HOM Mitigation

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Purpose and Audience

The purpose of the course is to enable students to become well-versed in the HOMs which give rise wakefield-beam interaction in accelerators and in particular to understand means of suppressing these HOMs. It is suitable for advanced undergraduates, graduate students and, active researchers in the field.

Objectives

This course will address the fundamentals of wakefields and their relation to the beam impedance. The features of both long-range and short-range wakefields will be discussed. Circuit models of relativistic electron beams coupled to multiple accelerator cavities will be developed to calculate the coupled modal frequencies. Practical methods to suppress the wakefields will be described with techniques taken from ongoing research (L-band and X-band linacs in particular). Throughout the course, basic physical principles such as superposition, energy conservation and causality will be emphasized.

Instructional Method

The will be two lectures. A homework assignment will be available.

Course Content

The progress of multiple bunches of electrons through a linear or circular accelerator gives rise to a trailing electromagnetic field. This wakefield can have catastrophic consequences if its progress is left unchecked as the beam can become unstable and develop a BBU (Beam Break Up) instability. This course discusses issues associated with wakefields and means of damping the fields to acceptable levels. Examples are taken from the recent international next generation linear colliders damping schemes.

Essential Reading

R. M. Jones, *Wake field Suppression in High Gradient Linacs for Lepton Linear Colliders*, Phys. Rev. ST Accel. Beams 12, 104801, 2009.

Background Reading

RF Linear Accelerators, Wiley & Sons Publishers (1998), by Thomas Wangler
"*RF Superconductivity for Accelerators*", Wiley Publishers (1998), by Hasan Padamsee, Jens Knobloch and Tom Hays
Physics of Collective Beam Instabilities in High Energy Accelerators (free pdf download) , Wiley & Sons Publishers (1993) by Alexander Chao
The Physics of Particle Accelerators: An Introduction, Oxford University Press (2000) by Klaus Wille
Fundamentals of Beam Physics, Oxford University Press (2003) by James Rosenzweig
"Particle Accelerator Physics I & II", (study edition) Springer-Verlag (2003) by Helmut Wiedeman
Impedances and Wakes in High Energy Particle Accelerators, World Scientific Publishers (1998), by Bruno W Zotter and Semyon Kheifets

2010 CAS Lecture Series:

HOM MITIGATION

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In this course, wakefields are analyzed and practical structures which limit emittance growth are demonstrated. A familiarity with fundamental concepts of accelerator physics is assumed. Basic features of wakefields are outlined and detailed results on wakefield minimization and beam diagnostics based on the ILC (International Linear Collider) superconducting L-band linacs are described together with the features of X-band normal conducting linacs.

1. **Part I**: Basic concepts and definitions are introduced. The features of short-range and long-range wakefields are sketched out. Resistive wall wake. Impedance and relation to wakefield.

OVERVIEW

- **The Transverse Wakefield Problem**
- **Wakefield Definitions**
- **Wakefield Examples and Methods of Calculation**
- **Wakefield Fundamentals - Panofsky-Wenzel Theorem**
- **Modal Sum Representation of Wakefield via Field Function Analysis (appendix)**
- **R-L-C Circuit Model of Single Mode and Impedance-Wake Relations**
- **Pill-Box Wake Function (short and Long-range wakes)**

Introduction

- **In this course, we will focus on ultra relativistic beams. The quantity under consideration has a finite value as the particle velocity approaches the velocity of light –often we will use $v = c$.**

- **The bunch lengths under consideration are sub-millimeter and even tens of microns (ILC, LCLS).**

- **I will try to maintain S.I units throughout –for the sake of consistency.**

General Properties of Wakefields

- **Longitudinal wakes give rise to energy spread over the train of bunches being accelerated.**

- **Transverse wakes in high frequency linear accelerators (linacs), if left unchecked, can readily *dilute the emittance of the beam.***

- **Can give rise to instability that causes the beam to oscillate transversely –**
  ***Beam Break Up (BBU) instability*.**

- **The instability that develops in a linac is a single pass instability.**
  - **In circular accelerators the effect is cumulative and the feedback mechanism amplifies the growth turn-by-turn. The growth is $\propto \exp(\Gamma t)$**
  - **We will analyze the growth effect in linacs.**

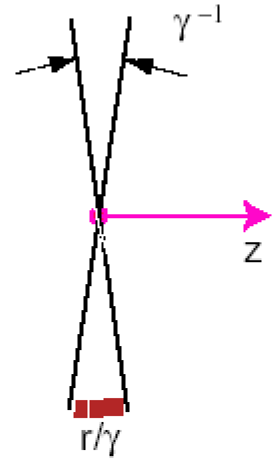
- **It is crucial to damp the wakefields such they are not an issue!**

