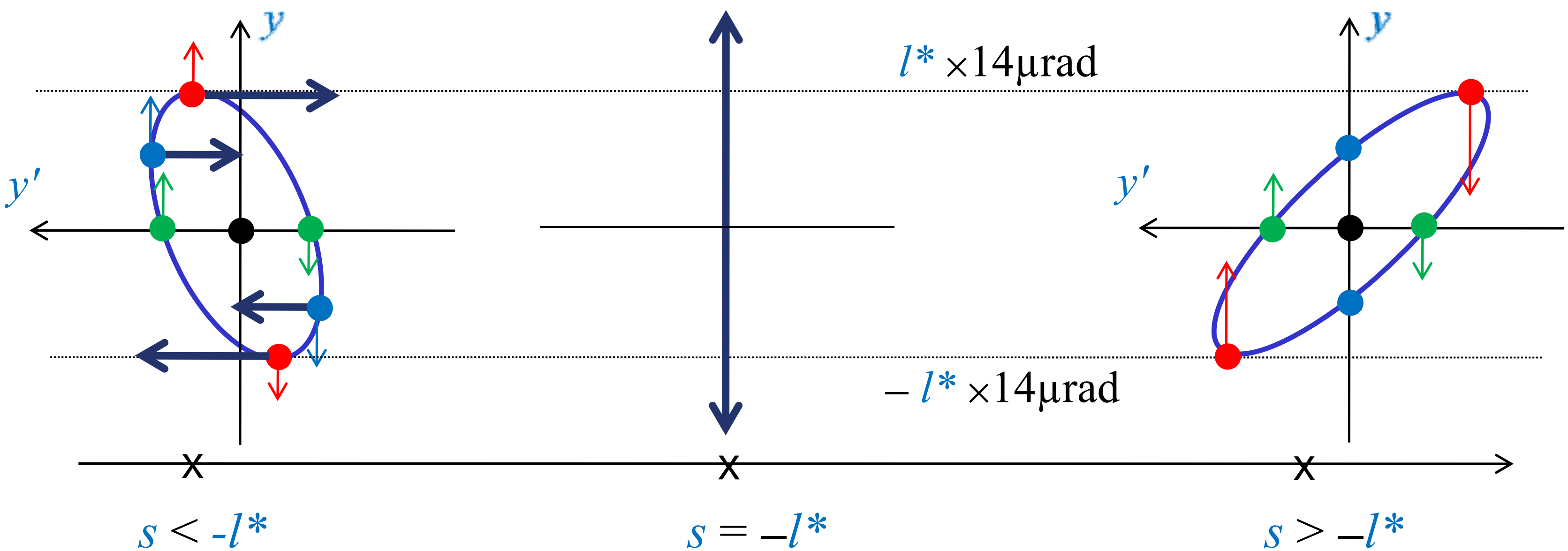
The Last Focusing Lens

The 14 µrad angular spread is the highest ever downstream of the Damping Rings. The ‘high’ beam convergence is created by a focusing lens, located at a distance l^* from the IP.

before the lens: diverging beam

focusing lens with focal length f

after the lens: converging beam



Linear motion in a focusing Thin Lens:

$$\begin{cases} y = y_0 \\ y' = y'_0 - y_0/f \end{cases} \Leftrightarrow \begin{pmatrix} y \\ y' \end{pmatrix}(s=-l^{*+}) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix}(s=-l^{*-})$$

The Final Lens: A Magnetic Quadrupole

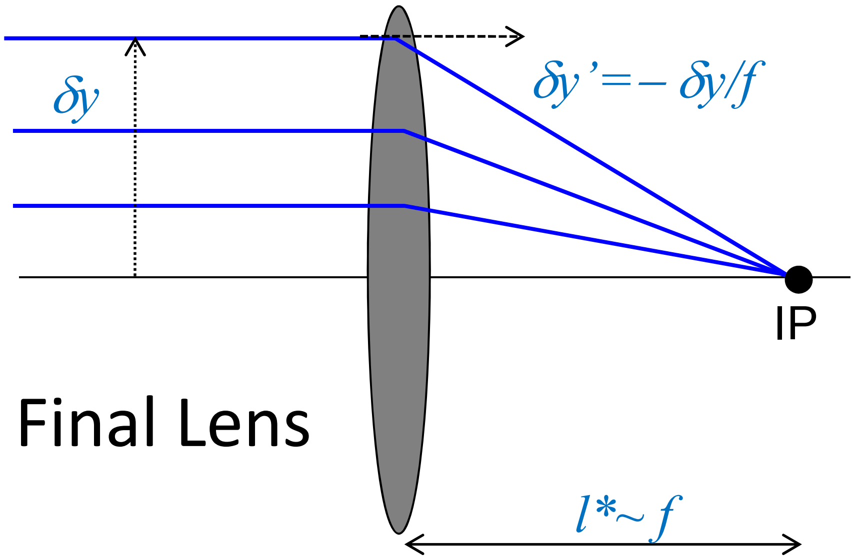
The angular kick $\delta y'$ created by the lens is proportional to the trajectory offset δy .
If it is generated by a magnet of length L:

