Fourth International Accelerator School for Linear Colliders

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Course A4: Damping Ring Design and Physics Issues

Lecture 2
Review of Linear Beam Dynamics and Radiation Damping

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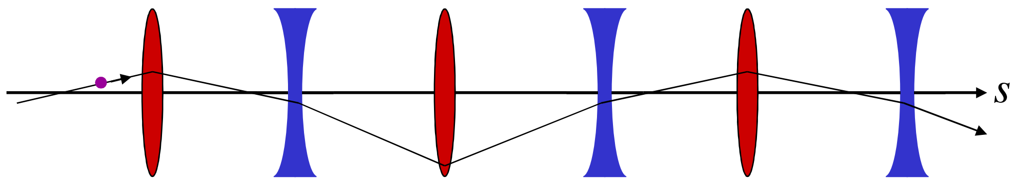
Linear beam dynamics and radiation damping

In this lecture, we shall discuss:

- The optical (lattice) functions: Twiss parameters and dispersion.
- The momentum compaction factor.
- Emittance, energy spread, bunch length.
- Synchrotron radiation: energy loss and damping times.
- Quantum excitation: equilibrium emittance, energy spread and bunch length.

Twiss parameters describe the local oscillation amplitudes of particles

As a particle moves along an accelerator beam line, it performs transverse (betatron) oscillations.



The horizontal motion can be described in terms of the horizontal coordinate x and the normalised horizontal momentum $p_x = \gamma m \dot{x} / P_0$

Alternatively, we can use action-angle variables: J_x, φ_x defined by:

$$2J_x = \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2$$

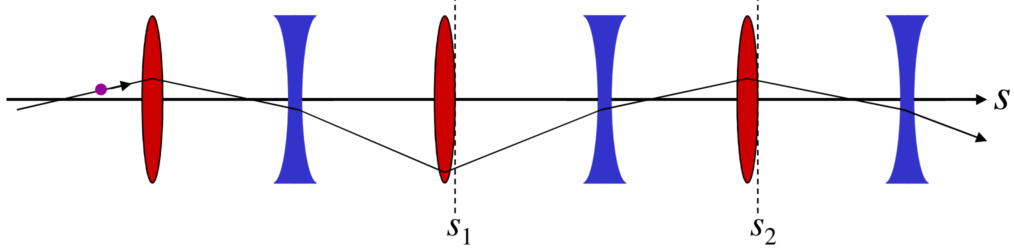
$$\tan\varphi_x = -\beta_x \frac{p_x}{x} - \alpha_x$$

The Twiss parameters α_x, β_x and γ_x are defined so that J_x is a constant of the motion (called the *betatron action*), and φ_x (the *betatron phase)* increases with s according to:

$$\frac{d\varphi_x}{ds} = \frac{1}{\beta_x}$$

Transfer matrix for one periodic cell

The Twiss parameters are determined from the transfer matrix for one periodic cell.



$$\begin{pmatrix} x \\ p_x \end{pmatrix}_{s_2} = M(s_2; s_1) \cdot \begin{pmatrix} x \\ p_x \end{pmatrix}_{s_1}$$

The transfer matrix M is parameterised as:

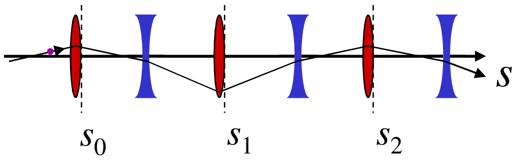
$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\mu_x + \begin{pmatrix} \alpha_x & \beta_x \\ -\gamma_x & -\alpha_x \end{pmatrix} \sin\mu_x$$

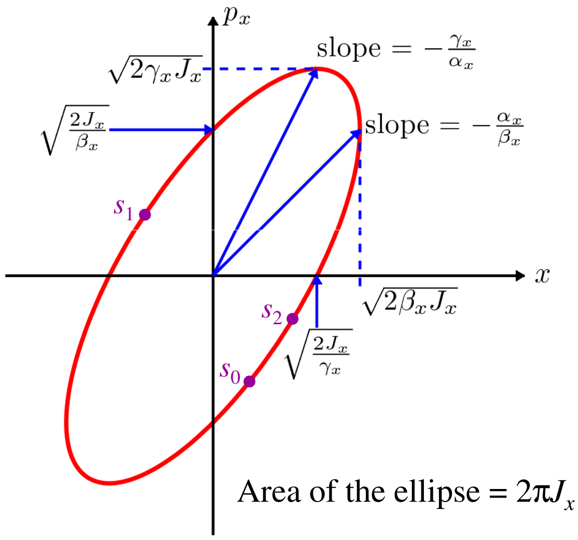
In transport through one periodic cell, the action-angle variables transform as:

$$J_x(s_2) = J_x(s_1) \qquad \varphi_x(s_2) = \varphi_x(s_1) + \mu_x$$

The Twiss parameters describe the shape of a phase space ellipse

If we observe the coordinate x and momentum p_x of a particle at the exit of each periodic cell and plot these variables on a phase space diagram, we construct an ellipse, the shape of which is determined by the Twiss parameters.





Symplecticity

If we ignore radiation (and some other) effects, then particle transport along a beamline is *symplectic*.

Mathematically, this means that any transfer matrix M satisfies:

$$M^{\mathrm{T}} \cdot S \cdot M = S$$

where the antisymmetric matrix S is given by:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The symplectic condition on the transfer matrix imposes a constraint on the Twiss parameters, which can be written:

$$\beta_x \gamma_x - \alpha_x^2 = 1$$

Physically, symplecticity means that the area of the phase space ellipse described by the motion of a particle is fixed. In other words, the amplitude J_x of the betatron oscillations is constant.

The Twiss parameters give the local oscillation amplitude

The action J_x is constant for a particle moving along the beamline, and the betatron phase φ_x increases monotonically.