- **Or, why not try to make use of the trailing wakefield to accelerate beam? Plasma wakefield accelerators are a possibility! Plasma wakefield acceleration/focusing is ongoing research at SLAC.**

- **In order to optimize the cost of acceleration high energy linacs accelerate multiple bunches of electrons/positrons within an rf pulse train.**

Relativistic Point Charge in Free Space

- **A point at rest has an isotropic distribution of electric field**



- **Consider point charge moving in the z-direction**
- **For v~c then: γ>>1 and the field is squeezed in the longitudinal direction**

- **The field is limited to a "pancake-like" region and in the limit of v=c the field is entirely transverse (the pancake has zero angular spread)**
- **The field for a particle moving with constant velocity is given by:**

$$\mathbf{E} = \frac{q\mathbf{R}}{4\pi\varepsilon_0 \gamma R_*^3}, \quad H = Y_0 \frac{\mathbf{v}}{c} \times \mathbf{E} \qquad \textbf{(1.1)}$$

where the vector R is drawn from the center of the charge q, to the observation point, $R_*^2 = z^2 + r^2/\gamma^2$, **and** $\gamma = \sqrt{1 - v^2/c^2}$ **.**

$$E_r(z,r) = \frac{q\gamma r}{4\pi\varepsilon_0 \left(z^2\gamma^2 + r^2\right)^{3/2}} \qquad \textbf{(1.2)}$$

- **In the pancake region, z~r/γ, or angle 1/γ:**

$$E_r = Z_0 H_\phi \sim \frac{\gamma q}{4\pi\varepsilon_0 r^2}, \quad E_z \sim \frac{q}{4\pi\varepsilon_0 r^2} \qquad \textbf{(1.3)}$$

- **No net power is transferred transverse to the particles motion, but there is a non-zero Poynting flux flowing parallel to the particle and attached to it**

Plane Wave Fourier Decomposition

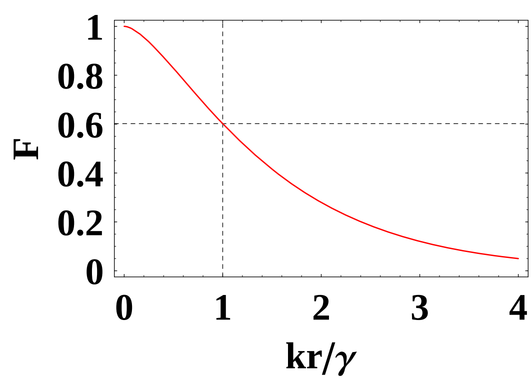
- **In the ultrarelativistic limit v->c the beam field is a plane wave electromagnetic field**
- **We decompose the field by means the Fourier transform:**

$$E_r(z,r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz \frac{q\gamma r}{4\pi\varepsilon_0\left(z^2\gamma^2+r^2\right)^{3/2}} e^{ikz} = \frac{qk}{4\pi^2\varepsilon_0\gamma r} K_1\left(\frac{kr}{\gamma}\right) = \frac{q}{4\pi^2\varepsilon_0 r^2} F\left(\frac{kr}{\gamma}\right)$$

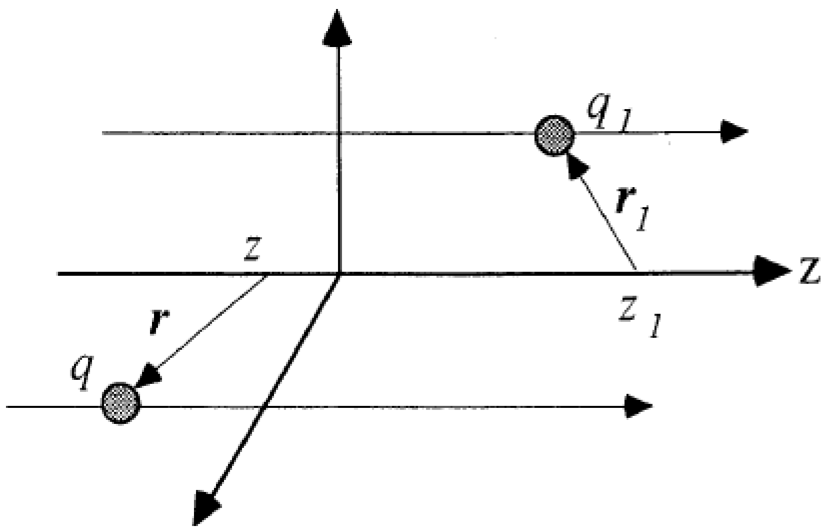
(1.4)

- **The beam field is thus a superposition of plane waves with the spectrum F. The spectral width is:**