$$BL = \delta y' \cdot B\rho = \frac{\delta y}{f} \cdot B\rho$$

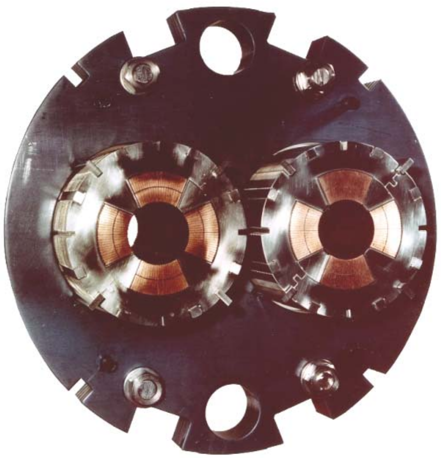
$$\Rightarrow GL \equiv \frac{B}{\delta y} L = \frac{B\rho}{f}$$

$$\Rightarrow GL = \frac{B\rho}{l^*} \approx 238\,\mathrm{T} \quad \text{for } l^* = 3.5\,\mathrm{m}$$

$$\Rightarrow G \approx 238\,\mathrm{T/m} \quad \text{for } L = 1\,\mathrm{m}$$



The LHC lattice quadrupoles are designed for $223\ \mathrm{T/m}$.
They are Superconducting Magnets (NbTi technology) with $56\ \mathrm{mm}$ bore diameter.



Long Quadrupole Magnet

Quadrupole magnets create gradient of bending field

$$B_y = Gx\,,B_x = Gy$$

From Maxwell's equation *in Vacuum*

$$(\nabla \wedge B)_z = \partial_x B_y - \partial_y B_x = 0$$

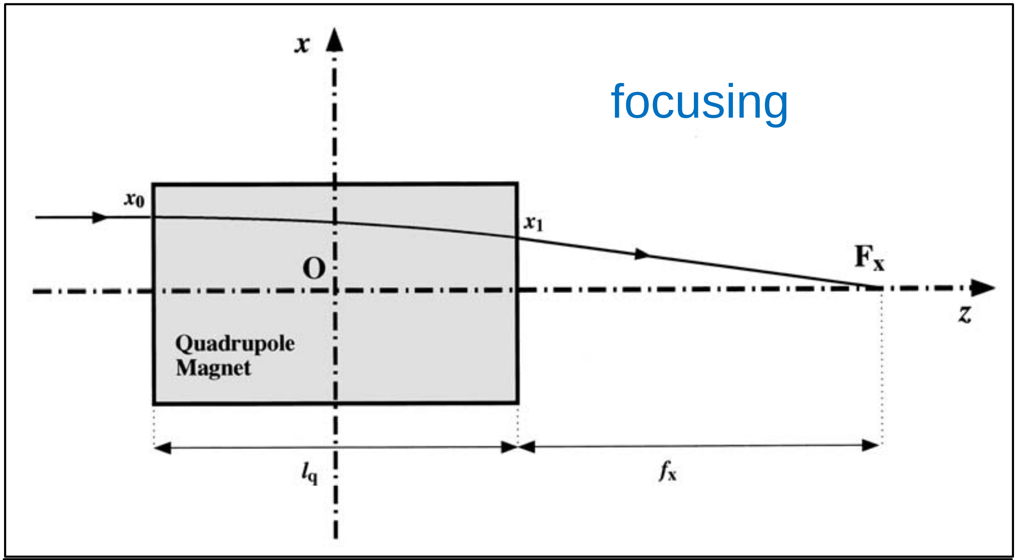
this gradient has the same value and same sign in both planes.

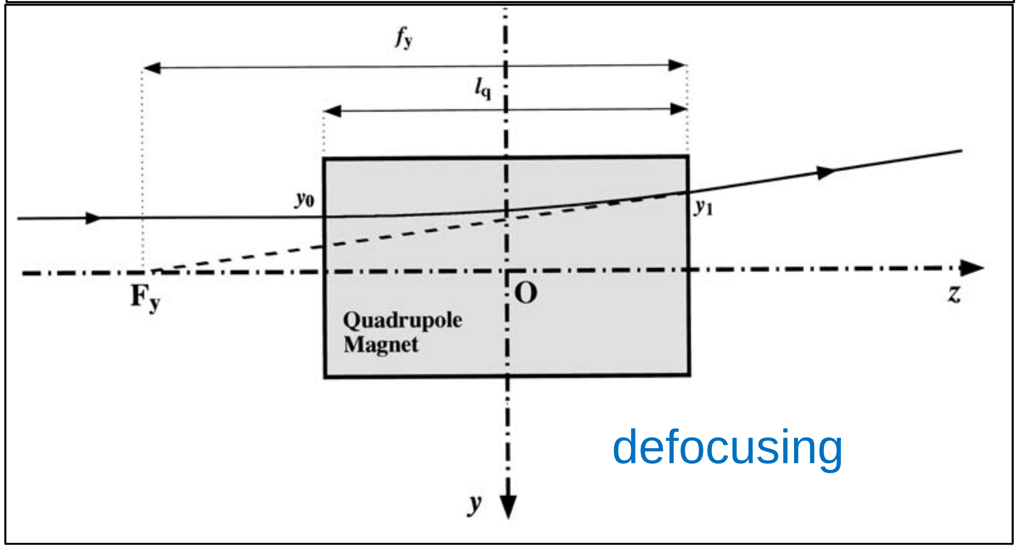
As a consequence, for $G > 0$, the Laplace force

$$\vec{F} = q\vec{v} \wedge \vec{B}$$

is focusing in the horizontal plane and defocusing in the vertical plane:

$$F_x = -qv_z Gx\,,F_y = +qv_z Gy$$



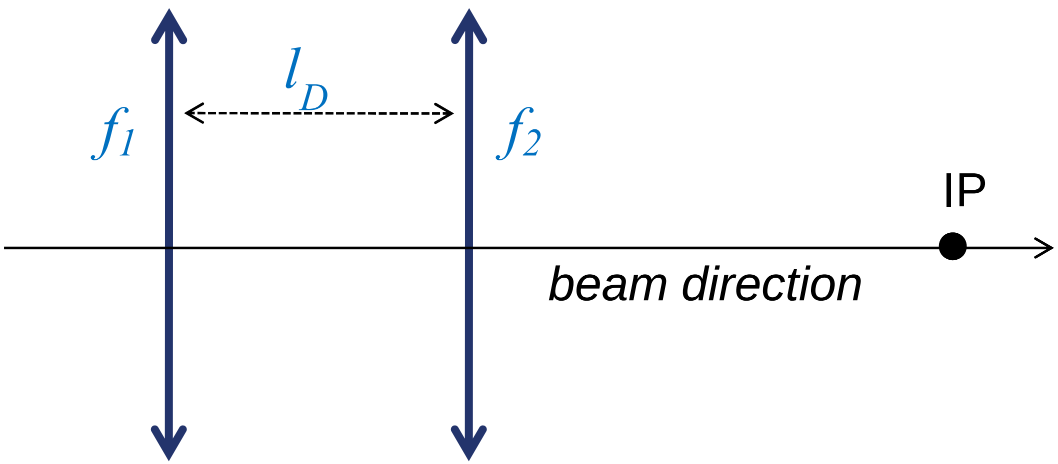


$\Rightarrow$ to focus in both planes, a quadrupole doublet, or triplet, is required.

Quadrupole Doublet: Thin Lens Approximation

A quadrupole doublet is potentially focusing the beam in the 2 planes.

As a first approach,
Short Quadrupoles can be treated like thin lenses.



The overall linear motion is given the following composition of elementary transfer matrices:

$$\begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & l_D \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} = \begin{pmatrix} 1-l_D/f_1 & l_D \\ (l_D - f_1 - f_2)/f_1 f_2 & 1-l_D/f_2 \end{pmatrix}$$

This transfer matrix is focusing in both planes if the 'R_{21}' terms are negative in both planes, i.e. for both signs of f_1 and f_2 .

$$\Rightarrow \begin{cases} f_1 f_2 < 0 \\ l_D > |f_1 + f_2| \end{cases}$$

Quadrupole Doublet: Long Quadrupoles

A long quadrupoles has the following transfer matrices (*cf. Beam Optics section*) :

$$F = \begin{pmatrix} \cos(\sqrt{K_1} l_Q) & \sin(\sqrt{K_1} l_Q)/\sqrt{K_1} \\ -\sqrt{K_1}\sin(\sqrt{K_1} l_Q) & \cos(\sqrt{K_1} l_Q) \end{pmatrix}$$

in its focusing plane

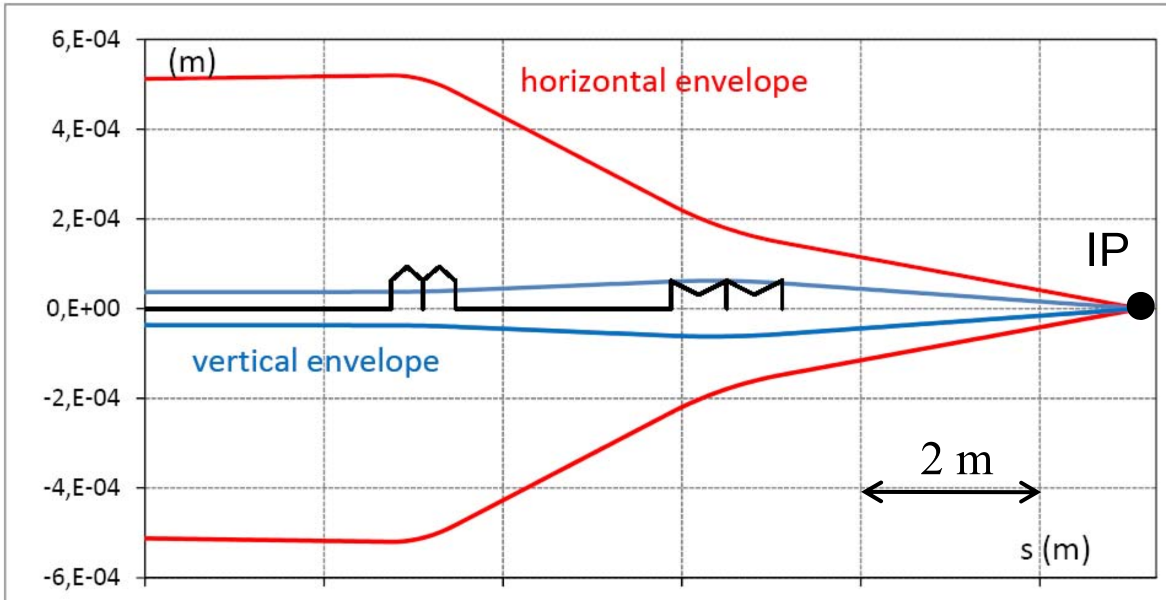
with $K_1 = G/B\rho$

$$D = \begin{pmatrix} \cosh(\sqrt{K_1} l_Q) & \sinh(\sqrt{K_1} l_Q)/\sqrt{K_1} \\ \sqrt{K_1}\sinh(\sqrt{K_1} l_Q) & \cosh(\sqrt{K_1} l_Q) \end{pmatrix}$$

in its defocusing plane

It can be treated as a Thin Lens in the limit $l_Q \to 0$ with $K_1 l_Q = 1/f$ constant.