By writing the Cartesian variables x and p_x in terms of the action-angle variables, we can represent the motion of the particle as similar to that of a simple harmonic oscillator, but with local variations in amplitude and frequency:

$$x = \sqrt{2J_x\beta_x}\cos\varphi_x$$

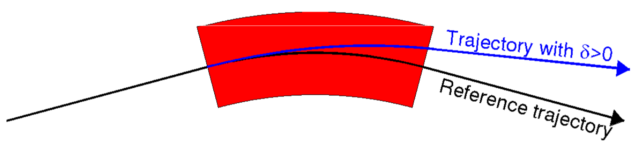
$$p_x = -\sqrt{\frac{2J_x}{\beta_x}}\left(\sin\varphi_x + \alpha_x\cos\varphi_x\right)$$

Dispersion describes the change in trajectory with energy

The energy E of a particle is usually specified in terms of the deviation δ from the reference energy (or the reference momentum P_0):

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0}$$

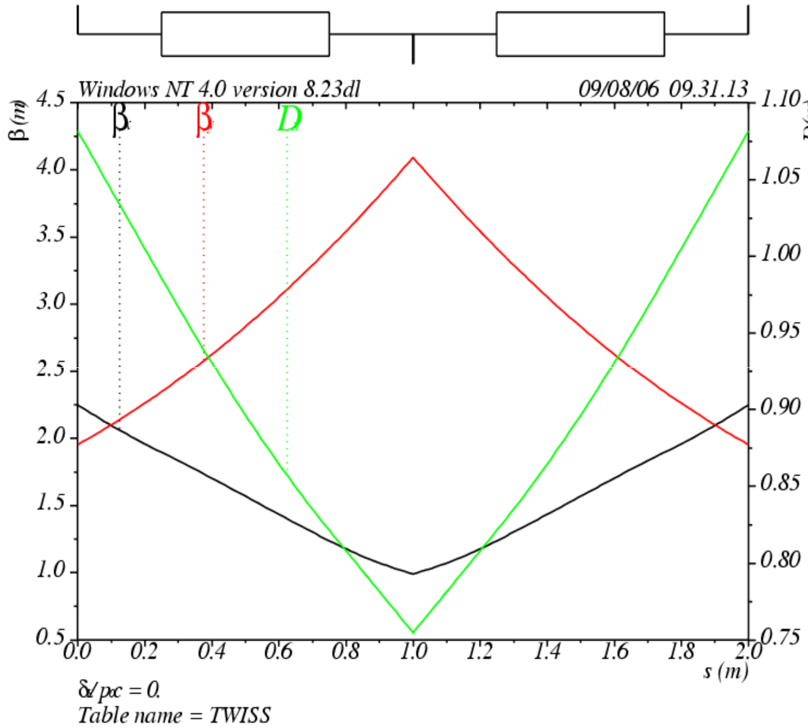
Particles with higher energy ($\delta > 0$) are deflected by a smaller angle in dipole magnets, compared to particles with the nominal energy.



In a storage ring, particles with non-zero energy deviation have a different closed orbit from particles with the reference energy. The dispersion function η_x is defined as change in closed orbit with respect to the energy deviation:

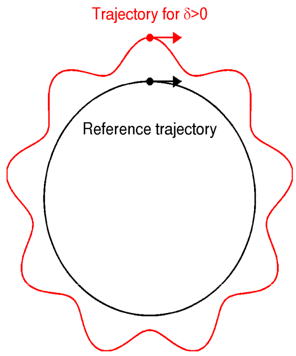
$$X_{CO} = \eta_x\delta$$

Example: beta functions and dispersion in a FODO cell



Momentum compaction factor

The presence of dispersion means that there is a change in the path length for a particle following a closed orbit, depending on the energy deviation.



The change in path length C for one complete revolution is described by the momentum compaction factor α_p:

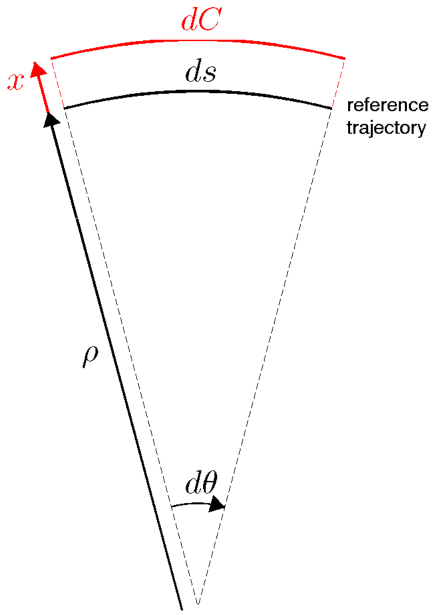
$$\frac{C}{C_0} = 1 + \alpha_p \delta + \alpha_p^{(2)} \delta^2 + O(\delta^3)$$

Momentum compaction factor

We can calculate the momentum compaction factor in a lattice from the dispersion in the dipoles.

With a curved trajectory, the path followed by a particle a horizontal distance x from the reference trajectory is:

$$dC = \left(1 + \frac{x}{\rho}\right) ds$$



If the offset is the result of an energy error:

$$x = \eta_x \delta$$

The total path length is:

$$C = \oint 1 + \frac{\eta_x}{\rho} \delta \, ds$$

Then the momentum compaction can be written:

$$\alpha_p = \frac{1}{C_0} \frac{dC}{d\delta}\bigg|_{\delta=0} = \frac{1}{C_0} \oint \frac{\eta_x}{\rho} ds$$

Momentum compaction factor and first synchrotron radiation integral

If we define:

$$I_1 = \oint \frac{\eta_x}{\rho} ds$$

then we can write the momentum compaction factor:

$$\alpha_p = \frac{I_1}{C_0}$$

I_1 is called the *first synchrotron radiation integral*: it is an integral over the circumference of a function that involves the lattice parameters (the dispersion) and the magnet parameters (the bending radius).

We shall define other synchrotron radiation integrals when we come to calculate the radiation damping times and the equilibrium beam sizes.

Momentum compaction, phase slip, and transition

The time taken for a particle to complete one revolution depends on:

- the path length (a function of the energy);
- the speed of the particle.

The speed of the particle does depend on the energy, but at ultra-relativistic velocities, the dependence is weak.