$\Delta k \sim \gamma / r$



Longitudinal Wake function And Loss Factor



The energy lost by the charge q_1 is given by the work done by the longitudinal e.m. force along the structure:

$$U_{11} = -\int_{-\infty}^{\infty} F(\mathbf{r}_1, z_1, \mathbf{r}_1, z_1; t).dz = -q_1 \int_{-\infty}^{\infty} \left[\mathbf{E}(\mathbf{r}_1, z_1, \mathbf{r}_1, z_1; t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}_1, z_1, \mathbf{r}_1, z_1; t)\right].dz$$

(1.5)

and this is evaluated at time $t=z_1/v$. The trailing charge changes its energy as it is influenced by field trailing the driving charge:

$$U_{21} = -\int_{-\infty}^{\infty} F(\mathbf{r}, z, \mathbf{r}_1, z_1; t).dz = -q \int_{-\infty}^{\infty} \left[\mathbf{E}(\mathbf{r}, z, \mathbf{r}_1, z_1; t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, z, \mathbf{r}_1, z_1; t)\right].dz \quad \textbf{(1.6)}$$

and this is evaluated at time $t=z_1/v+\tau$, where τ is the time delay between the driving and the witness bunch.

- **A physical accelerator (a linac or an accelerator ring) is not infinite in length!**
- **Providing the fields are confined within a given region and evanescent elsewhere then truncating the integral gives a very good approximation of the energy.**

- **A real vacuum chamber, standing wave accelerator, etc is formed by smooth transitions in the geometry and has various devices inserted such as RF cavities, kickers, diagnostic components, etc. These devices perturb the fields. Even with parallel computing computer codes with relatively large amounts of memory one is never able to model the full set of accelerator components simultaneously.**

- **In modeling the energy losses and impedance one models the individual components and sums the losses. This is usually quite accurate unless significant modal distortion takes place.**

- **Circuit models can account for mode distortion in CLIC, ILC, NLC accelerating structures for example (later lecture).**

The loss factor is defined as the energy lost by q_1 per unit charge squared:

$$k(\mathbf{r}_1) = \frac{U_{11}(\mathbf{r}_1)}{q_1^2} \qquad \textbf{(1.7)}$$

And the longitudinal wake can be defined in terms of the energy lost by the trailing charge per unit q_1 per unit q:

$$w_z(\mathbf{r},\mathbf{r}_1;\tau) = \frac{U_{21}(\mathbf{r},\mathbf{r}_1;\tau)}{q_1 q} \qquad \textbf{(1.8)}$$

Both the longitudinal wake and the loss factor have the same units, viz. Volts/Coulomb (V/C).

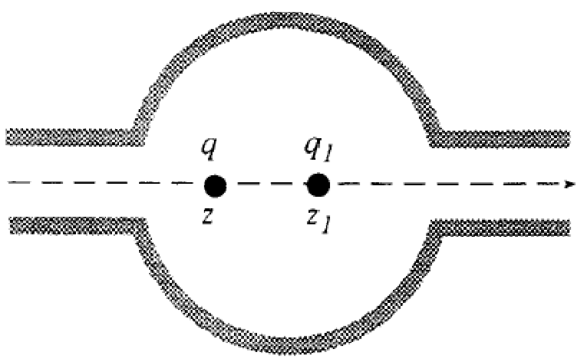
Often the wake per unit length is the practical quantity of interest (especially for periodic structures for example =>wake/unit period):

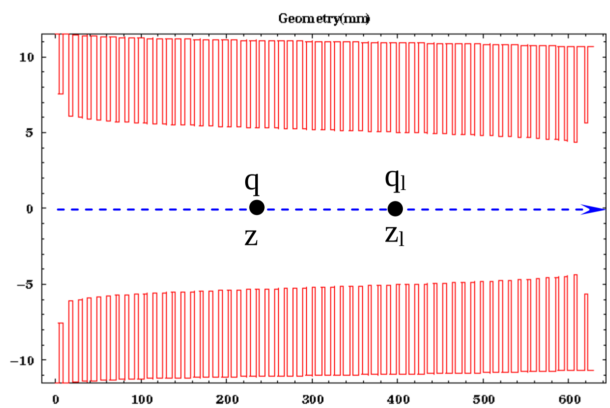
$$\frac{d}{dz} w_z(\mathbf{r},\mathbf{r}_1;\tau) = -\frac{1}{q_1 q} F_z(\mathbf{r},z,\mathbf{r}_1,z;\tau); \quad z = z_1 - v\tau$$

This wake has units of Volts/Coulomb/meter.