The ILC Final Doublet focuses the beam with the RMS envelopes shown here.



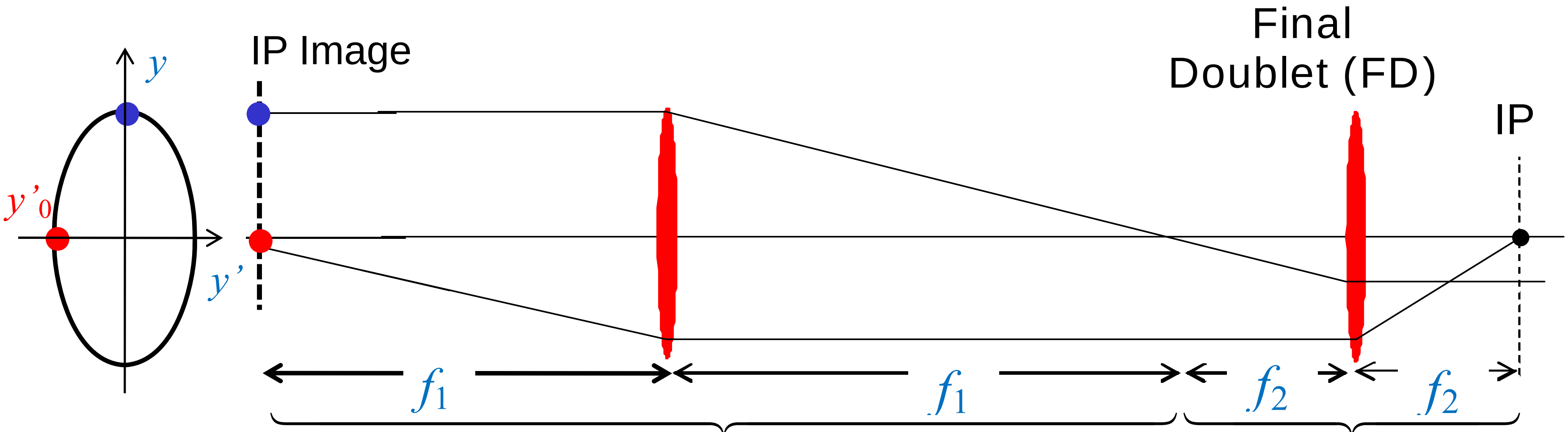
Nota Bene: final doublet quadrupoles are VERY far from being thin lenses.

The Final Focus System

The Final Focus System operates like an Optical Telescope:

• it uses 2 confocal lenses: a weak lens with a long f_1 focal distance, followed by a strong lens with a short f_2 focal distance;

• it realizes point-to-point and parallel-to-parallel imaging of the beam with a demagnification of the images by the factor $M = f_1/f_2 >> 1$



$$R_T = \begin{pmatrix} -\frac{f_2}{f_1} & 0 \\ 0 & -\frac{f_1}{f_2} \end{pmatrix} = \begin{pmatrix} 0 & f_1 \\ -\frac{1}{f_1} & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & f_2 \\ -\frac{1}{f_2} & 0 \end{pmatrix}$$

FFS Chromatic effect: the Momentum Bandwidth

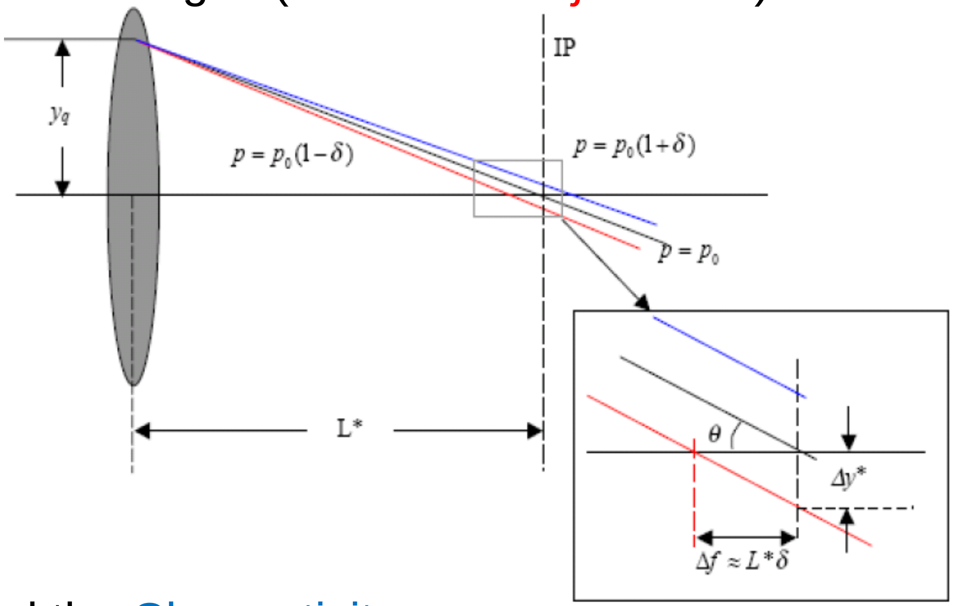
The Final Focus System generates chromatic aberrations of the IP beam image.

