Cornell Laboratory for Accelerator-based Sciences and Education (CLASSE)

Damping Rings

FIFTH INTERNATIONAL ACCELERATOR SCHOOL FOR LINEAR COLLIDERS

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Lecture A3, Part 1 – Damping Ring Basics

A. Introduction to Damping Rings

B. General Linear Beam Dynamics

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Damping Ring Lectures

Lecture A3, Part 1 – Damping Ring Basics
- Introduction to Damping Rings
- General Linear Beam Dynamics

Lecture A3, Part 2 – Low Emittance Ring Design
- Radiation Damping and Equilibrium Emittance
- Damping Ring Lattices

Lecture A3, Part 3 – Damping Ring Technical Systems
- Systems Overview
- Review of Selected Systems for ILC and CLIC
- R&D Challenges

Lecture A3, Part 4 – Beam Dynamics
- Overview of Impedance and Instability Issues
- Review of Selected Collective Effects
- R&D Challenges

Damping Rings Lecture – Part 1

Our objectives for today's lecture are to:

Examine the role of the damping rings in the ILC accelerator complex;

Review the parameters of the CLIC and ILC damping rings and identify *key challenges* in the design and construction of these machines;

Review the basic physics of storage rings including the linear beam dynamics;

Looking ahead to tomorrow:

Review radiation damping and equilibrium emittance;

Apply the above principles to the CLIC and ILC damping rings to begin to understand the major *design choices* that have been made

Outline of DR Lecture I, Part 1

Damping Rings Introduction

- Role of Damping Rings
- ILC Damping Ring Parameters and Design Issues
- The Issue of *Design Optimization*
- CLIC Damping Ring Parameters and Design Issues
- Summary

General Linear Beam Dynamics

- Storage Ring Equations of Motion
- Betatron Motion
- Twiss Parameters
- Emittance
- Coupling
- Dispersion
- Chromaticity

Some Introductory Comments I

Over the next few days we will be covering a significant amount of material

- The basic physics of storage/damping rings (unfortunately there is a great deal we won't have time to cover in detail)
- A significant amount of time will be spent on:
  - How design choices have been made
  - Looking at the technical challenges of both the CLIC and ILC damping ring designs

It is widely acknowledged that the sub-system in a linear collider complex with the *"most physics"* is the damping rings

- Some of the greatest technical and physics risks, for both the CLIC and ILC designs, reside in the DRs
- This has required a robust, ongoing R&D program for the damping rings
- It means that a great deal of what we will be discussing is not subject to simply calculating a correct solution, but rather going beyond existing solutions…

Some Introductory Comments II

My hope is to fully engage you in each of the issues we discuss!

So, please don't hesitate to jump in with questions during the lecture

Also don't hesitate to explore alternative concepts

- Many issues don't have unique solutions…
- Many issues are strongly influenced by the interfaces with the upstream and downstream sub-systems
- We need physicists/researchers in the field who are prepared to take on these types of challenges

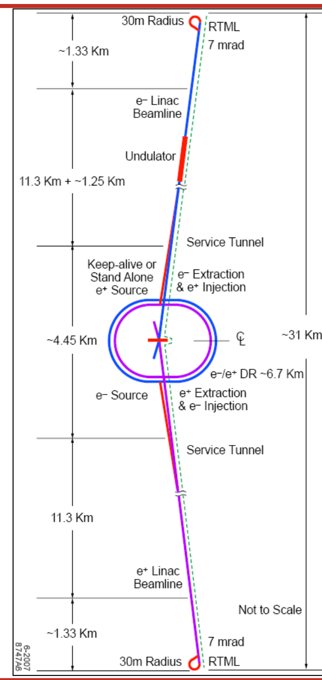
Ultimately I hope that a few of you will be sufficiently intrigued by these topics and challenges to join us in constructing a set of linear collider damping rings in a few years…

Role of the Damping Rings

The damping rings

- Accept e^+ and e^- beams with large transverse and longitudinal emittance and produce the ultra-low emittance beams necessary for high luminosity collisions at the IP
- Damp longitudinal and transverse jitter in the incoming beams to provide very stable beams for delivery to the IP
- Delay bunches from the source to allow feed-forward systems to compensate for pulse-to-pulse variations

ILC RDR Layout ⇨

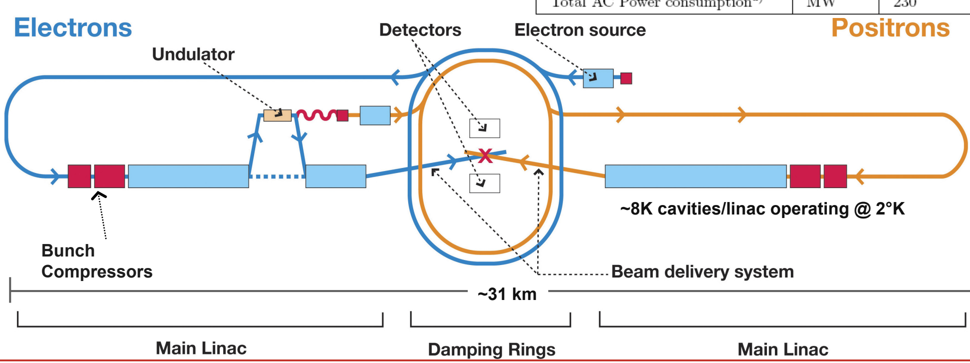


The ILC Reference Design

Machine Configuration

- Helical Undulator polarized e^+ source
- Two ~6.5 km damping rings in a central complex
- RTML running length of linac
- 2 ×11.2 km Main Linac
- Single Beam Delivery System
- 2 Detectors in Push-Pull Configuration

Parameter	Unit	
Center-of-mass energy range	GeV	200 - 500
Peak luminosity[a]	$cm^{-2}s^{-1}$	2×10^{34}
Average beam current in pulse	mA	9.0
Pulse rate	Hz	5.0
Pulse length (beam)	ms	~ 1
Number of bunches per pulse		1000 - 5400
Charge per bunch	nC	1.6 - 3.2
Accelerating gradient[a]	MV/m	31.5
RF pulse length	ms	1.6
Beam power (per beam)[a]	MW	10.8
Typical beam size at IP[a] ($h \times v$)	nm	640×5.7
Total AC Power consumption[a]	MW	230



DR Reference Design Parameters

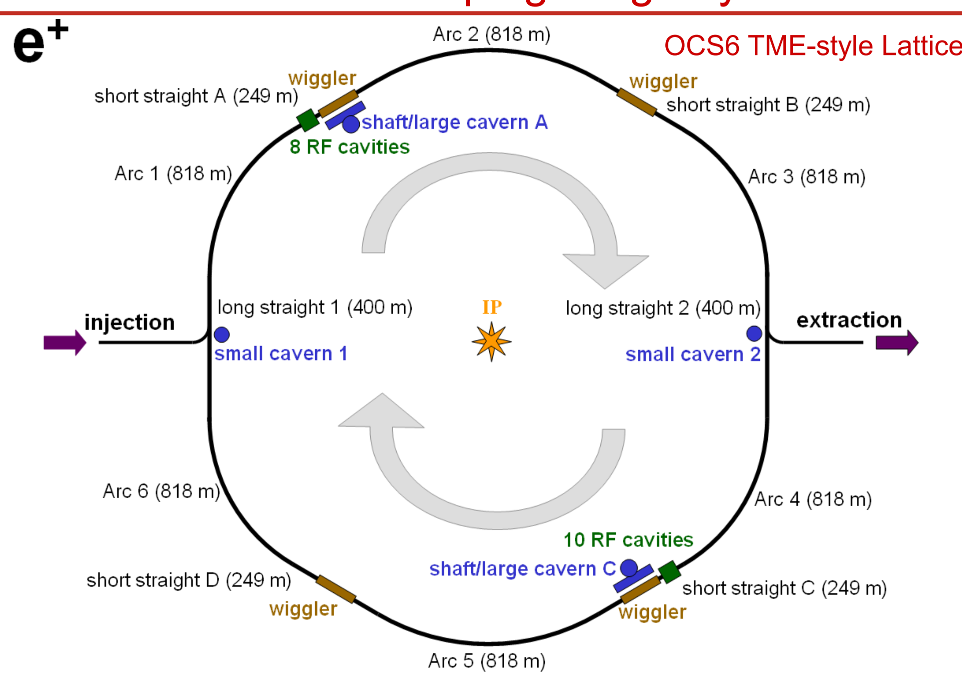
By the end of the first 2 days of lectures, the goal is for each of you to be able to explain the reasons that the parameters in this table have the values that are specified. *Caveat: Some parameters have already been changed*

By the end of the DR lectures, you should be able to identify and explain why several of these parameters are *(or already have been)* candidates for further optimization.