- **The wake function is, of course, no more than the force per unit charge acting on q.**

- **Important to note that in many practical cases the structures have some symmetry (circular, elliptical, rectangular) and the beam moves by a small amount from the electrical axis of symmetry.**



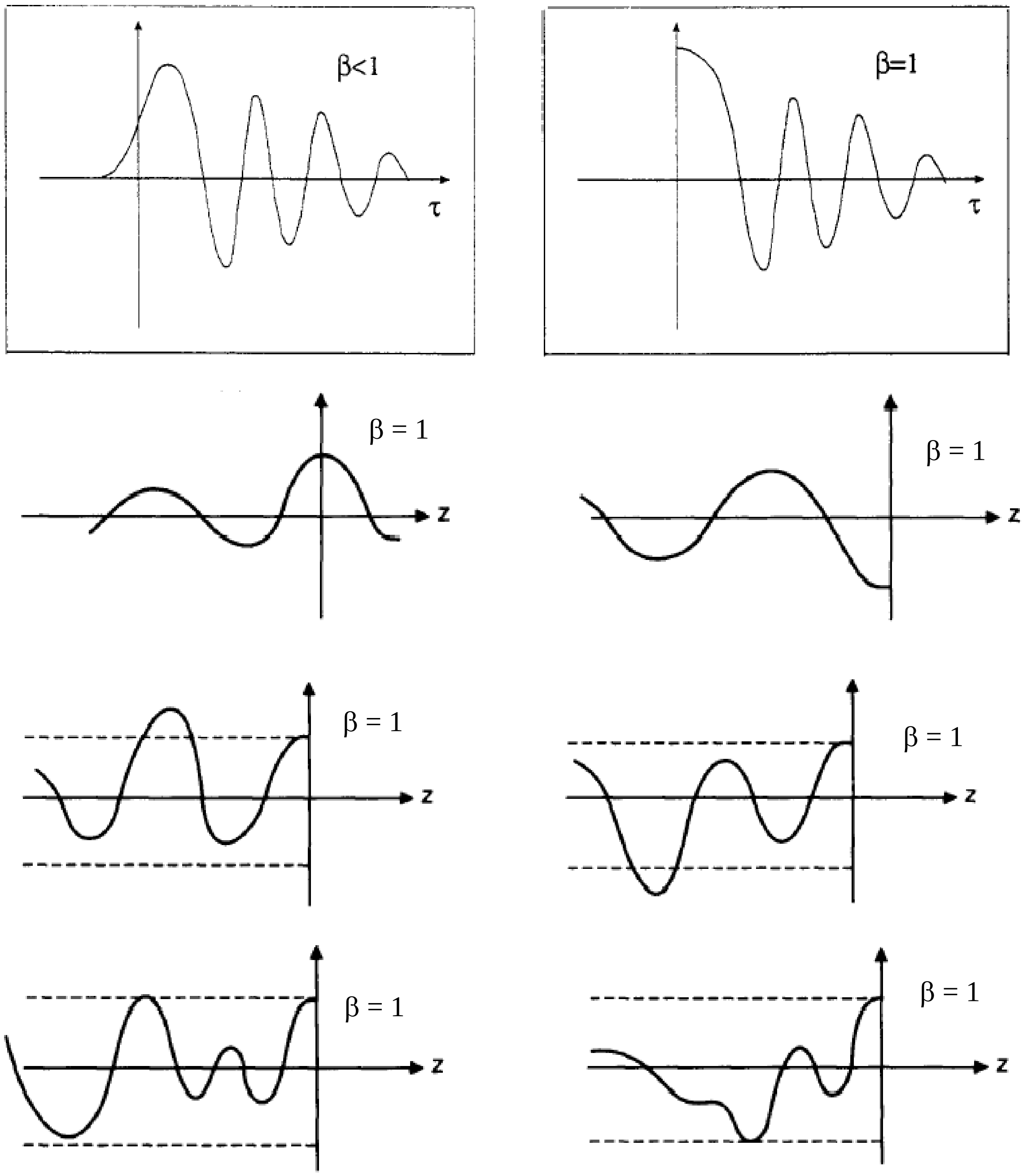


(a)

Leading and trailing charges in spherical cavity (a) and cylindrically symmetric circular cavities (b). The cavity on the left is representative of a superconducting TESLA cavity and the one on the right of the NLC X-band accelerators.

- **This means that in a multi-pole expansion of fields only the first few terms will be significant.**
- **Typically only monopole and dipole terms (and sometimes quadrupole) are meaningful in realistic simulations**

Physically Realizable Wakes?