Introducing the momentum dependence $f(p) = f(p_0)\dfrac{p}{p_0} = f(1+\delta)$ of the focal distances one can show that:,

- the aberration $R_T(p)_{11}$ for the parallel-to-parallel images (~ *cosine-like trajectories*) are of order δ^2 ;
- the aberration $R_T(p)_{12}$ for the point-to-point images (~ *sine-like trajectories*) are given by:

$$\Delta y^* = y'_0 (f_1 + f_2)\delta \approx (f_1 y'_0)\delta$$

$$= -(f_2 y'^*)\delta \approx l^* y'^* \delta$$



On average, one gets:

$$\Delta\sigma^*_y = \left\langle \Delta y^{*2} \right\rangle^{1/2} = l^* \sigma^*_{y'} \sigma_\delta$$

$$\Rightarrow \quad \frac{\Delta\sigma^*_y}{\sigma^*_y} = l^* \frac{\sigma^*_{y'}}{\sigma^*_y}\sigma_\delta = \frac{l^*}{\beta^*_y}\sigma_\delta \quad .$$

The parameter $\xi_y = l^* / \beta^*_y \approx 10^4$ is called the Chromaticity.

It sets the Momentum Bandwitdh (acceptance) of the FFS:

$$\xi_y \sigma_\delta^{\max} << 1$$

which demonstrates the need for Chromaticity Correction.

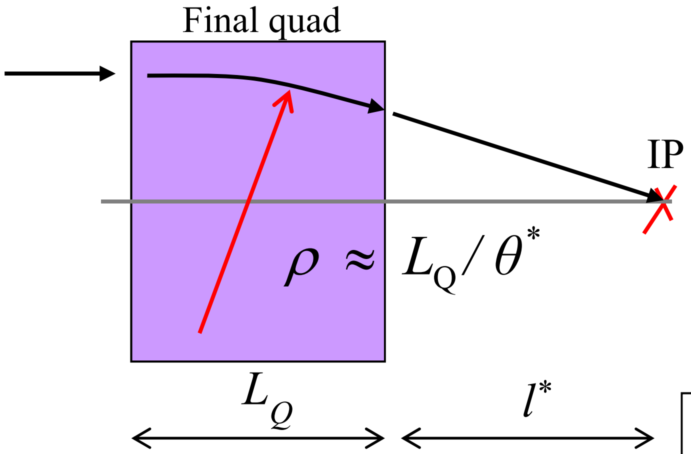
Synchrotron Radiation: *Oide Effect*

The energy spread generated by SR in the final quadrupole, given by

$$\left(\frac{\sigma_E}{E}\right)^2 \approx \frac{r_e\,\lambda_e\,\gamma^5\,L_Q}{\rho^3} \quad \text{with} \quad \rho\,\theta^* \approx L_Q$$

IP divergence & beamsize

$$\theta^* = \sqrt{\varepsilon/\beta^*}, \quad \sigma^* = \sqrt{\beta^*\varepsilon}$$



induces an irreducible growth of the IP beam size:

$$\sigma^2 \approx \sigma_0^2 + \left(l^*\,\theta^*\right)^2\,\left(\sigma_E/E\right)^2$$

$$\sigma^2 \approx \varepsilon\,\beta^* + C_1\,\left(l^*/L_Q\right)^2\,r_e\,\lambda_e\,\gamma^5\,\left(\varepsilon/\beta^*\right)^{5/2}$$

where C_1 is ~ 7 (depends on FD parameters).

The minimum beam size (*Oide limit*) :

$$\sigma_{\min} \approx 1.35\,C_1^{1/7}\,\left(l^*/L_Q\right)^{2/7}\left(r_e\,\lambda_e\right)^{1/7}\,\left(\gamma\varepsilon\right)^{5/7}$$

is realized for $\beta_{\text{optimal}} \approx 1.29\,C_1^{2/7}\,\left(l^*/L_Q\right)^{4/7}\left(r_e\,\lambda_e\right)^{2/7}\,\gamma\left(\gamma\varepsilon\right)^{3/7}$

Magnetism

(*or “Why four poles in a quadrupole magnet ?”*)

Quadrupole Magnets: the Optician Standpoint

• For the beam optician a Quadrupole is a device which creates a Dipole Gradient G, that is:

1) does not bend the reference trajectory,
2) soft kicks particles with small offsets from the reference trajectory
3) hard kicks particles with large offsets:

$$\delta y'\big|_Q = -g\delta y = -\frac{\delta y}{f}, \quad \text{with} \quad g = \frac{1}{f} = \frac{G l_Q}{B\rho}$$

Hence

Quadrupole = Dipole Gradient (*along a straight line*)

• In the same spirit, from the beam optics standpoint a Sextupole is a device which creates a Quadrupole Gradient G' in the transverse plane:

$$\delta y'\big|_S = (\delta y'\big|_Q)'\,\delta y = (-g\delta y)'\,\delta y = -g'\,\delta y^2 \quad \text{with} \quad g' = \frac{G' l_S}{B\rho}.$$

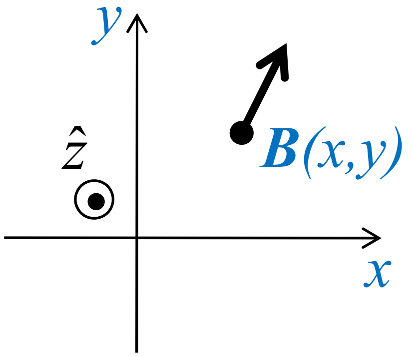
Hence

Sextupole = Quadrupole Gradient
Hexapôle = Gradient Quadripolaire (*in French*)

etc…

Quadrupole Magnets: the Mathematician Standpoint

Two Dimensional Static Magnetism
in an Electrical Current Free Region
(representing the accelerator beam pipe)



• We specialize on Time Independent Transverse Magnetic Fields,

$$\vec{B}(x,y) = B_x(x,y)\,\hat{x} + B_y(x,y)\,\hat{y}$$

which is realized by infinitely long magnets with a uniform field distribution along the longitudinal axis, and no end-fields.