The revolution period is expressed in terms of the energy deviation:

$$\frac{T}{T_0} = 1 + \eta_p \delta + \eta_p^{(2)} \delta^2 + O(\delta^3)$$

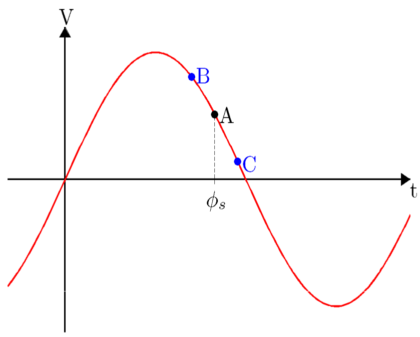
where η_p is the *phase slip factor*. It can be shown that:

$$\eta_p = \alpha_p - \frac{1}{\gamma_0^2}$$

$\eta_p < 0$	$\alpha_p < 0$ or $\gamma_0^2 < 1/\alpha_p$	Below transition: revolution frequency increases with energy
$\eta_p = 0$	$\gamma_0^2 = 1/\alpha_p$	At transition: revolution frequency independent of energy
$\eta_p > 0$	$\gamma_0^2 > 1/\alpha_p$	Above transition: revolution frequency decreases with energy

Synchrotron oscillations

If the momentum compaction factor is not zero, and there are RF cavities in the storage ring, then particles will perform *synchrotron oscillations*.



A particle initially at point B in the RF phase and zero energy deviation, will gain energy each time it passes the RF cavity. The momentum compaction means that the particle will take longer to go round the ring, and will start to move towards A.

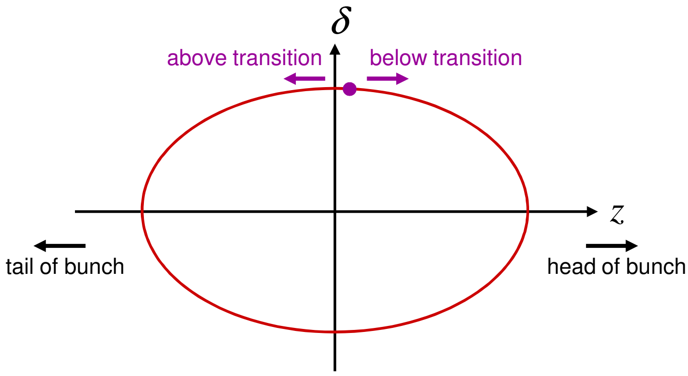
Similarly, a particle at point C gains insufficient energy from the RF to replace synchrotron radiation losses. Such a particle will start to take less time to go around the ring, and will also move towards A.

Synchrotron oscillations

The trajectory of a particle in longitudinal phase space is an ellipse.

For $+z$ towards the head of a bunch, particles above transition move round the ellipse *anticlockwise*.

The synchrotron tune is the number of revolutions of the phase space ellipse made in one revolution of the storage ring. For most lattices, the synchrotron tune is small, typically of order 0.01.



Beam emittance

The beam emittance is a measure of the phase space area covered by particles in the beam.

We define the Σ matrix using the second-order moments of the beam distribution:

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle x p_x \rangle & \langle p_x^2 \rangle \end{pmatrix}$$

Under a phase space transformation:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto M \cdot \begin{pmatrix} x \\ p_x \end{pmatrix}$$

the Σ matrix transforms as:

$$\Sigma \mapsto M \cdot \Sigma \cdot M^{\mathrm{T}}$$

So the matrix product $\Sigma \cdot S$ transforms as:

$$\Sigma \cdot S \mapsto M \cdot \Sigma \cdot M^{\mathrm{T}} \cdot S = M \cdot \Sigma \cdot S \cdot M^{-1}$$

where the last step follows from the symplecticity of M.

Beam emittance

We have found that $\Sigma{\cdot}S$ transforms as:

$$\Sigma \cdot S \mapsto M \cdot \Sigma \cdot S \cdot M^{-1}$$

But the eigenvalues of $\Sigma{\cdot}S$ are preserved by any similarity transformation of this type.

The eigenvalues of $\Sigma{\cdot}S$ are $\pm i\varepsilon_x$ where:

$$\varepsilon_x = \sqrt{\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2}$$

ε_x is called the emittance of the beam: the emittance is conserved under symplectic transport along a beam line.

Beam emittance

Recall that the particle coordinate and momentum are related to the action-angle variables by:

$$x = \sqrt{2J_x\beta_x}\cos\varphi_x$$

$$p_x = -\sqrt{\frac{2J_x}{\beta_x}}\left(\sin\varphi_x + \alpha_x\cos\varphi_x\right)$$

From these expressions, we find that:

$$\langle x^2\rangle = \beta_x\langle J_x\rangle$$
$$\langle xp_x\rangle = -\alpha_x\langle J_x\rangle$$
$$\langle p_x^2\rangle = \gamma_x\langle J_x\rangle$$

It then follows that:

$$\varepsilon_x = \sqrt{\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2} = \sqrt{\beta_x\gamma_x - \alpha_x^2}\,\langle J_x\rangle = \langle J_x\rangle$$

Beam emittance

Thus, the beam emittance is just the average value of the action variables over all particles in the beam:

$$\varepsilon_x = \langle J_x \rangle$$

The fact that the action of each particle is conserved under (symplectic) transport along the beam line is consistent with the fact that the emittance is conserved.

The beam distribution at any point can be written in terms of the emittance and the Twiss parameters:

$$\langle x^2 \rangle = \beta_x \varepsilon_x$$
$$\langle x p_x \rangle = -\alpha_x \varepsilon_x$$
$$\langle p_x^2 \rangle = \gamma_x \varepsilon_x$$

Note that a beam has three emittances: a horizontal, a vertical and a longitudinal emittance. The analysis presented here assumes that there is no coupling, i.e. that the three degrees of freedom are completely independent. However, in the general (coupled) case, we can still obtain the emittances as the eigenvalues of $\Sigma \cdot S$.