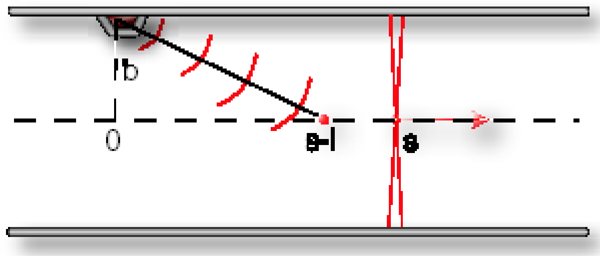


- **All wakes, apart from the upper pair are non-physical!**

- **Wakes that are non-zero ahead of the particle break causality.**
- **The solitary case of $\beta<1$ is allowed to have $W_z(s)$ finite for $s<0$**

  (the "pancake" of field has a finite width)

Characteristic "Catch-Up" Distance



- **In the limit v=c the field can only interact with the trailing particles. This is called the principle of causality.**

- **At v=c the field cannot precede the bunch**

- **Assume a discontinuity at s=0 scatters the field, and the leading particle passes this point at time t=0**

- **The scattered field reaches the point l behind the drive particle at time t and:** $ct=\sqrt{(s-l)^2+b^2}$**, where** $s(=vt)$ **is the coordinate of the leading particle at time t**

- **Assume** $l<<b$ **and** $b<<s$ **and thus:**

$$s=\sqrt{(s-l)^2+b^2}\sim s\left(1-\frac{2l}{s}+\frac{b^2}{s^2}\right)$$

- **The catch-up distance is then obtained:**

$$s\sim\frac{b^2}{2l}$$

- **Typically the length l is of the order of the bunch length: $l \sim \sigma_z$**

- **For example for the b=4mm and $\sigma_z \sim 100$ μm:**

$$s \sim \frac{b^2}{2l} \sim 8\text{cm}$$

- **Thus, in simulating the wake with a code such as ABCI or ECHO2D, it is important to ensure the simulation length is larger than the catch-up distance**

Longitudinal Wake Function and Loss Function of Bunch Distribution

- **The wake functions and loss factors we have defined have been for point charge, i.e. the wake function is a Green's function.**

- **Convolution of the Green's function with the actual current distribution gives the true wake function:**

$$W_z(\tau) = \frac{U(r,\tau)}{q_1 q} = \frac{1}{q_1}\int_{-\infty}^{\tau} i_b(\tau')w_z(\mathbf{r},\tau-\tau').d\tau' \qquad \textbf{(1.9)}$$

- **By superposition we also obtain:**

$$K(\mathbf{r}) = \frac{U(r,\tau)}{q_1^2} = \frac{1}{q_1}\int_{-\infty}^{\tau} W_z(r,\tau)i_b(\tau)d\tau \qquad \textbf{(1.10)}$$

For example, a rectangular bunch distribution, for a well-damped delta function wake, is readily integrated. The bunch current and point wake are of the form:

$$i_b(t) = q_1\frac{H[t+T]-H[t-T]}{2T} \qquad \textbf{(1.11)}$$

$$w_z(\tau) = w_0\cos[\omega_r\tau]H[\tau] \qquad \textbf{(1.12)}$$

In the region of the bunch we obtain (-T<τ<T):

$$W_z(\tau) = \frac{w_0}{2}\frac{\mathrm{Sin}\left[\omega_r\left(\tau+T\right)\right]}{\omega_r T}H\left[\tau+T\right] \tag{1.13}$$

In the limit of T->0 we obtain a delta function $i_q(\tau) = q_l\,\delta(\tau)$ and thus the point source wake is uncovered: $W_z(\tau)$->$w_z(\tau)$

Also, setting τ=0 in $W_z(\tau)$ and taking the limit of T->0:

$$\lim_{T->0} W_z(0) = \frac{w_0}{2} \tag{1.14}$$

Thus, the wake at the bunch is half that of the total wake function. The bunch loss factor is also obtained:

$$K = \frac{w_0}{2}\left[\frac{\mathrm{Sin}\left(\omega_r T\right)}{\omega_r T}\right]^2 \tag{1.15}$$

In the limit of T->0 we obtain the point charge loss factor:

$$k = \lim_{T->0} K = \frac{w_0}{2} \tag{1.16}$$

The wake function external to the distribution, i.e. for τ>=T:

$$W_z(\tau) = w_0\frac{\mathrm{Sin}\left(\omega_r T\right)\cos\left(\omega_r\tau\right)}{\omega_r T}H\left[\tau-T\right] \tag{1.17}$$

Thus, in the limit of T->0 and τ->0 in (1.17)

Longitudinal Coupling Impedance

The impedance is given by spectrum of the longitudinal point charge wake function:

$$Z_z(r,r_l;\tau)=\int_{-\infty}^{\infty} w_z(r,r_l;\tau)\exp(-j\omega\tau)d\tau \qquad \textbf{(1.18)}$$

and the point charge wake is given by the inverse Fourier transform of the impedance:

$$w_z(r,r_l;\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} Z_z(r,r_l;\omega)\exp(j\omega\tau)d\omega \qquad \textbf{(1.19)}$$

The Fourier transform possesses real and imaginary parts and they are related by the Kramers-Kronig, or Hilbert transform.