• Such a field $\vec{B}(x,y)$ derives from a longitudinal vector potential $\vec{A} = A(x,y)\,\hat{z}$

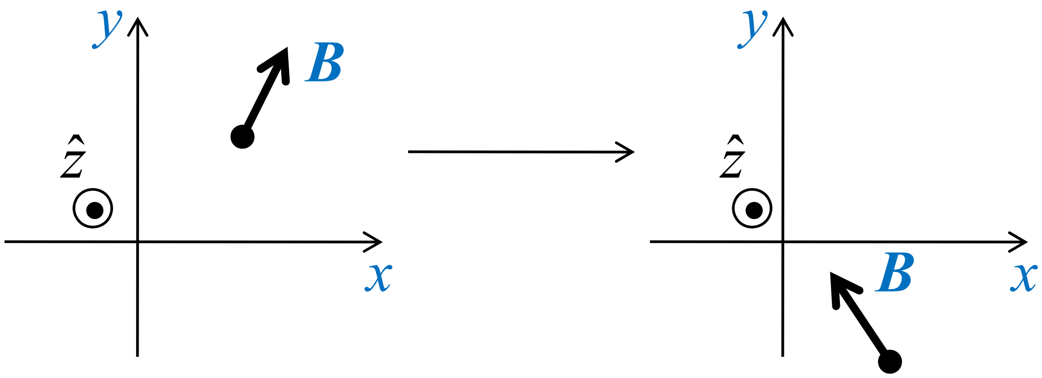
$$\vec{B} = \vec{\nabla} \wedge \vec{A} = -\partial_y A\,\hat{x} + \partial_x A\,\hat{y}$$

• Maxwell-Ampère's equation in the vacuum $\vec{\nabla} \wedge \vec{B}(x,y) = 0$ imposes that the scalar field $A(x,y)$ is an harmonic function obeying Laplace's equation:

$$\Delta\, A(x,y) = 0$$

Some Symmetry : Normal and Skew Magnetic Fields

• Symmetry with respect to the '*horizontal*' plane $(x, y) \rightarrow (x, -y)$:



$$B_x(x, y) \rightarrow -B_x(x, -y)$$

$$B_y(x, y) \rightarrow +B_y(x, -y)$$

$$A(x, y) \rightarrow A(x, -y)$$

• Transverse magnetic fields can be decomposed into the sum of symmetric (normal) and antisymetric (skew) components:

$$\vec{B}(x, y) = \vec{B}_N(x, y) + \vec{B}_S(x, y)$$

$$A(x, y) = A_N(x, y) + A_S(x, y)$$

such that

$$A_N(x, y) = +A_N(x, -y)$$

$$A_S(x, y) = -A_S(x, -y)$$

• The implementation of this symmetry simplifies the design of accelerator since it eliminates (x,y) transverse coupling by construction (*cf. Beam Optics section*).

Solving the Vector Potential A

- We introduce the complex variable $z = x + iy$ such that:

$$\begin{cases} \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \\ \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \end{cases} \Rightarrow \quad \Delta = 4\partial_z \partial_{\bar{z}}$$

- The solution of Laplace's equation $\Delta A(x,y) = 4\partial_z \partial_{\bar{z}} A(z,\bar{z}) = 0$ is

$$A(z,\bar{z}) = a(z) + a'(\bar{z})$$

and, for a real potential $A \in \mathrm{R}$:

$$A(z,\bar{z}) = a(z) + \bar{a}(\bar{z})$$

with a an analytic complex function (*essentially, an infinite polynomial*).

- By the symmetry $(x,y) \to (x,-y) \Leftrightarrow z \to \bar{z}$

$$\begin{cases} A_N(z,\bar{z}) = (A(z,\bar{z}) + A(\bar{z},z))/2 = \quad \mathrm{Re}\, a(z) + \mathrm{Re}\, a(\bar{z}) \\ A_S(z,\bar{z}) = (A(z,\bar{z}) - A(\bar{z},z))/2 = i(\mathrm{Im}\, a(z) - \mathrm{Im}\, a(\bar{z})) \end{cases}$$

Harmonic Expansion of the Vector Potential A

- Introducing the notation

$$a(z) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}(b_n - ia_n)\,z^n \qquad \text{with} \qquad (b_n, a_n) \quad \text{real coefficients}$$

the vector potential can be expanded as follows

$$\begin{cases} A_N(z,\bar{z}) = \dfrac{1}{2}\displaystyle\sum_{n=1}^{\infty}\frac{b_n}{n}(z^n + \bar{z}^n) \\ A_S(z,\bar{z}) = \dfrac{1}{2i}\displaystyle\sum_{n=1}^{\infty}\frac{a_n}{n}(z^n - \bar{z}^n) \end{cases}$$

- In polar coordinates (r, θ), the vector potential can be expanded in the multi-polar harmonic expansion:

$$\begin{cases} A_N(r,\theta) = \displaystyle\sum_{n=1}^{\infty}\frac{b_n}{n}r^n\cos(n\theta) \\ A_S(r,\theta) = \displaystyle\sum_{n=1}^{\infty}\frac{a_n}{n}r^n\sin(n\theta) \end{cases}$$