So, let's begin our tour of ring dynamics and what these parameters mean…

Parameter	Units	Value
Energy	GeV	5.0
Circumference	km	6.695
Nominal # of bunches & particles/bunch		$2625@2.0\times10^{10}$
Maximum # of bunches & particles/bunch		$5534@1.0\times10^{10}$
Average current	A	0.4
Energy loss per turn	MeV	8.7
Beam power	MW	3.5
Nominal bunch current	mA	0.14
RF Frequency	MHz	650
Total RF voltage	MV	24
RF bucket height	%	1.5
Injected betatron amplitude, $A_x + A_y$	m·rad	0.09
Equilibrium normalized emittance, $\gamma\varepsilon_x$	μm·rad	5.0
Chromaticity, χ_x/χ_y		-63/-62
Partition numbers, J_x		0.9998
J_y		1.0000
J_z		2.0002
Harmonic number, h		14,516
Synchrotron tune, ν_s		0.067
Synchrotron frequency, f_s	kHz	3.0
Momentum compaction, α_c		4.2×10^{-4}
Horizontal/vertical betatron tunes, ν_x/ν_y		52.40/49.31
Bunch length, σ_z	mm	9.0
Momentum spread, σ_p/p		1.28×10^{-3}
Horizontal damping time, τ_x	ms	25.7
Longitudinal damping time, τ_z	ms	12.9

The RDR Damping Ring Layout



ILC Damping Ring Design Inputs

A number of parameters in the previous table are (*essentially)* design *inputs* for the damping rings (or can be directly inferred from such inputs). The table below summarizes these critical interface issues.

We will examine these requirements from the perspective of the collision point first and then look at requirements coming from other sub-systems downstream and upstream of the DRs.

Particles per bunch	1×10^{10} - 2×10^{10}	Upper limit set by disruption at IP.
Max. Avg. current in main linac	~9 mA	Upper limit set by RF technology.
Machine repetition rate	5 Hz	Set by cryogenic cooling capacity. Partially determines required damping time.
Max. Linac RF pulse length	~1 ms	Upper limit set by RF technology.
Min. Particles per machine pulse	$\sim5.6\times10^{13}$	Lower limit set by luminosity goal.
Injected normalized emittance	0.01 m-rad	Set by positron source. Partially determines required damping time.
Injected energy spread	±0.5%	Set by positron source.
Injected betatron amplitude (A_x+A_y)	0.09 m-rad	Set by positron source.
Extracted normalized emittances	8 μm horizontally 20 nm vertically	Set by luminosity goal.
Max. Extracted bunch length	9 mm (⇨6 mm)	Upper limit set by bunch compressors.
Max. Extracted energy spread	0.15%	Upper limit set by bunch compressors.

Don't forget, however, that these parameters are the result of a great deal of back-and-forth negotiation between sub-systems and between accelerator and HEP physicists. Thus they represent a mix of technological limits and physics desires…

Downstream Requirements

The principle parameter driver is the production of luminosity at the collision point

$$\mathcal{L} = \frac{N^2 f_{coll}}{4\pi\sigma_x\sigma_y}\mathcal{H}_D$$

where

N is the number of particles per bunch (*assumed equal for all bunches*)

f_{coll} is the overall collision rate at the interaction point (IP)

σ_x and σ_y are the horizontal and vertical beam sizes (*assumed equal for all bunches*)

$\mathcal{H}_D$ is the luminosity enhancement factor

Ideally we want:

- High intensity bunches
- High repetition rate
- Small transverse beam sizes

Parameters at the Interaction Point

The parameters at the interaction point have been chosen to provide a nominal luminosity of 2×10^{34} $cm^{-2}s^{-1}$. With

- $N = 2 \times 10^{10}$ particles/bunch
- $\sigma_x \sim 640$ nm $\Leftrightarrow$ $\beta_x^* = 20$ mm, $\varepsilon_x = 20$ pm-rad
- $\sigma_y \sim 5.7$ nm $\Leftrightarrow$ $\beta_y^* = 0.4$ mm, $\varepsilon_y = 0.08$ pm-rad
- $\mathcal{H}_D \sim 1.7$

$$\mathcal{L} = \frac{N^2 f_{coll}}{4\pi\sigma_x\sigma_y}\mathcal{H}_D = \left(1.4 \times 10^{30} cm^{-2}\right) \times f_{coll}$$

In order to achieve the desired luminosity, an average collision rate of ~14kHz is required (we will return to this parameter shortly). The beam sizes at the IP are determined by the strength of the final focus magnets and the emittance, phase space volume, of the incoming bunches.

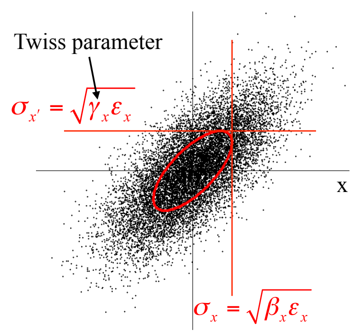
A number of issues impact the choice of the final focus parameters. For example, the beam-beam interaction as two bunches pass through each other can enhance the luminosity, however, it also disrupts the bunches. If the beams are too badly disrupted, safely transporting them out of the detector to the beam dumps becomes quite difficult. Another effect is that of beamstrahlung which leads to significant energy losses by the particles in the bunches and can lead to unacceptable detector backgrounds. Thus the above parameter choices represent a complicated optimization.

Emittance Transport from the DR to the IP

The geometric emittances required at the IP are:

- $\varepsilon_x = 20$ pm-rad
- $\varepsilon_y = 0.08$ pm-rad

We need to use the relativistic invariant quantity, the *normalized emittance*, in order to project this to the requirements for the damping ring.

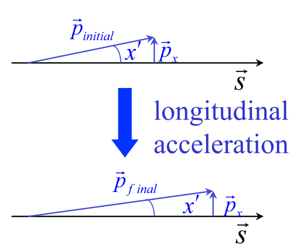


Note: We will take a more detailed look at emittance in the DR in tomorrow's lecture

Normalized Emittance:

Use of the conjugate phase-space coordinates (x, p_x) from the Hamiltonian instead of (x, x') gives:

$$p_x = px' = mc\beta\gamma x'$$



Thus we define the normalized emittance as

$\varepsilon_n = \beta\gamma\varepsilon_{geo} \approx \gamma\varepsilon_{geo}$ for a relativistic electron

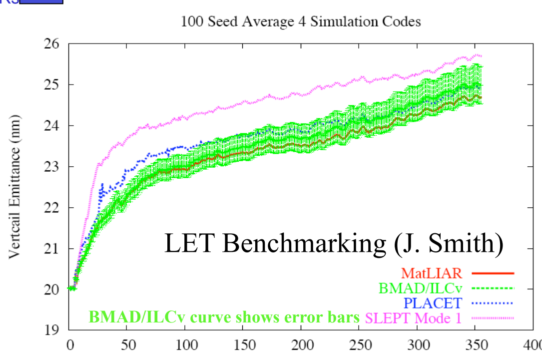
Emittance Transport from the DR to the IP

We can now infer the requirements for the equilibrium emittance requirements for the ILC DRs

	ε_{geo} @ IP (250 GeV)	ε_n @ IP	Equilibrium ε_n @ DR	Equilibrium ε_{geo} @ DR (5 GeV)
x	20 pm-rad	10 μm-rad	½ × (10 μm-rad)	0.5 nm
y	0.08 pm-rad	40 nm-rad	½ × (40 nm-rad)	2 pm

Allow for 100% vertical emittance growth downstream of DRs

DR extracted emittances must allow for downstream emittance growth during transport as well as for the finite damping time during the machine pulse cycle



ILC Main Linac (ML) Parameters

The bunch-train structure is largely determined by the design of the superconducting RF system of the main linac (ML)

<u>Primary Limitation</u>

- 1 ms RF pulse
- 9 mA average current in each pulse } RF power system
- 5 Hz repetition rate } Cryogenic load

This leads to the nominal bunch train parameters:

n_b = 2625 bunches per pulse

Δt_b ~ 380 ns for uniform loading through pulse

The resulting collision rate at the IP is then

f_{coll} = 13.1 kHz

consistent with the target luminosity. The 5 Hz repetition rate places the primary constraint on the DR damping times. In order for the bunches in each pulse to experience 8 full damping cycles, a transverse damping time of ≤25 ms is required.