Also, as we have seen in the case of the rectangular distribution:

$$k=W_z(r,r_l;0)/2 \qquad \textbf{(1.20)}$$

and in terms of impedance :

$$k=W_z(r,r_l;0)/2=\frac{1}{\pi}\int_{0}^{\infty}\mathrm{Re}\{Z_z(r,r_l;\omega)\}d\omega \qquad \textbf{(1.21)}$$

Thus, we can think of the real part of the impedance as the power spectrum of the energy loss. This can be generalized to the complex impedance being related to the complex power spectrum of the energy loss.

In practice a Gaussian profile is often used:

$$I(t) = q_1 c \frac{\exp(-\frac{c^2 t^2}{2\sigma^2})}{\sqrt{2\pi}\sigma}, \qquad I(\omega) = q_1 \exp(-\frac{\omega^2\sigma^2}{2c^2}) \qquad \textbf{(1.22)}$$

The current is normalized such that: $\int_{-\infty}^{\infty} I(t)dt = q_1$

Wakefunctions, Bunch Distributions and Impedance

- **The objective is to obtain the total bunch wake in terms of the current and impedance**

Recall the fact that the wake due to a distribution is:

$$W_z(\tau) = \frac{U(r,\tau)}{q_l q} = \frac{1}{q_l}\int_{-\infty}^{\tau} i_b(\tau')w_z(\mathbf{r},\tau-\tau').d\tau'$$

and take the Fourier transform:

$$W_z(\tau) = \frac{U(r;\tau)}{q_l q} = \frac{1}{2\pi q_l}\int_{-\infty}^{\infty} Z(r;\omega)I(\omega)\exp(j\omega\tau)d\omega \qquad \textbf{(1.23)}$$

The loss factor:

$$K(\mathbf{r}) = \frac{U(r,\tau)}{q_l^2} = \frac{1}{q_l}\int_{-\infty}^{\tau} W_z(r,\tau)i_b(\tau)d\tau$$

Again, performing a Fourier transform gives:

$$K = \frac{U(r)}{q_l^2} = \frac{1}{\pi}\int_0^{\infty} Z(r;\omega)\left|I(\omega)\right|^2 d\omega \qquad \textbf{(1.24)}$$

For a Gaussian beam we have:

$$K(r) = \frac{1}{\pi}\int_0^{\infty} Z(r;\omega)\exp(-\frac{\omega^2\sigma^2}{c^2})d\omega \qquad \textbf{(1.25)}$$

- **As σ->0 then the current goes to infinity and, as expected, the loss factor (in (1.25)) becomes that of the point source k.**

For example, taking the case of a parallel R-L-C circuit the impedance is given by:

$$Z_l(\omega) = \frac{2k/\omega}{\omega_0/(Q\omega) - I(\omega_0/\omega - 1)(\omega_0/\omega + 1)} \qquad \textbf{(1.26)}$$

where the losses are represented by the quality factor Q ($=\omega_0 RC$). The bunch wake and loss factors are quite complicated expressions for this case. For low losses (high Q), the impedance simplifies to:

$$Z_l(\omega) = 2I\frac{k\omega}{\omega_0^2 - \omega^2} \qquad \textbf{(1.27)}$$

The bunch wake function, using (1.23) is now:

$$W_z(\tau) = 2k\cos(\omega_0\tau)e^{-\frac{\sigma^2\omega_0^2}{2c^2}} \qquad \textbf{(1.28)}$$

and the bunch loss factor for a Gaussian beam, using (1.25), becomes:

$$K = ke^{-\frac{\sigma^2\omega_0^2}{c^2}} \qquad \textbf{(1.29)}$$

Both of these results ((1.28) and (1.29)) are only valid for τ>3σ because this allows the infinite limits to be used in the convolution of the bunch distribution with the point source wake.

Note that for an R-L-C parallel resonance circuit the loss factor can be written:

$$k = \frac{\omega_0 R}{2Q} \qquad \textbf{(1.30)}$$

In accelerator physics the shunt impedance is usually defined such that a factor of 4 (rather than 2) occurs in the denominator of (1.30) –the context should make it clear as to whether or not 4 is used. Thus

$$k = \frac{\omega_0 R_{acc}}{4Q}$$

where R_{acc} = 2R (= V.V*/P).

