Problems Lecture 1: Linac Basics

1) Calculate the relative longitudinal motion of two particles with an energy of $9\,\mathrm{GeV}$ and a difference of 3% over a distance of $21\,\mathrm{km}$.

2) Calculate the solutions to Hill's equation for $K(s) = K_0 > 0$.

3) Calculate the solutions to Hill's equation for $K(s) = 0$ assuming $\beta(s = 0) = \beta_0$ and $\beta'(s = 0) = 0$.

4) How much energy is roughly stored in one ILC cavity at nominal gradient?

Solutions: Linac Basics

1) We calculate

$$\gamma = \frac{E_0}{mc^2} \approx \frac{9\,\mathrm{GeV}}{0.511\,\mathrm{MeV}} \approx 18000$$

then we use

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

to find an approximation for β

$$\beta \approx 1 - \frac{1}{2\gamma^2} \approx 1 - 1.5 \times 10^{-9}$$

over the length of a linac $21\,\mathrm{km}$ the longitudinal delay compared to light is $\approx 32\,\mu\mathrm{m}$ for two particles which have a energy difference of $\Delta\gamma$ the relative longitudinal motion would be

$$\beta_1 - \beta_2 \approx \frac{1}{2\gamma^2} - \frac{1}{2(\gamma+\Delta\gamma)^2} \approx \frac{\Delta\gamma}{\gamma^2}$$

for an example of 3% the motion is $\approx 1\,\mu\mathrm{m} \ll \sigma_z$

Note: Due to the acceleration the effect is even smaller

Solutions: Linac Basics

2) We use $K(s) = \mathrm{const} > 0$.

- We know the solution is a harmonic oszillation with a fixed amplitude, which we call β_0
- We now need to check that this fulfills the differential equation for β

- Ansatz: $\beta = \beta_0$, $\beta' = 0$:

$$\frac{\beta''\beta}{2} - \frac{\beta'^2}{4} + K\beta^2 = 1$$
$$\Rightarrow K\beta_0^2 = 1$$

Hence

$$\beta_0 = \frac{1}{\sqrt{K}}$$

Solutions: Linac Basics

3) $K = 0$

Ansatz: β is a polynom of second order

- We use $\beta'(s=0) = 0$, $\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$

$$\frac{\beta''\beta}{2} - \frac{\beta'^2}{4} + K\beta^2$$

$$\Rightarrow \frac{\beta''\beta}{2} - \frac{\beta'^2}{4} = 1$$

$$\Rightarrow \frac{1}{2}\left(\beta_0 + \frac{s^2}{\beta_0}\right)\left(\frac{2}{\beta_0}\right) - \frac{1}{4}\left(\beta_0 + \frac{2s}{\beta_0}\right)\left(\frac{2s}{\beta_0}\right) = 1$$

$$\Rightarrow 1 + \frac{s^2}{\beta_0^2} - \frac{s^2}{\beta_0^2} = 1$$

4) Assuming $R/Q = 1\,\mathrm{k\Omega/m}$ we find approximately $120\,\mathrm{J}$

Problems Lecture 2: Lattice Design

1) A transport lattice with no acceleration consists of FODO cells with quadrupole spacing $L = 10\,\mathrm{m}$ and focal distance $f = 10\,\mathrm{m}$. How large is the phase advance?

2) Estimate the RMS beam jitter at a position with $\beta(s_2) = 1\,\mathrm{m}$ if one quadrupole jitters 450° upstream with a focal length $f = 7\,\mathrm{m}$ and $\beta(s_1) = 10\,\mathrm{m}$. The quadrupole jitter amplitude has an RMS of $1\,\mu\mathrm{m}$.

3) Calculate the average beta-function in a thin lens FODO lattice as a function of $\hat{\beta}$, $\check{\beta}$ and L/f

How much does a cavity with tilt $\theta \ll 1$ deflect the beam?