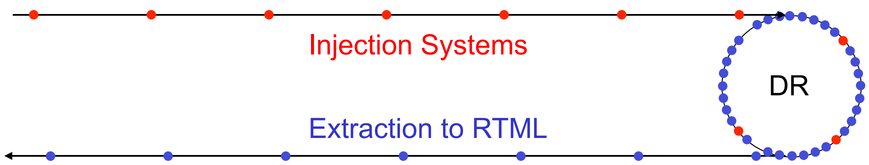
Baseline Bunch Train

From the discussion on the preceding page, we can now see the basic bunch train structure

- 1 msec pulse
- ~3000 uniformly spaced bunches
- ~350 ns between bunches

⇨ Train Length of $\sim$300km $\gg$ ML length > DR Circumference

Thus, the damping rings must act as a *reservoir* to store the full train. Because we cannot afford to build a 300+ km ring, we must *fold* the long bunch train into a much shorter ring ⇨ key trade-offs between bunch spacing and ring circumference.



Note that (for the RDR baseline) there will be significant overlap between the injection and extraction cycles:

– Structure of machine
– Maintain relatively constant beam loading

Bunch Compressors

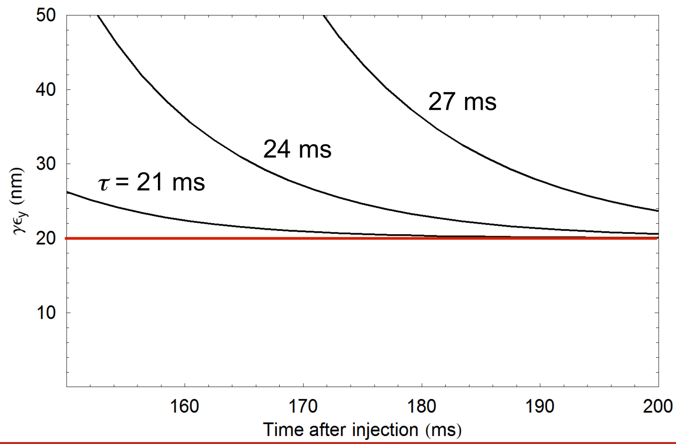
Shortly after extraction from the damping ring, the bunches will traverse the bunch compressors. These devices take the relatively long bunches of the damping rings ($\sigma_z \sim$ fraction of a centimeter) and manipulate the longitudinal phase space to provide bunches that are compatible with the very small focal point at the IP ($\sigma_z \sim$ 200-500 microns). Technical and cost limitations place serious constraints on how long the bunch from the DR can be and the maximum energy spread.

RDR DR Bunch length: 9 mm ⇨ 2-stage bunch compressor
Extracted energy spread within the bunch compressor acceptance

From the downstream point of view, lowering the bunch length to 6mm would allow the cheaper and simpler solution of using a single stage bunch compressor. From the DR point of view, shorter bunches require smaller values of the ring momentum compaction (impacts sensitivity to collective effects) or higher RF voltage (more RF units, hence greater cost).

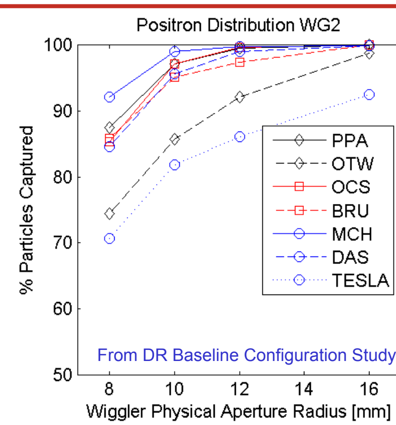
Upstream Requirements

The key upstream requirement is the emittance of the beams produced by the injectors. Positron production via a heavy metal target results in much larger emittances due to scattering in the target for positrons than for electrons whose emittance can be controlled by the design of the injector gun and its cathode. The approach to the target extraction emittance is shown for various DR damping times assuming the target e+ injected emittance ($\varepsilon_n = 0.01$ m-rad).



Upstream Requirements

In addition to the need to damp the large emittance beams that are injected from the positron source, the injected beams are expected to have potentially large betatron amplitudes and energy errors. This requires that the acceptance of the damping ring to be sufficiently large to accommodate these oscillations immediately after injection. It places important constraints on the minimum aperture of the vacuum system and the minimum good field regions of all of the magnets (including the damping wigglers).



Particle capture rates assuming that the limiting physical aperture in the damping rings is due to the vacuum chambers in the wiggler regions. The choice of a superferric wiggler design, with large physical aperture, allows for a DR design with full acceptance.

Arriving at a design

We have now looked at several interface issues between the damping rings and the rest of the accelerator complex

- Train structure
- Equilibrium emittance requirements
- Bunch length requirements
- Acceptance of ring
- Timing structure

There are various choices that can be made to design a ring at this point

- The choices typically have a myriad of trade-offs
- Will look at a few examples to understand the design evaluations that are required
- Design choices must be carefully matched to likely paths of evolution of the overall machine design

Optimization Issues - I

Optimization is complicated. Many decisions are tightly coupled and many trade-offs are required.

Example 1: Ring Circumference

- Large circumference ⇨ space charge effects are more severe
- If space charge effects are significant ⇨ a higher energy is desirable
- Higher energy ⇨ larger equilibrium emittance
- Control of equilibrium emittance ⇨ significant impacts on ring design

- Small circumference ⇨ fewer components and smaller tunnel so cheaper and potentially better net hardware reliability
- Small circumference ⇨ folding of linac bunch train into ring requires more closely spaced bunches
- Closely spaced bunches ⇨ more challenging bunch-by-bunch injection and extraction
- Closely spaced bunches ⇨ electron cloud and fast ion effects more severe

Optimization Issues - II

Example 2: Beam Energy

- Higher energy ➪ sensitivity to collective effects is lessened (beam instabilities, intrabeam scattering, space charge, etc)
- Higher energy ➪ damping rates increase from the increased synchrotron radiation
- Higher energy ➪ for a given normalized emittance from the sources, the beam is smaller due to adiabatic damping from the initial beam acceleration and the ring acceptance issues are eased

- Lower energy ➪ in the limit of small enough bunch charge, this provides a smaller equilibrium emttance
- Lower energy ➪ weaker magnets and lower field RF cavities to focus the beam, hence cheaper (and often more reliable) hardware

Optimization Issues - III

Example 3: Technical Contraints: High Voltage Kickers

- Wide kicker pulse ➪ typically more stable, hence better for uniform injection/extraction
- Wide kicker pulse ➪ requires a large ring circumference to allow bunch-by-bunch injection and extraction (bunch spacing)
- Wide kicker pulse ➪ relatively fewer kicker structures (matched to pulse width) will be required in the ring (minimize impedance issues, improve reliability, minimize cost)
- Wide kicker pulse ➪ works well in a scenario with full train injection/extraction