- **The loss factor is, in general a function of the r.m.s. length of the bunch. For Gaussian bunches, in general one finds:**

$$Z_r(\omega) \propto \omega^a \Leftrightarrow K \propto \sigma_t^{-(a+1)}$$

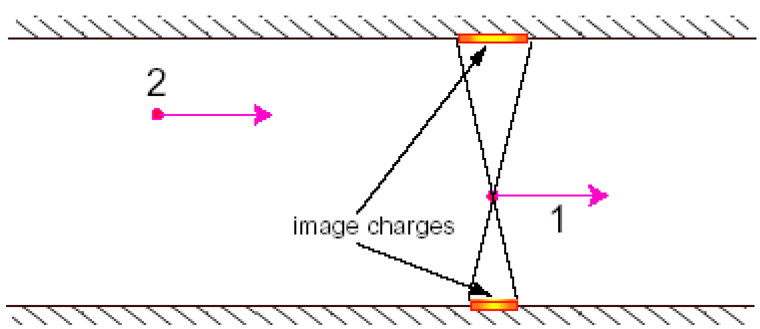
- **Note the "r" dependence may be dropped as it will be understood to be present according to the context of the wake function and impedance.**

Wakefield For Perfectly Conducting Structures

- **For ultra-relativistic beams in perfectly conducting accelerator structures the longitudinal and transverse forces on a beam vanish**
  - **No wakefield in limit β (= v/c)->1 (γ->infinity)**
  - **Only true with no obstacles to reflect the field**

- **Why?**

  - **A particle traveling in a perfectly conducting cylindrical pipe induces image charges on the surface of the wall. These image charges travel with the same velocity c.**

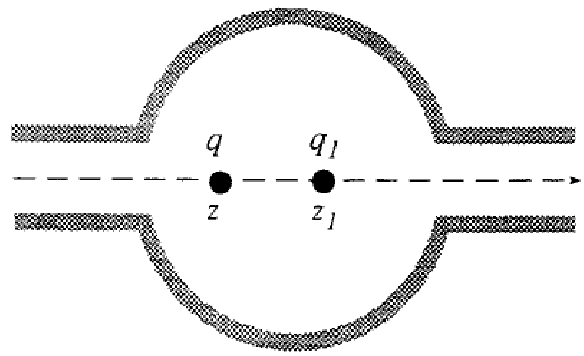


  - **Since the particle and image charge move on parallel paths, in the limit of v=c they do not interact with each other.**

Longitudinal Wake as a Summation of Multipoles

- **For accelerators and microwave components with cylindrical symmetry it is natural to assume that the wake functions can be expanded over modes exhibiting the symmetry**

- **We consider charges moving on axis.**
- **The coordinates of the driving charge and witness charge are (r_l,ϕ_l=0,z_l) and (r,ϕ=0,z) respectively.**



- **The charge density can be represented as a superposition of multipole moments in cyclindrical coordinates:**

$$\rho_l = q_l \frac{\delta(r-r_l)}{r_l}\delta(\phi)\delta(z-z_l) = \frac{q_l}{2\pi}\frac{\delta(r-r_l)}{r_l}\delta(z-z_l)\sum_{m=0}^{\infty}\left(2-\delta_m^0\right)\cos m\phi$$

(1.31)

- **Where the azimuthal symmetry of the geometry has been utilized and δ_m^0=0 for m>0, δ_0^0=1 , and $z_l = \beta c\tau$.**

- **Thus, the charge can be envisaged the summation of a series of charged rings with angular dependence cos(mϕ).**
- **m=0 for example represents a charges ring with uniform density up to r=r$_l$**

- **The wake is nothing more than the solution to Maxwell's equations with a charge source driving the differential equations and hence we make a superposition of multipole moments:**

$$w_z(\mathbf{r},\mathbf{r}_1;\tau)=\sum_{m=0}^{\infty} w_{z,m}(\mathbf{r},\mathbf{r}_1;\tau)=\sum_{m=0}^{\infty} \bar{w}_{z,m}(\mathbf{r},\mathbf{r}_1;\tau)\cos m\phi \qquad \mathbf{(1.32)}$$

Radial Expansion of Wake Function in the Ultra-relativistic Limit

- **The e.m. fields produced by charges traveling down an accelerator structure are driven solutions of Maxwell's equations subject to the boundary conditions imposed at the walls.**

- **The longitudinal electric field is produced by the bunch of charged particles, and by the currents induced in the walls. Considering only the induced fields, it can shown that**

$$\left[\nabla_{\perp}^{2}-\left(\frac{\omega}{\beta\gamma c}\right)^{2}\right]\tilde{E}_{z}=0 \qquad \textbf{(1.33)}$$

- **In the ultra-relativistic limit (v=c) we clearly have:**

$$\nabla_{\perp}^{2}\tilde{E}_{z}=0 \qquad \textbf{(1.34)}$$

- **The solution in cylindrical coordinates is:**

$$w_{z}(r,r_{l};\tau)=\sum_{m=0}^{\infty}\bar{w}_{z,m}(r,r_{l};\tau)=\sum_{m=0}^{\infty}r^{m}r_{l}^{m}\bar{\bar{w}}_{z,m}(\tau) \qquad \textbf{(1.35)}$$

- **The monopole mode, m=0, does not depend on radial position. Use is often made of this fact when calculating the wake by placing the evaluation**

point at the radius of the beam tube where the field is zero.