Damping of emittances

Particle sources (particularly positron sources) produce beams with relatively large emittances. One of the jobs of the damping rings is to reduce the emittances, to make a beam that can be used to produce luminosity.

	Injected emittance	Extracted emittance
Horizontal e^+	1 μm	0.8 nm
Vertical e^+	1 μm	0.002 nm
Longitudinal e^+	> 30 μm	10 μm

But… if the emittance is conserved in transporting a beam along a beam line, how do the damping rings reduce the beam emittances?

Radiation damping

Whenever charged particles undergo acceleration, they emit radiation (in the case of relativistic particles, this is called synchrotron radiation).

In a storage ring, the radiation emission is dominated by the bending fields of the dipole magnets.

Synchrotron radiation is a non-symplectic effect; in some respects, it is analogous to a frictional force that will steadily damp the motion of an oscillator.

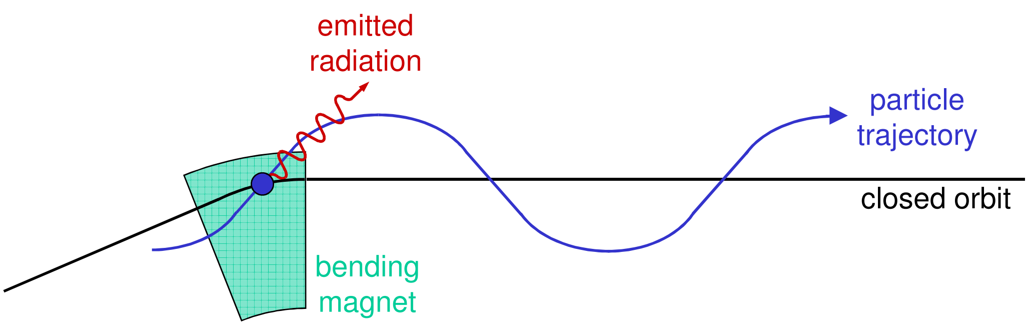
In the case of particles in a storage ring, the combination of radiation emission in the dipoles and the restoration of this energy in the RF cavities leads to gradual reduction in the emittances, towards some equilibrium values.

Radiation damping

The majority of photons are emitted within a cone of angle $1/\gamma$ around the instantaneous direction of motion of the particle.

For ultra-relativistic particles, γ is very large, so nearly all the radiation is emitted directly along the instantaneous direction of motion of the particle.

Particles lose longitudinal *and transverse* momentum in bending magnets.

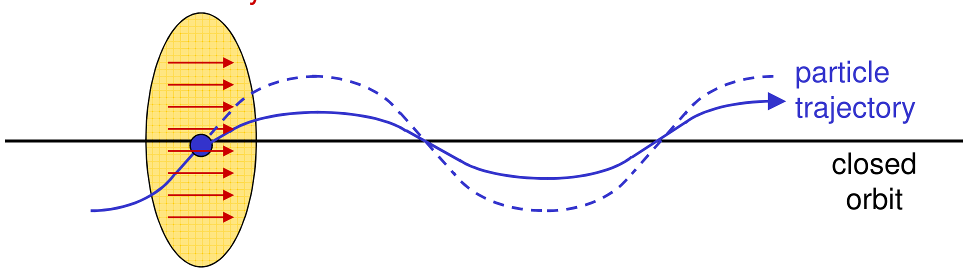


Neglecting dispersion and chromaticity, the trajectory of the particle after emitting a photon is the same as it would be if no photon were emitted: however the vertical momentum p_y is reduced.

Radiation damping

In an RF cavity, the particle sees an accelerating electric field parallel to the closed orbit: the RF cavities in a storage ring restore the energy lost by synchrotron radiation.

The increase in momentum of a particle in an RF cavity is parallel to the closed orbit. This leads to a reduction in the amplitude of the betatron oscillations of the particle: however, the vertical momentum p_y is not changed.



Damping of vertical emittance

The vertical motion is usually easier to deal with than the horizontal motion, because most storage rings have (in the absence of alignment or steering errors) zero vertical dispersion.

Consider a particle travelling round a storage ring. When the particle travels through bending magnets, it emits radiation within a cone of opening angle $1/\gamma$ around the instantaneous direction of motion of the particle. For $\gamma >> 1$, the direction of the radiation is approximately along the direction of motion of the particle, so the direction of motion is unchanged by the "recoil".

The total momentum of the particle changes with the emission of radiation:

$$p' = p - dp \approx p\left(1 - \frac{dp}{P_0}\right)$$

where dp is the momentum of the radiation, and the total momentum of the particle p is close to the reference momentum P_0. Since the direction of the particle is unchanged, the vertical emittance after emitting the radiation is:

$$p'_y = p_y\left(1 - \frac{dp}{P_0}\right)$$

Damping of vertical emittance

After emission of radiation, the vertical momentum of the particle is:

$$p'_y = p_y\left(1 - \frac{dp}{P_0}\right)$$

Now we substitute this into the expression for the vertical betatron action:

$$2J_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2$$

to find the change in the action resulting from the emission of radiation:

$$dJ_y = -\left(\alpha_y y p_y + \beta_y p_y^2\right)\frac{dp}{P_0}$$

We average over all particles in the beam, to find:

$$\langle dJ_y \rangle = d\varepsilon_y = -\frac{dp}{P_0}\varepsilon_y$$

where we have used:

$$\langle y p_y \rangle = -\alpha_y \varepsilon_y \qquad \langle p_y^2 \rangle = \gamma_y \varepsilon_y \qquad \text{and} \qquad \beta_y\gamma_y - \alpha_y^2 = 1$$

Damping of vertical emittance

Finally, we integrate the loss in momentum around the ring. The emittance is conserved under symplectic transport; so if the non-symplectic effects (in this case, the radiation effects) are slow, we can write:

$$\frac{d\varepsilon_y}{dt} = -\frac{\varepsilon_y}{T_0}\oint\frac{dp}{P_0} \approx -\frac{U_0}{E_0 T_0}\varepsilon_y$$

where T_0 is the revolution period, and U_0 is the energy loss in one turn. We define the damping time τ_y:

$$\tau_y = 2\frac{E_0}{U_0}T_0$$

so the evolution of the emittance is:

$$\varepsilon_y(t) = \varepsilon_y(0)\exp\left(-2\frac{t}{\tau_y}\right)$$

Typically, the damping time in a synchrotron storage ring is measured in tens of milliseconds (whereas the revolution period is measured in microseconds).