- Narrow kicker pulse ➪ higher bandwidth requires careful impedance matching with kicker structure
- Narrow kicker pulse ➪ many short kicker structures required (reliability and cost concerns)
- Narrow kicker pulse ➪ high voltage pulses beyond state-of-the-art when the ILC RDR was published

Optimization Issues - IV

Example 4: Technical Constraints: Damping Wigglers

- Competing technologies:
  - Permanent magnet
  - Normal conducting electromagnet
  - Superconducting electromagnet
- Performance issues:
  - Aperture
  - Allowable field strength
  - Field quality
  - Sensitivity to radiation damage
  - Operating cost
- ILC design choice:

  Employ only a damping ring with no pre-damping ring
  - Places significant weight on aperture and field quality issues in order to handle the large input beams from the positron source

Optimization Issues - V

Example 5: Physics Requests

- Provide wider energy range for producing luminosity ⇨ for the ILC, this affects the positron production mode
- Positron production at fixed energy point in main linac ⇨ if want to explore a lower energy, need to produce positrons on one pulse and then change the acceleration in the ML for collisions on a separate pulse
- Two pulse configurations ⇨ positron damping ring only filled 50% of time
- 50% duty cycle ⇨ new RF system design
- 50% duty cycle ⇨ increase damping rate so that 5Hz pulses for collision can be maintained
- Lower positron production energy ⇨ poorer production and inability to achieve desired standard operating parameters
- Lower positron production energy ⇨ potentially unacceptable impact on the positron target design

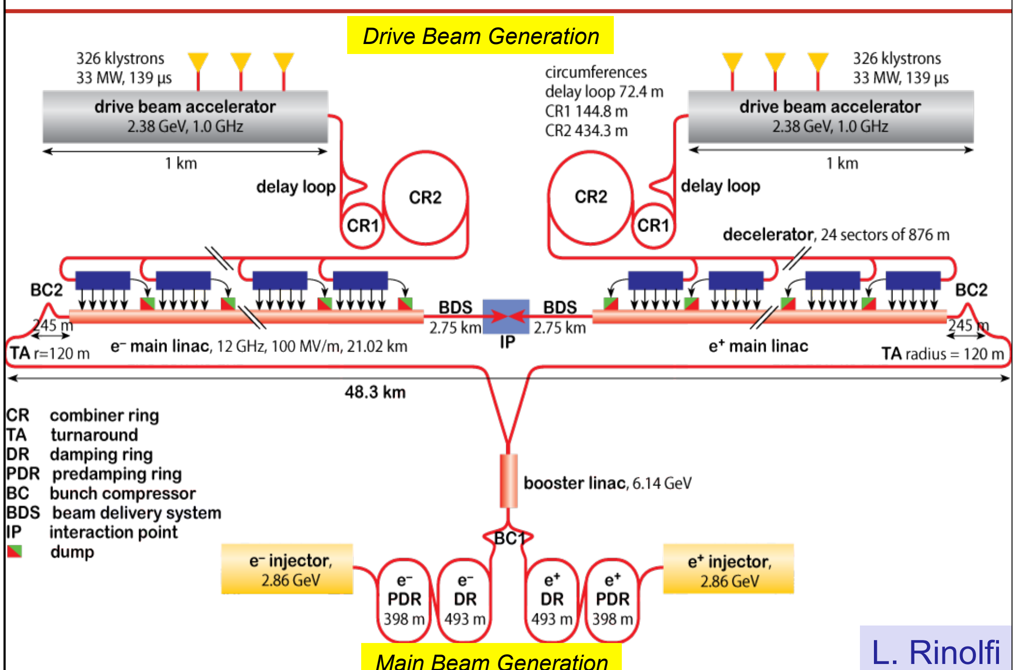
ILC DR Design

The ILC DR baseline configuration is able to meet the key design parameters required for the baseline design

- Validation of the various design choices continues
- Major limiting areas of operational concern identified for further R&D included
  - Achievement of 2pm vertical emittance
  - Electron Cloud effects
  - Fast Ion effects
  - Ability to stably inject and extract closely spaced bunches
- An aggressive R&D program has been underway for the past 2 years to try to address these issues
- The design continues to evolve as we iterate the overall ILC machine design to achieve maximum value…

Before going any further, however, let's look at the CLIC damping ring design…

General CLIC Layout for 3 TeV



CLIC DR Design

Present state of the CLIC DR Design

As you may already be able to see, these parameters are very different from the ILC DR case

This is driven primarily by the differences between the hot and cold main linac RF design

Parameters	Value
Energy [GeV]	2.86
Circumference [m]	420.56
Coupling	0.0013
Energy loss/turn [MeV]	4.2
RF voltage [MV]	4.9
Natural chromaticity x / y	-168/-60
Momentum compaction factor	8e-5
Damping time x / s [ms]	1.9/ 0.96
Dynamic aperture x / y [σ_{inj}]	30 / 120
Number of dipoles/wigglers	100/52
Cell /dipole length [m]	2.36 / 0.43
Dipole/Wiggler field [T]	1.4/2.5
Bend gradient [1/m^2]	-1.10
Max. Quad. gradient [T/m]	73.4
Max. Sext. strength [kT/m^2]	6.6
Phase advance x / z	0.452/0.056
Bunch population, [10^9]	4.1
IBS growth factor	1.4
Hor./ Ver Norm. Emittance [nm.rad]	400 / 4.5
Bunch length [mm]	1.6

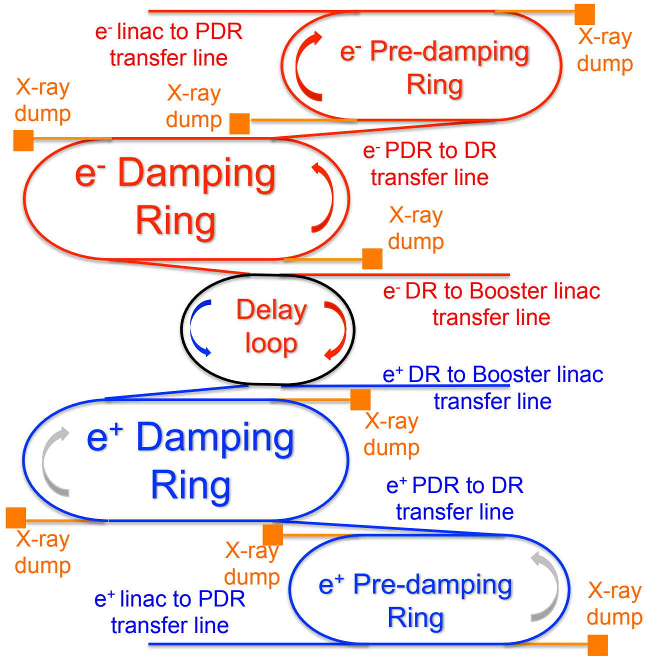
Some ILC-CLIC Comparisons

Parameter	Units	ILC DR (RDR)	CLIC DR
Energy	GeV	5.0	2.86
Circumference	km	6.695	0.42056
Nominal # of bunches & particles/bunch		2625@2.0×10^{10}	312@0.41×10^{10}
Macropulse Repetition Rate	Hz	5	50
Average current	A	0.4	0.15
Energy loss per turn	MeV	8.7	4.2
RF Frequency	MHz	650	2000
Total RF voltage	MV	24	4.9
Equilibrium normalized emittance, $\gamma\varepsilon_x$	μm·rad	5.0	0.4
Natural Chromaticity, χ_x/χ_y		-63/-62	-168/-60
Momentum compaction, α_c		4.2×10^{-4}	8×10^{-5}
Bunch length, σ_z	mm	9.0	1.6
Momentum spread, σ_p/p		1.3×10^{-3}	1.4×10^{-3}
Horizontal damping time, τ_x	ms	25.7	1.9
Longitudinal damping time, τ_z	ms	12.9	0.96