Solutions

1) We use

$$\cos\mu = 1 - \frac{L^2}{2f^2}$$
$$\Rightarrow \cos\mu = 1 - \frac{1}{2}$$
$$\Rightarrow \mu = \arccos\left(\frac{1}{2}\right) = 60^\circ$$

2) The angular deflection is given by the offset δ and the focal strength f

$$y' = \frac{\delta}{f}$$

we transform into nromalised phase space

$$y'_N = \sqrt{\beta(s_1)}\frac{\delta}{f}$$

450° downstream this is

$$y_N = \sqrt{\beta(s_1)}\frac{\delta}{f}$$

which translates into

$$y = \sqrt{\beta(s_1)\beta(s_2)}\frac{\delta}{f}$$

inserting number we find

$$y \approx 0.45\delta$$

hence the RMS jitter $\sigma_{y,jitt} = 0.45\,\mu\mathrm{m}$.

Solutions

3) We will integrate from the centre of a defocusing quadrupole (at $s = 0$) to the centre of the next focusing quadrupole (at $s = L$). In the centre of the defocusing quadrupole we have $\beta = \check{\beta}$ and $\alpha = 0$. We calculate the Twiss parameters immediately after the quadrupole (at $\epsilon \to 0$):

$$\begin{pmatrix} \beta(\epsilon) & -\alpha(\epsilon) \\ -\alpha(\epsilon) & \gamma(\epsilon) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/(2f) & 1 \end{pmatrix} \begin{pmatrix} \check{\beta} & 0 \\ 0 & 1/\check{\beta} \end{pmatrix} \begin{pmatrix} 1 & 1/(2f) \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \beta(\epsilon) & -\alpha(\epsilon) \\ -\alpha(\epsilon) & \gamma(\epsilon) \end{pmatrix} = \begin{pmatrix} \check{\beta} & \check{\beta}/(2f) \\ \check{\beta}/(2f) & 1/\check{\beta} + \check{\beta}/(2f)^2 \end{pmatrix}$$

now we calculate beta along a drift using

$$\begin{pmatrix} \beta(s) & -\alpha(s) \\ -\alpha(s) & \gamma(s) \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \check{\beta} & \check{\beta}/(2f) \\ \check{\beta}/(2f) & 1/\check{\beta} + \check{\beta}/(2f)^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ L & 1 \end{pmatrix}$$

$$\beta(s) = \check{\beta} + \frac{\check{\beta}}{f} s + \left(\frac{1}{\check{\beta}} + \frac{\check{\beta}}{4f^2} \right) s^2$$

$$\langle \beta \rangle = \frac{1}{L} \int_0^L \beta(s) ds = \check{\beta} + \frac{\check{\beta}}{2f} L + \frac{L^2}{3} \left(\frac{1}{\check{\beta}} + \frac{\check{\beta}}{4f^2} \right)$$

to avoid to much calculation we exploit

$$\beta(L) = \hat{\beta} = \check{\beta} + \frac{\check{\beta}}{f} L + \left(\frac{1}{\check{\beta}} + \frac{\check{\beta}}{4f^2} \right) L^2$$

hence

$$\langle \beta \rangle = \frac{2}{3} \check{\beta} + \frac{1}{3} \check{\beta} + \frac{L}{6f} \check{\beta}$$

Solutions

4) The deflection of the beam by a single structure of length L and gradient G with tilt $\theta \ll 1$ is

$$\delta y' = \frac{eGL}{2}\frac{1}{E}\theta = \frac{\delta}{2}\theta$$

δ is the relative acceleration by the cavity.

Relevant is the kick in the normalised coordinates

$$\sqrt{\gamma}\begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} 0 \\ \delta y' \end{pmatrix} = \sqrt{\gamma}\begin{pmatrix} 0 \\ \sqrt{\beta}\delta y' \end{pmatrix} = \sqrt{\beta}\frac{\delta}{2}\theta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For the emittance the square of the kick is important