Synchrotron radiation energy loss

To complete our calculation of the vertical damping time, we need to find the energy lost by a particle through synchrotron radiation on each turn through the storage ring. The power radiated by a particle of charge e and energy E in a magnetic field B is given by:

$$P_\gamma = \frac{C_\gamma}{2\pi} c^3 e^2 B^2 E^2$$

C_γ is a constant, given by:

$$C_\gamma = \frac{e^2}{3\varepsilon_0 \left(mc^2\right)^4} \approx 8.846 \times 10^{-5} \text{ m/GeV}^3$$

A charged particle in a magnetic field follows a circular trajectory with radius ρ, given by:

$$B\rho = \frac{E}{ec}$$

Hence the synchrotron radiation power can be written:

$$P_\gamma = \frac{C_\gamma}{2\pi} c \frac{E^4}{\rho^2}$$

Synchrotron radiation energy loss

For a particle with the nominal energy, and travelling at (close to) the speed of light around the closed orbit, we can find the energy loss simply by integrating the radiation power around the ring:

$$U_0 = \oint P_\gamma \, dt = \oint P_\gamma \frac{ds}{c}$$

Using the previous expression for P_γ, we find:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 \oint \frac{1}{\rho^2} ds$$

Conventionally, we define the *second synchrotron radiation integral*, I_2:

$$I_2 = \oint \frac{1}{\rho^2} ds$$

In terms of I_2, the energy loss per turn U_0 is written:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2$$

Damping of horizontal emittance

The effect of radiation on the horizontal emittance is complicated by the presence of dispersion. When a particle emits some radiation (resulting in a change in energy deviation $\Delta\delta$) at a location where there is dispersion, the closed orbit changes by:

$$\Delta X = \eta_x \Delta\delta$$

$$\Delta P_X = \eta_{px} \Delta\delta$$

This means that the coordinate and momentum (with respect to the dispersive closed orbit) of a particle after emitting radiation with momentum dp are:

$$x' = x - \eta_x \frac{dp}{P_0}$$

$$p_x' = p_x\left(1 - \frac{dp}{P_0}\right) - \eta_{px}\frac{dp}{P_0}$$

Damping of horizontal emittance

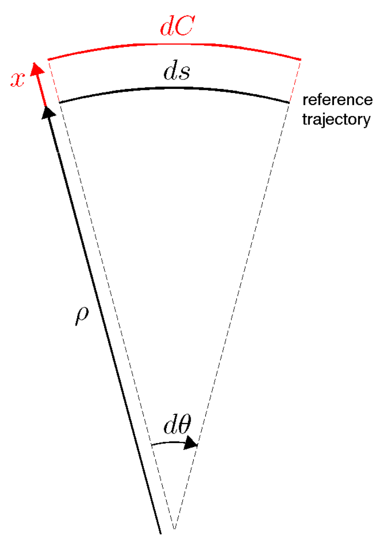
There are two further complications that we need to consider when calculating the effects of synchrotron radiation on the horizontal motion.

The first is that, because of the curvature of the trajectory, the path length of a particle in a dipole magnet is a function of the horizontal coordinate. Thus the integral of the power loss:

$$\oint \frac{dp}{P_0} = \oint \frac{P_\gamma}{P_0 c}\, dt$$

becomes, in terms of the path length:

$$\oint \frac{dp}{P_0} = \oint \frac{P_\gamma}{P_0 c}\frac{dC}{c} = \oint \frac{P_\gamma}{P_0 c}\left(1 + \frac{x}{\rho}\right)\frac{ds}{c}$$



Damping of horizontal emittance

The second complication for horizontal motion is that dipole magnets are sometimes constructed with a gradient, so the vertical field changes as a function of the horizontal coordinate:

$$B = B_0 + \frac{\partial B}{\partial x} x$$

Taking all these effects into account, we can essentially proceed the same way as we did for the vertical betatron motion; that is:

- Write down the changes in coordinate x and momentum p_x resulting from an emission of radiation with momentum dp (taking into account the additional effects of dispersion).
- Substitute expressions for the new coordinate and momentum into the expression for the horizontal betatron action, to find the change in action resulting from the radiation emission.
- Average over all particles in the beam, to find the change in the emittance resulting from radiation emission from each particle.
- Integrate around the ring (taking account of changes in path length and field strength with x in the bends) to find the change in emittance over one turn.

Damping of horizontal emittance

The algebra is rather more complicated than for the case of the vertical emittance. The result is that the horizontal emittance evolves according to:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x \qquad \varepsilon_x(t) = \varepsilon_x(0)\exp\left(-2\frac{t}{\tau_x}\right)$$

where the horizontal damping time is given by:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0$$

The horizontal damping partition number j_x is given by:

$$j_x = 1 - \frac{I_4}{I_2}$$

where the fourth synchrotron radiation integral I_4 contains the effects of the variation in path length and field strength with x:

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}$$

Damping of synchrotron oscillations

The change in energy deviation δ and longitudinal coordinate z for a particle in one turn around a storage ring are given by:

$$\Delta\delta = \frac{eV_{RF}}{E_0}\sin\left(\varphi_s - \frac{\omega_{RF} z}{c}\right) - \frac{U}{E_0}$$

$$\Delta z = -\alpha_p C_0 \delta$$

where V_{RF} is the RF voltage and ω_{RF} the RF frequency, E_0 is the reference energy of the beam, φ_s is the nominal RF phase, and U is the energy lost by the particle through synchrotron radiation.