CLIC DR Complex

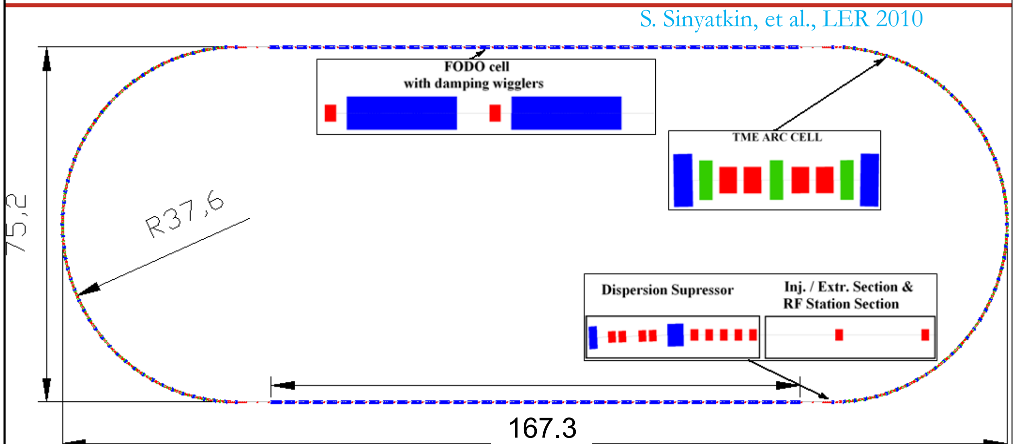
Normal conducting RF design in ML allows train-level injection/extraction instead of bunch-by-bunch

The use of pre-damping rings relaxes the dynamic aperture and energy acceptance requirements in the damping rings



CLIC DR Layout

S. Sinyatkin, et al., LER 2010



Racetrack shape with

- 96 TME arc cells (4 half cells for dispersion suppression)
- 26 Damping wiggler FODO cells in the long straight sections (LSS)
- Space reserved upstream in the LSS for injection/extraction elements and RF cavities

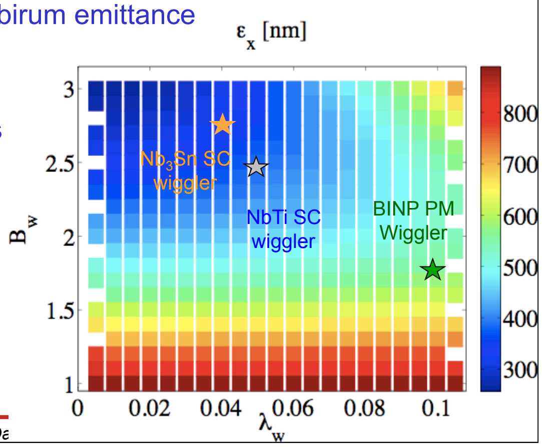
CLIC Damping Ring Challenges I

In the presence of the pre-damping rings, the major challenges for dynamic aperture and energy acceptance move upstream of the damping rings

Major issues for damping rings include:

- The repetition rate (50 Hz) requires very short damping times
- High charge density in each bunch means Intrabeam Scattering has a significant impact on the equilibirum emittance

⇨ Both of these issues drive the damping wiggler specifications to a very high field design which can only be achieved with superconducting technology



CLIC Damping Ring Challenges II

Achieving the necessary equilibrium emittance requires careful lattice design

- Target Emittance sensitive to:
  - IBS, which must be directly taken into account – it's not a small perturbation which is unlike any other rings of this type
  - E_{ring}
  - Achievable wiggler parameters
- A very strongly focusing lattice requires particular care with:
  - Magnet strengths
  - Alignment tolerances

Collective Instabilities

- Electron Cloud in the positron ring
- Fast Ion Instability in the electron ring
- Space charge plays a major role in the energy and circumference choice

CLIC Damping Ring Challenges III

Repetition rate and bunch structure

- 0.5 ns bunch spacing to match main linac structure
- 2 GHz RF System – Examples of other rings:
  - SLAC-PEPII LER – 476 MHz
  - LBNL-ALS – 500 MHz
  - KEKB – 500 MHz SC
  - CESR – 500 MHz SC
  - KEK-ATF – 714 MHz
  - ILC DR – 650 MHz SC
  - CLIC DR – 2000 MHz
- Requires
  - New power source design
  - Demonstrated capability to handle high peak and average currents

CLIC Damping Ring Challenges IV

With the extremely small beam sizes at the IP, exquisite pulse stability, $O(10^{-4})$ is required

- Similar to ILC DR
- However the pulser requirements, which must inject/extract the whole train in each pulse, is conceptually simpler
  - ILC DR pulse width ~6ns
  - CLIC DR pulse width ~160ns

So, while the design challenges involve many of the same issues for the ILC DR and the CLIC DR, the actual operating parameters give rise to distinctly different designs with different issues being the dominant ones.

Summary

At this point we have completed an overview of some of the key design issues for the CLIC and ILC damping rings

These rings offer a range of challenges both to the lattice designers as well as the technical designers who must come up with reliable implementations of hardware that meet the design specifications

I hope that you walk away from this portion of the lecture with an appreciation for how complicated trade-offs are required to meet aggressive physics specifications

In the next part of this lecture we will spend some time looking at the basic physics of storage rings in order to provide further insight into the details of such decisions

Storage Ring Basics

Now we will begin our review of storage ring basics. In particular, we will cover:

- Ring Equations of Motion
- Betatron Motion
- Emittance
- Transverse Coupling
- Dispersion and Chromaticity
- Momentum Compaction Factor
- Radiation Damping and Equilibrium Beam Properties

Equations of Motion

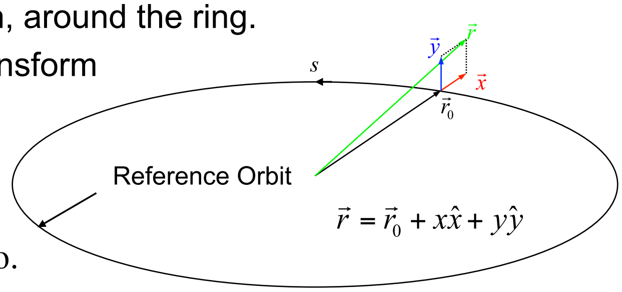
Particle motion in electromagnetic fields is governed by the Lorentz force: $\frac{d\vec{p}}{dt} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$

with the corresponding Hamiltonian: $\mathcal{H} = c\left[m^2c^2 + \left(\vec{P} - e\vec{A}\right)^2\right]^{1/2} + e\Phi$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial P_x},\ \dot{P}_x = -\frac{\partial \mathcal{H}}{\partial x},\ldots$$

For circular machines, it is convenient to convert to a curvilinear coordinate system and change the independent variable from time to the location, s-position, around the ring.

In order to do this we transform to the *Frenet-Serret* coordinate system. The local radius of curvature is denoted by ρ.



Equations of Motion

With a suitable canonical transformation, we can re-write the Hamiltonian as:

$$\tilde{\mathcal{H}} = -\left(1 + \frac{x}{\rho}\right)\left[\frac{\left(\mathcal{H} - e\Phi\right)^2}{c^2} - m^2c^2 - \left(p_x - eA_x\right)^2 - \left(p_y - eA_y\right)^2\right]^{1/2} - eA_s$$

Using the relations $E = \mathcal{H} - e\Phi\,,\quad p = \sqrt{\frac{E^2}{c^2} - m^2c^2}$

and expanding to 2nd order in p_x and p_y yields:

$$\tilde{\mathcal{H}} \approx -p\left(1 + \frac{x}{\rho}\right) + \frac{1 + x/\rho}{2p}\left[\left(p_x - eA_x\right)^2 - \left(p_y - eA_y\right)^2\right] - eA_s$$

which is now periodic in s.