Nota Bene

A Skew Field derives from a Rotation of the Normal Field:

$$A_S^{(n)}(r,\theta) \propto A_N^{(n)}\left(r,\theta + \frac{\pi}{2n}\right)$$

Harmonic Expansion of the Magnetic Field B

- Introduce the complex magnetic field $B = B_y + iB_x$ such that

$$B = (\partial_x - i\partial_y)A = 2\partial_z A \;,$$

the complex field can be expanded as follows

$$\begin{cases} B_N(z) = \sum_{n=1}^{\infty} b_n z^{n-1} \\ B_S(z) = -i\sum_{n=1}^{\infty} a_n z^{n-1} \end{cases}$$

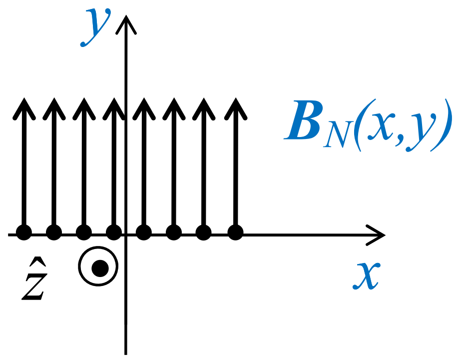
- In polar coordinates (r, θ), $B = e^{-i\theta}(B_\theta + iB_r)$ and the magnetic field can be expanded in the multi-polar harmonic expansion:

$$\begin{cases} B_r^N(r,\theta) = \sum_{n=1}^{\infty} b_n r^{n-1}\sin(n\vartheta) \;,\; B_\theta^N(r,\theta) = \sum_{n=1}^{\infty} b_n r^{n-1}\cos(n\vartheta) \\ B_r^S(r,\theta) = -\sum_{n=1}^{\infty} a_n r^{n}\cos(n\vartheta) \,,\; B_\theta^S(r,\theta) = \sum_{n=1}^{\infty} a_n r^{n-1}\sin(n\vartheta) \end{cases}$$

The Ideal Dipole Magnetic Field : $n = 1$

- **Normal Component**

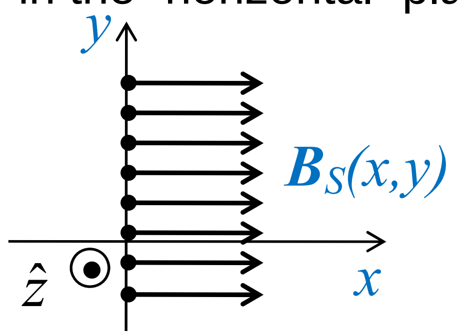
$$\vec{B}_N = \begin{pmatrix} 0 \\ b_1 \end{pmatrix}$$



B_N is a uniform “vertical” field which bends trajectory in the “horizontal” plane

- **Skew Component**

$$\vec{B}_S = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}$$



B_S is a uniform “horizontal” field which bends trajectory in the “vertical” plane

- Unlike this mathematical 2D model, most of real accelerator dipole magnets are not straight, but are bent around the reference trajectory with a curvature radius ρ given by

$$B\rho = p_{\text{ref}} / q$$

Dipole magnet transfer maps are the most complex elementary map to modelize !!