If the revolution period is T_0, then we can write the longitudinal equations of motion for the particle:

$$\frac{d\delta}{dt} = \frac{eV_{RF}}{E_0 T_0}\sin\left(\varphi_s - \frac{\omega_{RF} z}{c}\right) - \frac{U}{E_0 T_0}$$

$$\frac{dz}{dt} = -\alpha_p c \delta$$

Damping of synchrotron oscillations

Let us assume that z is small compared to the RF wavelength, i.e. $\omega_{RF} z/c \ll 1$.

Also, the energy loss per turn is a function of the energy of the particle (particles with higher energy radiate higher synchrotron radiation power), so we can write (to first order in the energy deviation):

$$U = U_0 + \Delta E \left.\frac{dU}{dE}\right|_{E=E_0} = U_0 + E_0 \delta \left.\frac{dU}{dE}\right|_{E=E_0}$$

Further, we assume that the RF phase φ_s is set so that for $z = \delta = 0$, the RF cavity restores exactly the amount of energy lost by synchrotron radiation. The equations of motion then become:

$$\frac{d\delta}{dt} = -\frac{eV_{RF}}{E_0 T_0}\cos\varphi_s \frac{\omega_{RF}}{c} z - \frac{1}{T_0}\delta \left.\frac{dU}{dE}\right|_{E=E_0}$$

$$\frac{dz}{dt} = -\alpha_p c \delta$$

Damping of synchrotron oscillations

Combining these equations gives:

$$\frac{d^2\delta}{dt^2} + 2\alpha_E \frac{d\delta}{dt} + \omega_s^2 \delta = 0$$

This is the equation for a damped harmonic oscillator, with frequency ω_s and damping constant α_E given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0} \cos\varphi_s \frac{\omega_{RF}}{T_0} \alpha_p$$

$$\alpha_E = \frac{1}{2T_0} \left. \frac{dU}{dE} \right|_{E=E_0}$$

Damping of synchrotron oscillations

If $\alpha_E << \omega_s$, the energy deviation and longitudinal coordinate damp as:

$$\delta(t) = \hat{\delta} \exp(-\alpha_E t) \sin(\omega_s t - \theta_0)$$

$$z(t) = \frac{\alpha_p c}{\omega_s} \hat{\delta} \exp(-\alpha_E t) \cos(\omega_s t - \theta_0)$$

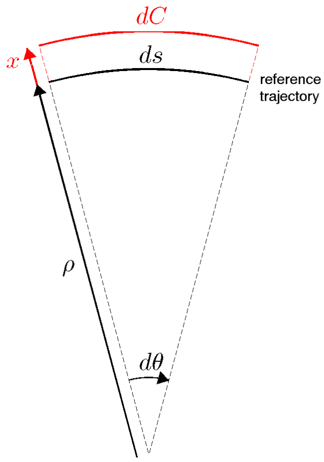
To find the damping constant α_E, we need to know how the energy loss per turn U depends on the energy deviation δ…

Damping of synchrotron oscillations

We can find the total energy lost by integrating over one revolution period:

$$U = \oint P_\gamma \, dt$$

To convert this to an integral over the circumference, we should recall that the path length depends on the energy deviation; so a particle with a higher energy takes longer to travel round the lattice.



$$dt = \frac{dC}{c}$$

$$dC = \left(1 + \frac{x}{\rho}\right) ds = \left(1 + \frac{\eta_x \delta}{\rho}\right) ds$$

$$U = \frac{1}{c} \oint P_\gamma \left(1 + \frac{\eta_x \delta}{\rho}\right) ds$$

Damping of synchrotron oscillations

With the energy loss per turn given by:

$$U = \frac{1}{c} \oint P_\gamma \left(1 + \frac{\eta_x}{\rho} \delta\right) ds$$

and the synchrotron radiation power given by:

$$P_\gamma = \frac{C_\gamma}{2\pi} c^3 e^2 B^2 E^2 = \frac{C_\gamma}{2\pi} c \frac{E^4}{\rho^2}$$

we find, after some algebra:

$$\left.\frac{dU}{dE}\right|_{E=E_0} = j_E \frac{U_0}{E_0}$$

where:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2 \qquad j_E = 2 + \frac{I_4}{I_2}$$

I_2 and I_4 are the same synchrotron radiation integrals that we saw before:

$$I_2 = \oint \frac{1}{\rho^2} ds \qquad I_4 = \oint \frac{\eta_x}{\rho} \left(\frac{1}{\rho^2} + 2k_1\right) ds \qquad k_1 = \frac{e}{P_0} \frac{\partial B_y}{\partial x}$$

Damping of synchrotron oscillations

Finally, we can write the longitudinal damping time:

$$\tau_z = \frac{1}{\alpha_E} = \frac{2}{j_z}\frac{E_0}{U_0}T_0$$

U_0 is the energy loss per turn for a particle with the reference energy E_0, following the reference trajectory. It is given by:

$$U_0 = \frac{C_\gamma}{2\pi}E_0^4 I_2$$

j_z is the longitudinal damping partition number, given by:

$$j_z = 2 + \frac{I_4}{I_2}$$

Damping of synchrotron oscillations

The longitudinal emittance is given by a similar expression to the horizontal and vertical emittances:

$$\varepsilon_z = \sqrt{\langle z^2\rangle\langle\delta^2\rangle - \langle z\delta\rangle^2}$$

In most storage rings, the correlation $\langle z\delta\rangle$ is negligible, so the emittance becomes:

$$\varepsilon_z \approx \sigma_z\sigma_\delta$$

Hence, the damping of the longitudinal emittance can be written:

$$\varepsilon_z(t) = \varepsilon_z(0)\exp\left(-2\frac{t}{\tau_z}\right)$$

Summary: synchrotron radiation damping

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2 \qquad C_\gamma = 8.846 \times 10^{-5} \text{ m/GeV}^3$$

The radiation damping times are given by:

$$\tau_x = \frac{2}{j_x} \frac{E_0}{U_0} T_0 \qquad \tau_y = \frac{2}{j_y} \frac{E_0}{U_0} T_0 \qquad \tau_z = \frac{2}{j_z} \frac{E_0}{U_0} T_0$$

The damping partition numbers are:

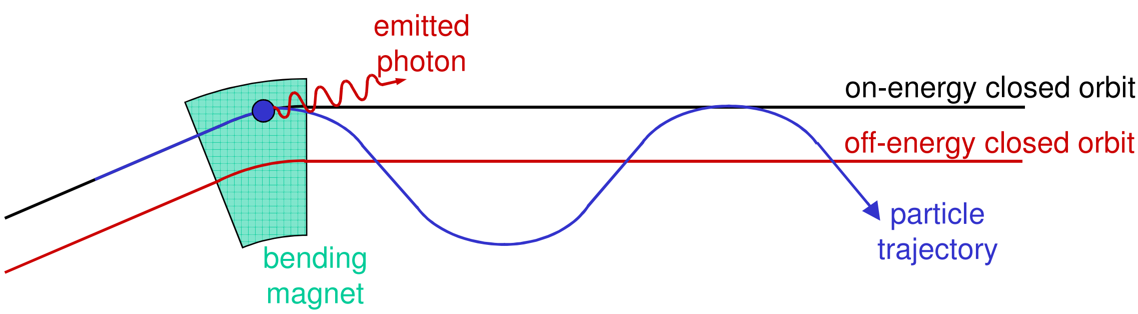
$$j_x = 1 - \frac{I_4}{I_2} \qquad j_y = 1 \qquad j_z = 2 + \frac{I_4}{I_2}$$

The second and fourth synchrotron radiation integrals are:

$$I_2 = \oint \frac{1}{\rho^2} ds \qquad I_4 = \oint \frac{\eta_x}{\rho} \left(\frac{1}{\rho^2} + 2k_1 \right) ds$$

Quantum excitation

If radiation were a purely classical process, the emittances would damp to nearly zero. However radiation is emitted in discrete units (photons), which induces some “noise” on the beam. The effect of the noise is to increase the emittance. The beam eventually reaches an equilibrium determined by a balance between the radiation damping and the quantum excitation.



Quantum excitation of betatron motion

Consider a particle that emits a photon of energy u at a location with dispersion. The horizontal coordinate x and horizontal momentum p_x after the photon emission are:

$$x' = x - \eta_x \frac{u}{E_0}$$

$$p_x' = p_x - \eta_{px} \frac{u}{E_0}$$

Substituting into the expression for the action, and averaging over all particles in the beam, we find the change in the emittance is:

$$\Delta\varepsilon_x = \frac{1}{2}\frac{\langle u^2 \rangle}{E_0^2}\mathcal{H}_x$$

where the "curly H" function is:

$$\mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$

Quantum excitation of betatron motion

Including both quantum excitation and radiation damping, the horizontal emittance evolves as:

$$\frac{d\varepsilon_x}{dt} = \frac{1}{2E_0^2 C_0}\oint \mathcal{N}\langle u^2 \rangle \mathcal{H}_x \, ds - \frac{2}{\tau_x}\varepsilon_x$$

where $\mathcal{N}$ is the number of photons emitted per unit path length. We quote the result (from synchrotron radiation theory):

$$\mathcal{N}\langle u^2 \rangle = 2C_q \gamma^2 E_0 \frac{P_\gamma}{\rho}$$

where the "quantum constant" C_q is:

$$C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc} \approx 3.832\times10^{-13}\ \text{m}$$

Quantum excitation of betatron motion

Using the expression for $\mathcal{N}\langle u^2\rangle$, and expressions used previously for the radiation power P_γ, we find:

$$\frac{d\varepsilon_x}{dt} = C_q \gamma^2 \frac{2}{j_x \tau_x} \frac{I_5}{I_2} - \frac{2}{\tau_x} \varepsilon_x$$

where the fifth synchrotron radiation integral I_5 is:

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho^3|} ds$$

The equilibrium horizontal emittance, ε_0, is reached when the quantum excitation balances the radiation damping, $d\varepsilon_x/dt = 0$:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}$$

Quantum excitation of betatron motion

The equilibrium horizontal emittance is given by:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}$$

ε_0 is determined by the beam energy, the lattice functions (Twiss parameters and dispersion) in the dipoles, and the bending radius in the dipoles.

ε_0 is sometimes called the “natural emittance” of the lattice, since it is the horizontal emittance that will be achieved in the limit of zero bunch charge: as the current is increased, interactions betweens particles in a bunch can increase the emittance above the equilibrium determined by radiation effects.

In many storage rings, the vertical dispersion in the absence of alignment, steering and coupling errors is zero, so $\mathcal{H}_y = 0$. However, the equilibrium vertical emittance is larger than zero, because the vertical opening angle of the radiation excites some vertical betatron oscillations.

For more discussion about vertical emittance, see Lecture 5.

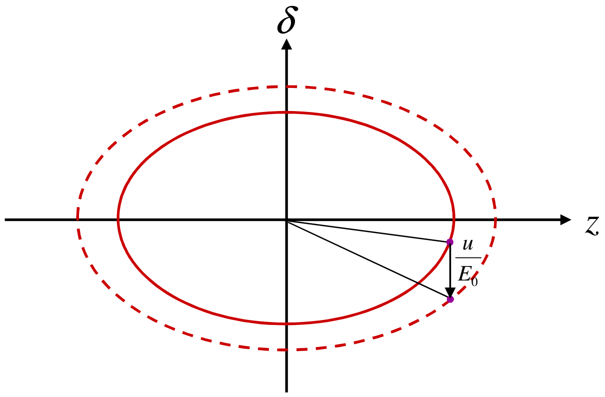
Quantum excitation of synchrotron oscillations

Quantum effects excite longitudinal emittance as well as transverse emittance. Consider a particle with longitudinal coordinate z and energy deviation δ, which emits a photon of energy u.

$$\delta' = \hat{\delta}' \sin\theta' = \hat{\delta}\sin\theta - \frac{u}{E_0}$$

$$z' = \frac{\alpha_p c}{\omega_s}\hat{\delta}'\cos\theta' = \frac{\alpha_p c}{\omega_s}\hat{\delta}\cos\theta$$

$$\therefore \quad \hat{\delta}'^2 = \hat{\delta}^2 - 2\hat{\delta}\frac{u}{E_0}\sin\theta + \frac{u^2}{E_0^2}$$