Equations of Motion

Thus, in the absence of synchrotron motion, we can generate the equations of motion with:

$$x' = \frac{\partial \tilde{\mathcal{H}}}{\partial p_x}, \quad p_x' = -\frac{\partial \tilde{\mathcal{H}}}{\partial x}, \quad y' = \frac{\partial \tilde{\mathcal{H}}}{\partial p_y}, \quad p_y' = -\frac{\partial \tilde{\mathcal{H}}}{\partial y}$$

which yields:

$$x'' - \frac{\rho + x}{\rho^2} = \pm \frac{B_y}{B\rho} \frac{p_0}{p} \left(1 + \frac{x}{\rho}\right)^2, \quad \text{top / bottom sign for + / - charges}$$

and

$$y'' = \mp \frac{B_x}{B\rho} \frac{p_0}{p} \left(1 + \frac{x}{\rho}\right)^2$$

Note: $1/B\rho$ is the *beam rigidity* and is taken to be positive

Specific field configurations are applied in an accelerator to achieve the desired manipulation of the particle beams. Thus, before going further, it is useful to look at the types of fields of interest via the multipole expansion of the transverse field components.

Magnetic Field Multipole Expansion

Magnetic elements with 2-dimensional fields of the form

$$\vec{B} = B_x(x,y)\hat{x} + B_y(x,y)\hat{y}$$

can be expanded in a complex multipole expansion:

$$B_y(x,y) + iB_x(x,y) = B_0 \sum_{n=0}^{\infty} (b_n + ia_n)(x+iy)^n$$

$$\text{with } b_n = \frac{1}{n!B_0} \frac{\partial^n B_y}{\partial x^n}\bigg|_{(x,y)=(0,0)} \quad \text{and } a_n = \frac{1}{n!B_0} \frac{\partial^n B_x}{\partial x^n}\bigg|_{(x,y)=(0,0)}$$

In this form, we can normalize to the main guide field strength, $-B\hat{y}$, by setting $b_0=1$ to yield:

$$\frac{1}{B\rho}(B_y + iB_x) = \frac{e}{p_0}(B_y + iB_x) = \mp \frac{1}{\rho} \sum_{n=0}^{\infty} (b_n + ia_n)(x+iy)^n \quad \text{for } \pm q$$

Multipole Moments

Upright Fields	Skew Fields
Dipole:	Dipole ($\theta = 90°$):
$\frac{e}{p_0}B_x = 0$ $\frac{e}{p_0}B_y = \kappa_x$	$\frac{e}{p_0}B_x = -\kappa_y$ $\frac{e}{p_0}B_y = 0$
Quadrupole:	Quadrupole ($\theta = 45°$):
$\frac{e}{p_0}B_x = ky$ $\frac{e}{p_0}B_y = kx$	$\frac{e}{p_0}B_x = -k_{skew}x$ $\frac{e}{p_0}B_y = k_{skew}y$
Sextupole:	Sextupole ($\theta = 30°$):
$\frac{e}{p_0}B_x = mxy$ $\frac{e}{p_0}B_y = \frac{1}{2}m\left(x^2 - y^2\right)$	$\frac{e}{p_0}B_x = -\frac{1}{2}m_{skew}\left(x^2 - y^2\right)$ $\frac{e}{p_0}B_y = m_{skew}xy$
Octupole:	Octupole ($\theta = 22.5°$):
$\frac{e}{p_0}B_x = \frac{1}{6}r\left(3x^2y - y^3\right)$ $\frac{e}{p_0}B_y = \frac{1}{6}r\left(x^3 - 3xy^2\right)$	$\frac{e}{p_0}B_x = -\frac{1}{6}r_{skew}\left(x^3 - 3xy^2\right)$ $\frac{e}{p_0}B_y = \frac{1}{6}r_{skew}\left(3x^2y - y^3\right)$

Equations of Motion (Hill's Equation)

We next want to consider the equations of motion for a ring with only guide (dipole) and focusing (quadrupole) elements:

$$B_y = \mp B_0 + \frac{p_0}{e}kx = B_0\left(\rho kx \mp 1\right) \quad \text{and} \quad B_x = \frac{p_0}{e}ky = B_0\rho kx$$

Taking $p=p_0$ and expanding the equations of motion to first order in x/ρ and y/ρ gives:

$$x'' + K_x(s)x = 0, \qquad K_x(s) = \frac{1}{\rho^2(s)} \mp k(s)$$

$$y'' + K_y(s)y = 0, \qquad K_y(s) = \pm k(s)$$

also commonly denoted as k_1

where the upper/low signs are for a positively/negatively charged particle.

The focusing functions are periodic in s: $K_{x,y}(s+L) = K_{x,y}(s)$

Solutions to Hill's Equation

Some introductory comments about the solutions to Hill's equations:

- The solutions to Hill's equation describe the particle motion around a reference orbit, the *closed orbit*. This motion is known as betatron motion. We are generally interested in small amplitude motions around the closed orbit (as has already been assumed in the derivation of the preceding pages).
- Accelerators are generally designed with discrete components which have locally uniform magnetic fields. In other words, the focusing functions, $K(s)$, can typically be represented in a piecewise constant manner. This allows us to locally solve for the characteristics of the motion and implement the solution in terms of a *transfer matrix*. For each segment for which we have a solution, we can then take a particle's initial conditions at the entrance to the segment and transform it to the final conditions at the exit.

Solutions to Hill's Equation

Let's begin by considering constant $K=k$:

$$x'' + kx = 0$$

where x now represents either x or y. The 3 solutions are:

$x(s) = a\sin\left(\sqrt{k}s\right) + b\cos\left(\sqrt{k}s\right),$	$k > 0$	Focusing Quadrupole
$x(s) = as + b,$	$k = 0$	Drift Region
$x(s) = a\sinh\left(\sqrt{\lvert k\rvert}s\right) + b\cosh\left(\sqrt{\lvert k\rvert}s\right),$	$k < 0$	Defocusing Quadrupole

For each of these cases, we can solve for initial conditions and recast in 2×2 matrix form:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\vec{x} = \mathbf{M}\left(s \middle| s_0\right)\vec{x}_0$$

Transfer Matrices

We can now re-write the solutions of the preceding page in transfer matrix form:

$$\mathbf{M}(s|s_0) = \begin{cases} \begin{pmatrix} \cos\left(\sqrt{k}\,\ell\right) & \frac{1}{\sqrt{k}}\sin\left(\sqrt{k}\,\ell\right) \\ -\sqrt{k}\sin\left(\sqrt{k}\,\ell\right) & \cos\left(\sqrt{k}\,\ell\right) \end{pmatrix} & \text{Focusing Quadrupole} \\ \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} & \text{Drift Region} \\ \begin{pmatrix} \cosh\left(\sqrt{|k|}\,\ell\right) & \frac{1}{\sqrt{|k|}}\sinh\left(\sqrt{|k|}\,\ell\right) \\ \sqrt{|k|}\sinh\left(\sqrt{|k|}\,\ell\right) & \cosh\left(\sqrt{|k|}\,\ell\right) \end{pmatrix} & \text{Defocusing Quadrupole} \end{cases}$$

where $\ell = s - s_0.$

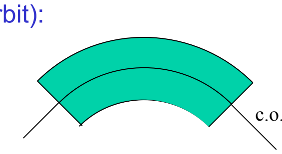
Transfer Matrices

Examples:

- Thin lens approximation: $\ell \to 0, \qquad f = \lim_{\ell \to 0} \frac{1}{|K|\,\ell}$

$$\mathbf{M}_{\text{focusing}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \qquad \mathbf{M}_{\text{defocusing}} = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

- Sector dipole (entrance and exit faces ⊥ to closed orbit):



$$\mathbf{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix} \approx \begin{pmatrix} 1 & \ell \\ -\frac{\ell}{\rho^2} & 1 \end{pmatrix}$$

where $\theta = \frac{\ell}{\rho}$

Transfer Matrices

Transport through an interval $s_0 \rightarrow s_2$ can be written as the product of 2 transport matrices for the intervals $s_0 \rightarrow s_1$ and $s_1 \rightarrow s_2$:

$$\mathbf{M}(s_2|s_0) = \mathbf{M}(s_2|s_1)\mathbf{M}(s_1|s_0)$$

and the determinant of each transfer matrix is: $|\mathbf{M}_i| = 1$

Many rings are composed of repeated sets of identical magnetic elements. In this case it is particularly straightforward to write the *one-turn matrix* for P superperiods, each of length L, as:

$$\mathbf{M}_{ring} = \left[\mathbf{M}(s+L|s)\right]^P$$

with the boundary condition that: $\mathbf{M}(s+L|s) = \mathbf{M}(s)$

The multi-turn matrix for m revolutions is then: $\left[\mathbf{M}(s)\right]^{mP}$