The Ideal Quadrupole Magnetic Field : n = 2

- **Normal Component**

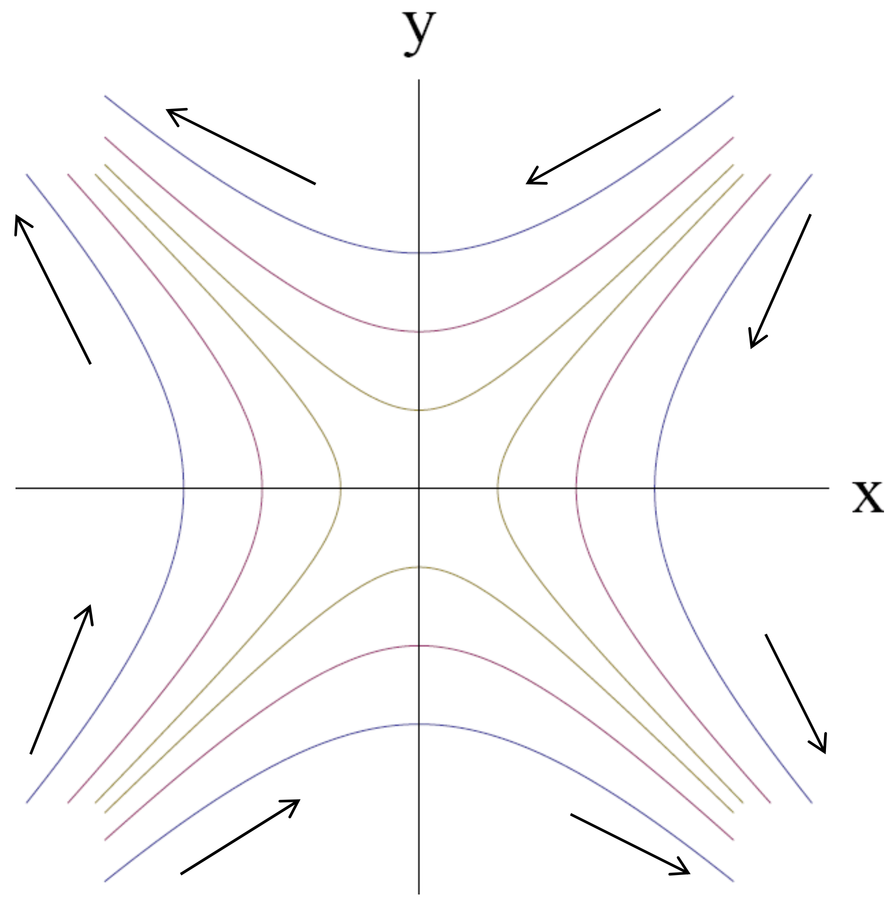
$$B_N^{(n=2)}(z,\bar{z}) = b_2 z$$

$$\Rightarrow \quad \vec{B}_N^{(2)} = b_2 \begin{pmatrix} y \\ x \end{pmatrix} \quad \text{and} \quad \boxed{b_2 \equiv G}$$

- **Skew Component**

$$B_S^{(n=2)}(z,\bar{z}) = -i a_2 z$$

$$\Rightarrow \quad \vec{B}_S^{(2)} = a_2 \begin{pmatrix} -x \\ y \end{pmatrix}$$



Magnetic field lines of a quadrupole magnet

$\boldsymbol{B}_S$ is derived $\boldsymbol{B}_N$ from by a rotation of $\dfrac{\pi}{4}$.

The Ideal Sextupole Magnetic Field : n = 3

- **Normal Component**

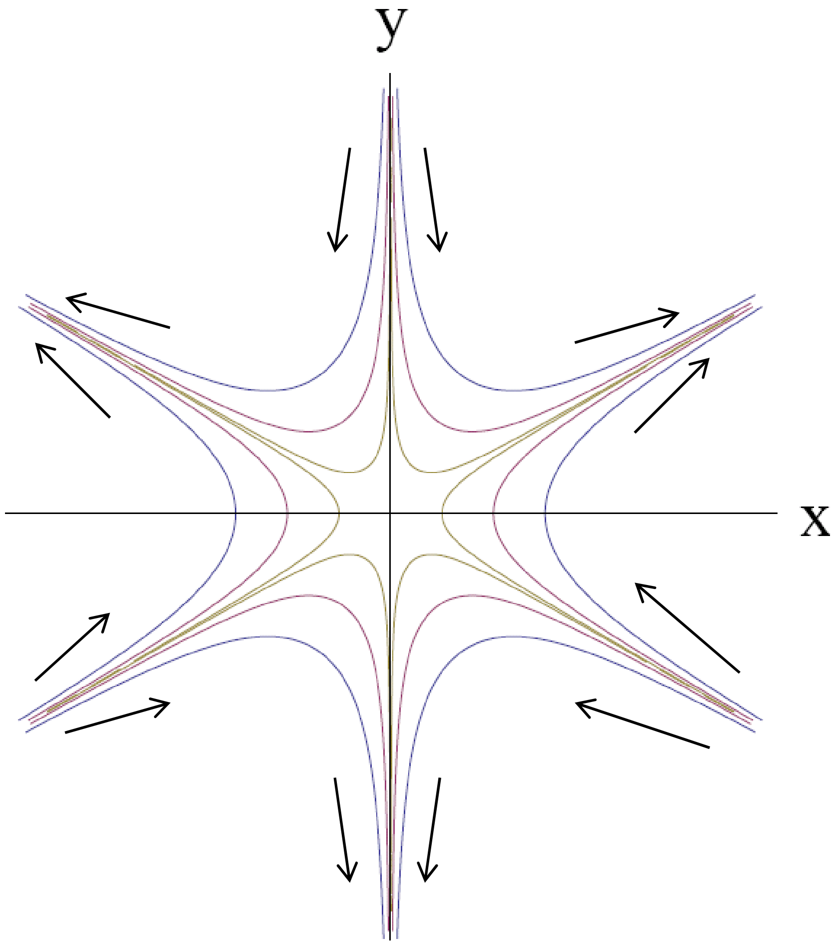
$$B_N^{(n=3)}(z,\bar{z}) = b_3 z^2$$

$$\Rightarrow \quad \vec{B}_N^{(3)} = b_3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

- **Skew Component**

$$B_S^{(n=3)}(z,\bar{z}) = -i a_3 z^2$$

$$\Rightarrow \quad \vec{B}_S^{(3)} = a_3 \begin{pmatrix} -x^2 + y^2 \\ 2xy \end{pmatrix}$$



Magnetic Field lines of a sextupole magnet

B_S is derived B_N from by a rotation of $\dfrac{\pi}{6}$.

Misaligned Sextupole Magnet

We consider only the Normal Sextupole Magnet component.

As expected, the sextupole field can be expressed as a superposition of gradients of quadrupole :

$$\vec{B}_N^{(3)} = b_3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} = b_3 x \begin{pmatrix} y \\ x \end{pmatrix} - b_3 y \begin{pmatrix} -x \\ y \end{pmatrix} = \vec{B}_N^{(2)}\Big|_{b_2 = b_3 x} + \vec{B}_s^{(2)}\Big|_{a_2 = -b_3 y}$$

but the 'bad' surprise is that it includes both normal and skew components.

As a consequence:

- the effect of a sextupole on a horizontally misaligned particle is that of a normal quadrupole ,
- the effect of a sextupole on a vertically misaligned particle is that of a skew quadrupole .

Conversely:

- an horizontally misaligned sextupole acts as a normal quadrupole,
- a vertically misaligned sextupole acts as a skew quadrupole.

End of Part 1