Averaging over the bunch gives:

$$\Delta\sigma_\delta^2 = \frac{\langle u^2\rangle}{2E_0^2} \qquad \text{where} \qquad \sigma_\delta^2 = \frac{1}{2}\langle\hat{\delta}^2\rangle$$

Quantum excitation of synchrotron oscillations

Including the effects of radiation damping, the evolution of the energy spread is:

$$\frac{d\sigma_\delta^2}{dt} = \frac{1}{2E_0^2 C_0}\oint \mathcal{N}\langle u^2\rangle\, ds - \frac{2}{\tau_z}\sigma_\delta^2$$

From the previous expression for $\mathcal{N}\langle u^2\rangle$, we find:

$$\frac{d\sigma_\delta^2}{dt} = C_q\gamma^2\frac{2}{j_z\tau_z}\frac{I_3}{I_2} - \frac{2}{\tau_z}\sigma_\delta^2$$

We find the equilibrium energy spread from $d\sigma_\delta^2/dt = 0$:

$$\sigma_{\delta 0}^2 = C_q\gamma^2\frac{I_3}{j_z I_2}$$

The third synchrotron radiation integral I_3 is given by:

$$I_3 = \oint \frac{1}{|\rho^3|}\, ds$$

Natural energy spread

The equilibrium energy spread determined by radiation effects is:

$$\sigma_{\delta 0}^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}$$

This is often referred to as the “natural” energy spread, since collective effects can often lead to an increase in the energy spread with increasing bunch charge.

The natural energy spread is determined essentially by the beam energy and by the bending radii of the dipoles. Note that the natural energy spread *does not depend on the RF parameters (either voltage or frequency)*.

The corresponding equilibrium bunch length is:

$$\sigma_z = \frac{\alpha_p c}{\omega_s} \sigma_\delta$$

We can increase the synchrotron frequency ω_s, and hence reduce the bunch length, by increasing the RF voltage, or by increasing the RF frequency.

Summary: radiation damping

Including the effects of radiation damping and quantum excitation, the emittances vary as:

$$\varepsilon(t) = \varepsilon(0)\exp\left(-\frac{2t}{\tau}\right) + \varepsilon(\infty)\left[1 - \exp\left(-\frac{2t}{\tau}\right)\right]$$

The damping times are given by:

$$j_x \tau_x = j_y \tau_y = j_z \tau_z = 2\frac{E_0}{U_0} T_0$$

The damping partition numbers are given by:

$$j_x = 1 - \frac{I_4}{I_2} \qquad j_y = 1 \qquad j_z = 2 + \frac{I_4}{I_2}$$

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2 \qquad C_\gamma = 8.846 \times 10^{-5} \text{ m/GeV}^3$$

Summary: equilibrium beam sizes

The natural emittance is:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2} \qquad C_q = 3.832 \times 10^{-13} \text{ m}$$

The natural energy spread and bunch length are given by:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{j_z I_2} \qquad \sigma_z = \frac{\alpha_p c}{\omega_s} \sigma_\delta$$

The momentum compaction factor is:

$$\alpha_p = \frac{I_1}{C_0}$$

The synchrotron frequency and synchronous phase are given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0} \frac{\omega_{RF}}{T_0} \alpha_p \cos\varphi_s \qquad \sin\varphi_s = \frac{U_0}{eV_{RF}}$$

Summary: synchrotron radiation integrals

The synchrotron radiation integrals are:

$$I_1 = \oint \frac{\eta_x}{\rho} ds$$

$$I_2 = \oint \frac{1}{\rho^2} ds$$

$$I_3 = \oint \frac{1}{|\rho|^3} ds$$

$$I_4 = \oint \frac{\eta_x}{\rho} \left(\frac{1}{\rho^2} + 2k_1 \right) ds \qquad k_1 = \frac{e}{P_0} \frac{\partial B_y}{\partial x}$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3} ds \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$

Damping times in the ILC damping rings

Let us consider the damping time that we need in the ILC. The shortest damping time is set by the vertical emittance of the positron beam.

	Injected emittance	Extracted emittance
Horizontal e^+	1 μm	0.8 nm
Vertical e^+	1 μm	0.002 nm
Longitudinal e^+	> 30 μm	10 μm

We must reduce the injected emittance to the extracted emittance in the store time of 200 ms (set by the repetition rate of the main linac).

Using:

$$\varepsilon(t) = \varepsilon(0)\exp\left(-2\frac{t}{\tau}\right)$$

we find that we need a vertical damping time of 30 ms. In practice, the damping time must be less than this, to allow for a non-zero equilibrium.

Damping times in the ILC damping rings

The ILC damping rings are 6.7 km in circumference and have a beam energy of 5 GeV.

The energy loss per turn is:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2$$

If the only dipole fields are those that determine the ring geometry, and have field strength B, then we can write:

$$I_2 = \oint \frac{1}{\rho^2}\, ds = \frac{B}{B\rho}\oint \frac{ds}{\rho} = 2\pi \frac{eB}{P_0}$$

Hence:

$$U_0 = C_\gamma E_0^3 ecB$$

For a dipole field of 0.15 T, we find U_0 = 500 keV.

Damping times in the ILC damping rings

The beam energy is 5 GeV; the energy loss per turn from the dipoles is 500 keV. This means that the vertical damping time is:

$$\tau_y = 2\frac{E_0}{U_0}T_0 = 450\ \text{ms}$$

We need a damping time of less than 30 ms; the radiation from the dipoles provides a damping time of 450 ms!

To reduce the damping time, we need to increase the energy loss per turn. Increasing the dipole field can help, but has adverse impact on other aspects of the dynamics (the momentum compaction factor is reduced, which lowers some of the instability thresholds).

The other option is to use a *damping wiggler*, which consists of a sequence of dipoles bending in opposite directions… we will discuss wigglers used to enhance radiation damping in Lecture 4.