Twiss Parameters

The generalized one turn matrix can be written as:

$$\mathbf{M} = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix} = \mathbf{I}\cos\Phi + \mathbf{J}\sin\Phi$$

Identity matrix

This is the most general form of the matrix. α, β, and γ are known as either the Courant-Snyder or Twiss parameters (note: they have nothing to do with the familiar relativistic parameters) and Φ is the betatron phase advance. The matrix **J** has the properties:

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \quad \mathbf{J}^2 = -\mathbf{I} \Leftrightarrow \beta\gamma = 1 + \alpha^2$$

The n-turn matrix can be expressed as: $\mathbf{M}^n = \mathbf{I}\cos(n\Phi) + \mathbf{J}\sin(n\Phi)$ which leads to the stability requirement for betatron motion:

$$\left|\mathrm{Trace}(\mathbf{M})\right| = 2\cos\Phi \le 2$$

The Envelope Equations

We will look for 2 independent solutions to Hill's Equation of the form: $x(s) = aw(s)e^{i\psi(s)}$ and $x^*(s) = aw(s)e^{-i\psi(s)}$

Then w and ψ satisfy:

$$w'' + Kw - \frac{1}{w^3} = 0$$

$$\psi' = \frac{1}{w^2}$$

Betatron envelope and phase equations

Since any solution can be written as a superposition of the above solutions, we can write [with $w_i = w(s_i)$]:

$$\mathbf{M}(s_2|s_1) = \begin{pmatrix} \frac{w_2}{w_1}\cos\psi - w_2 w_1' \sin\psi & w_1 w_2 \sin\psi \\ -\frac{(1 + w_1 w_1' w_2 w_2')}{w_1 w_2}\sin\psi - \left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1}\right)\cos\psi & \frac{w_1}{w_2}\cos\psi + w_1 w_2' \sin\psi \end{pmatrix}$$

The Envelope Equations

Application of the previous transfer matrix to a full turn and direct comparison with the Courant-Snyder form yields:

$$w^2 = \beta$$

$$\alpha = -ww' = -\frac{\beta'}{2}$$

the betatron envelope equation becomes

$$\frac{1}{2}\beta'' + K\beta - \frac{1}{\beta}\left[1 + \frac{\beta'^2}{4}\right] = 0$$

and the transfer matrix in terms of the Twiss parameters can immediately be written as:

$$\mathbf{M}(s_2|s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\Delta\psi + \alpha_1 \sin\Delta\psi) & \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ -\frac{1 + \alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\Delta\psi + \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1\beta_2}}\cos\Delta\psi & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\Delta\psi - \alpha_2 \sin\Delta\psi) \end{pmatrix}$$

General Solution to Hill's Equation

The general solution to Hill's equation can now be written as:

$$x(s) = A\sqrt{\beta_x(s)}\cos\left[\psi_x(s)+\phi_0\right] \quad \text{where} \quad \psi_x(s) = \int_0^s \frac{ds}{\beta_x(s)}$$

We can now define the *betatron tune* for a ring as:

$$Q_x = \nu_x = \frac{\Phi_{turn}}{2\pi} = \frac{1}{2\pi}\int_s^{s+C}\frac{ds}{\beta_x(s)} \quad \text{where} \quad C = \text{ring circumference}$$

If we make the coordinate transformation:

$$z = \frac{x}{\sqrt{\beta_x}} \quad \text{and} \quad \xi(s) = \frac{1}{\nu_x}\int_0^s \frac{ds}{\beta_x(s)}$$

we see that particles in the beam satisfy the equation for simple harmonic motion:

$$\frac{d^2 z}{d\xi^2} + \nu_x^2 z = 0$$

The Courant-Snyder Invariant

With K real, Hill's equation is conservative. We can now take

$$x(s) = A\sqrt{\beta_x(s)}\cos\left[\psi_x(s)+\phi_0\right] \quad \text{and}$$

$$x'(s) = -\frac{A}{\sqrt{\beta_x(s)}}\left\{\alpha(s)\cos\left[\psi_x(s)+\phi_0\right] + \sin\left[\psi_x(s)+\phi_0\right]\right\}$$

After some manipulation, we can combine these two equations to give:

Conserved quantity

$$A^2 = \varepsilon = \frac{x^2}{\beta_x(s)} + \left[\frac{\alpha_x(s)}{\sqrt{\beta_x(s)}}x + \sqrt{\beta_x(s)}x'\right]^2$$

Recalling that $\beta\gamma = 1+\alpha^2$ yields:

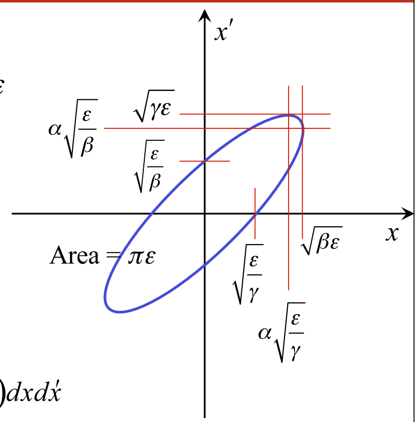
$$A^2 = \varepsilon = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s)$$

Emittance

The equation

$$\gamma(s)x^2(s)+2\alpha(s)x(s)x'(s)+\beta(s)x'^2(s)=\varepsilon$$

describes an ellipse with area $\pi\varepsilon$.



For an ensemble of particles, each following its own ellipse, we can define the moments of the beam as:

$$\langle x\rangle=\int x\rho(x,x')dxdx' \qquad \langle x'\rangle=\int x'\rho(x,x')dxdx'$$

$$\sigma_x^2=\int\left(x-\langle x\rangle\right)^2\rho(x,x')dxdx' \qquad \sigma_{x'}^2=\int\left(x'-\langle x'\rangle\right)^2\rho(x,x')dxdx'$$

$$\sigma_{xx'}^2=\int\left(x-\langle x\rangle\right)\left(x'-\langle x'\rangle\right)\rho(x,x')dxdx'=r\sigma_x\sigma_{x'}$$

The rms emittance of the beam is then $\varepsilon_{rms}=\sqrt{\sigma_x^2\sigma_{x'}^2-\sigma_{xx'}^2}=\frac{\langle A^2\rangle}{2}$

which is the area enclosed by the ellipse of an *rms particle*.

Coupling

Up to this point, the equations of motion that we have considered have been independent in x and y. An important issue for all accelerators, and particularly for damping rings which attempt to achieve a very small vertical emittance, is coupling between the two planes. For the damping ring, we are primarily interested in the coupling that arises due to small rotations of the quadrupoles. This introduces a *skew quadrupole* component to the equations of motion.

$$x''+K_x(s)x=0 \;\Rightarrow\; x''+K_x(s)x+k_{skew}y=0$$

$$y''+K_y(s)y=0 \;\Rightarrow\; y''+K_y(s)y+k_{skew}x=0$$

Another skew quadrupole term arises from “feed-down” when the closed orbit is displaced vertically in a sextupole magnet. In this case the effective skew quadrupole moment is given by the product of the sextupole strength and the closed orbit offset

$$k_{skew}=my_{co}$$

Coupling

For uncoupled motion, we can convert the 2D (x,x') and (y,y') transfer matrices to 4D form for the vector (x,x',y,y'):

$$\mathbf{M}_{4\mathrm{D}}\left(s\middle|s_0\right)=\begin{pmatrix}\mathbf{M}_{\mathrm{focusing}} & 0\\ 0 & \mathbf{M}_{\mathrm{defocusing}}\end{pmatrix}=\begin{pmatrix}\mathbf{M}_{\mathrm{F}} & 0\\ 0 & \mathbf{M}_{\mathrm{D}}\end{pmatrix}$$

where we have arbitrarily chosen this case to be focusing in x. The matrix is block diagonal and there is no coupling between the two planes. If the quadrupole is rotated by angle θ, the transfer matrix becomes:

$$\mathbf{M}_{\mathrm{skew}}=\begin{pmatrix}\mathbf{M}_{\mathrm{F}}\cos^2\theta+\mathbf{M}_{\mathrm{D}}\sin^2\theta & \sin\theta\cos\theta\left(\mathbf{M}_{\mathrm{D}}-\mathbf{M}_{\mathrm{F}}\right)\\ \sin\theta\cos\theta\left(\mathbf{M}_{\mathrm{D}}-\mathbf{M}_{\mathrm{F}}\right) & \mathbf{M}_{\mathrm{D}}\cos^2\theta+\mathbf{M}_{\mathrm{F}}\sin^2\theta\end{pmatrix}$$

and motion in the two planes is coupled.

Coupling and Emittance

Later in this lecture series we will look in greater detail at the sources of vertical emittance for the damping rings.

In the absence of coupling and ring errors, the vertical emittance of a ring is determined by the the radiation of photons and the fact that emitted photons are randomly radiated into a characteristic cone with half-angle $\theta_{1/2}\sim 1/\gamma$. This quantum limit to the vertical emittance is generally quite small and can be ignored for presently operating storage rings.

Thus the presence of betatron coupling becomes one of the primary sources of vertical emittance in a storage ring.

Dispersion

In our initial derivation of Hill's equation, we assumed that the particles being guided had the design momentum, p_0, thus ignoring longitudinal contributions to the motion. We now want to address off-energy particles. Thus we take the equation of motion:

$$x'' - \frac{\rho + x}{\rho^2} = \pm \frac{B_y}{B\rho} \frac{p_0}{p} \left(1 + \frac{x}{\rho}\right)^2$$

and expand to lowest order in $\delta = \frac{\Delta p}{p_0}$ and $\frac{x}{\rho}$ which yields:

$$x'' + K(s)x = \frac{\delta}{\rho}$$

We have already obtained a homogenous solution, $x_\beta(s)$. If we denote the particular solution as $D(s)\delta$, the general solution is:

$$x = x_\beta(s) + D(s)\delta$$

Dispersion Function and Momentum Compaction

The dispersion function satisfies: $D'' + K(s)D = 1/\rho$

with the boundary conditions: $D(s+L) = D(s);\ D'(s+L) = D'(s)$

The solution can be written as the sum of the solution to the homogenous equation and a particular solution:

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = \mathbf{M}(s_2|s_1) \begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix}$$

which can be expressed in a 3×3 matrix form as:

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{M}(s_2|s_1) & \vec{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}, \quad \textbf{where} \quad \vec{d} = \begin{pmatrix} d \\ d' \end{pmatrix}$$

Momentum Compaction

We can now consider the difference in path length experienced by such an off-momentum particle as it traverses the ring. The path length of an on-momentum particle is given by: $C = \oint \frac{x_{c.o.}}{\rho} ds$

For the off-momentum case, we then have: $\Delta C = \delta \times \oint \frac{D(s)}{\rho} ds = I_1 \delta$

I_1 is the first *radiation integral*.

The momentum compaction factor, α_c, is defined as:

$$\alpha_c = \frac{\Delta C / C}{\delta} = \frac{I_1}{C}$$

The Synchrotron Radiation Integrals

I_1 is the first of 5 "radiation integrals" that we will study in this lecture. These 5 integrals describe the key properties of a storage ring lattice including:

- Momentum compaction
- Average power radiated by a particle on each revolution
- The radiation excitation and average energy spread of the beam
- The *damping partition numbers* describing how radiation damping is distributed among longitudinal and transverse modes of oscillation
- The natural emittance of the lattice

In later sections of this lecture we will work through the key aspects of radiation damping in a storage ring

Chromaticity

An off-momentum particle passing through a quadrupole will be under/over-focused for positive/negative momentum deviation. This is chromatic aberration. Hill's equation becomes:

$$x'' + \left[K_0(s)(1-\delta)\right]x = 0$$

We will evaluate the chromaticity by first looking at the impact of local gradient errors on the particle beam dynamics.

Effect of a Gradient Error

We consider a local perturbation of the focusing strength $K = K_0 + \Delta K$. The effect of ΔK can be represented by including a thin lens transfer matrix in the one-turn matrix. Thus we have

$$\mathbf{M}_{\Delta K} = \begin{pmatrix} 1 & 0 \\ -\Delta K\ell & 1 \end{pmatrix}$$

and

$$\mathbf{M}_{1-turn} = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\Phi_0 + \alpha\sin\Phi_0 & \beta\sin\Phi_0 \\ -\gamma\sin\Phi_0 & \cos\Phi_0 - \alpha\sin\Phi_0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\Delta K\ell & 1 \end{pmatrix}$$

With $\Phi=\Phi_0+\Delta\Phi$, we can take the trace of the one-turn matrix to give:

$$\cos(\Phi_0 + \Delta\Phi) = \cos\Phi_0 - \frac{1}{2}\beta\Delta K\ell\sin\Phi_0$$

Effect of a Gradient Error

Using the relation: $\cos(\Phi_0 + \Delta\Phi) = \cos\Delta\Phi \cos\Phi_0 - \sin\Delta\Phi \sin\Phi_0$

we can identify: $\Delta\Phi \approx \frac{1}{2}\beta\Delta K \ell$

Thus we can write: $\Delta Q = \frac{1}{4\pi}\beta\Delta K \ell$

and we see that the result of gradient errors is a shift in the betatron tune. For a distributed set of errors, we then have:

$$\Delta Q = \frac{1}{4\pi}\oint \beta \Delta K ds$$

which is the result we need for evaluating chromatic aberrations. *Note that the tune shift will be positive/negative for a focusing/ defocusing quadrupole.*

Chromaticity

We can now write the betatron tune shift due to chromatic aberration as:

$$\Delta Q = \frac{1}{4\pi}\oint \beta \Delta K ds \approx -\frac{\delta}{4\pi}\oint \beta\, K ds$$

The chromaticity is defined as the change in tune with respect to the momentum deviation:

$$C = \frac{\partial Q}{\partial \delta}$$

Because the focusing is weaker for a higher momentum particle, the natural chromaticity due to quadrupoles is always negative. This can be a source of instabilities in an accelerator. However, the fact that a momentum deviation results in a change in trajectory (the dispersion) as well as the change in focusing strength, provides a route to mitigate this difficulty.

Sextupoles

Recall that the magnetic field in a sextupole can be written as:

$$\frac{e}{p_0}B_x = mxy \qquad \frac{e}{p_0}B_y = \frac{1}{2}m\left(x^2 - y^2\right)$$

Using the orbit of an off-momentum particle $x = x_\beta(s) + D(s)\delta$

we obtain $$\frac{e}{p_0}B_x = mD(s)\delta y_\beta(s) + mx_\beta(s)y$$

and $$\frac{e}{p_0}B_y = mD(s)\delta x_\beta(s) + \frac{1}{2}mD^2(s)\delta^2 + \frac{1}{2}m\left[x_\beta^2(s) - y_\beta^2(s)\right]$$

where the first terms in each expression are a quadrupole feed-down term for the off-momentum particle. Thus the sextupoles can be used to compensate the chromatic error. The change in tune due to the sextupole is

$$\Delta Q = \frac{\delta}{4\pi}\oint mD(s)\beta(s)ds$$

Summary

During the last portion of today's lecture, we have begun our walk through the basics of storage/damping ring physics.

We will pick up this discussion tomorrow with the effect known as radiation damping which is central to the operation of all lepton collider, storage and damping rings.

Once we have completed that discussion we will look in greater detail at the lattice choices that have been made for the damping rings and how these lattices are presently being forced to evolve.

In the first part of today's lecture we had an overview of the key design issues impacting the damping ring lattice. The homework problems will provide an opportunity to become more familiar with some of these issues.

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