- **The expansion concerns the secondary fields induced by the beam. The space charge fields show a different dependence.**

Multipole Longitudinal Impedance

- **As the impedance is no more than the Fourier transform of the wake function then it too can be expanded in a multipole expansion:**

$$Z(r,r_1;\omega)=\sum_{m=0}^{\infty} Z_m(r,r_1;\tau)=\sum_{m=0}^{\infty}\bar{Z}_m(r,r_1;\tau)\cos m\phi \qquad \textbf{(1.36)}$$

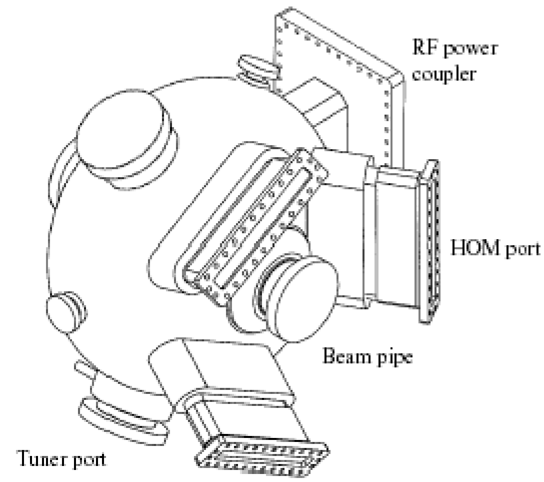
- **For ultra relativistic charges the radial dependence is known:**

$$\bar{Z}_m(r,r_1;\omega)=r^m r_1^m \bar{\bar{Z}}_m(\omega) \qquad \textbf{(1.37)}$$

where $\bar{\bar{Z}}_m$ has dimensions Ω/m^{2m}

Synchronous Beam Fields

- **We have seen that the fields scattered from the obstacles (HOM ports, couplers, tuners, kickers, etc) give rise to non-zero wake functions.**
- **Some of these fields are localized around the bunch (resistive wall for example), others are localized in resonant structures such as the r.f. cavities.**
- **All these fields interact with the beam**



PEP II Cavity: Beam pipe, HOM and tuner ports are illustrated

- ***Only the fields synchronous with the charges can change the energy of the charges***

Making an expansion of the longitudinal field in a series of plane waves:

$$E_z(z,t) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\kappa \tilde{E}_z(\omega,\kappa) e^{j(\omega t - \kappa z)} \qquad \textbf{(1.38)}$$

- **The explicit dependence on (r, r_l, z_l) has been omitted**

In terms of the wake function we have:

$$w_z(\tau) = -\frac{1}{q_1}\int_{-\infty}^{\infty} E_z(z, t = \frac{z}{v} + \tau)dz =$$
$$-\left(\frac{1}{2\pi}\right)^2 \frac{1}{q_1}\int_{-\infty}^{\infty} e^{j\omega\tau}d\omega \int_{-\infty}^{\infty} d\kappa \tilde{E}_z(\omega,\kappa)\int_{-\infty}^{\infty} e^{-jz(\kappa-\kappa_0)}dz \quad \textbf{(1.39)}$$

where $\kappa_0 = \omega / v$. The point source delta function is given by:

$$\delta(\kappa - \kappa_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jz(\kappa-\kappa_0)}dz \quad \textbf{(1.40)}$$

Thus, only those components of the fields propagating with the same phase velocity can effect the charges energy. The fields from all other phases do not contribute as they average out to zero. We are left with:

$$w_z(\tau) = -\frac{1}{2\pi q_1}\int_{-\infty}^{\infty} e^{j\omega\tau}\tilde{E}_z(\omega, \kappa = \kappa_0)d\omega \quad \textbf{(1.41)}$$

Now, as the wake function is defined in terms of the longitudinal coupling impedance is defined as:

$$w_z(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_z(\omega)\exp(j\omega\tau)d\omega \quad \textbf{(1.42)}$$

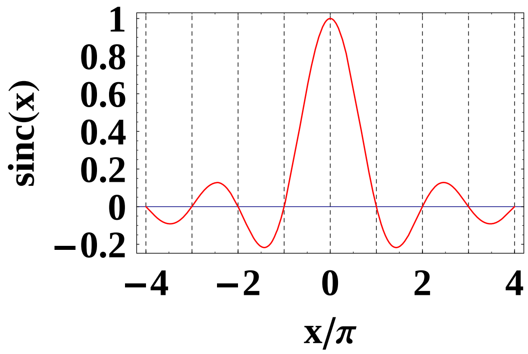
then it is clear that the impedance may also be written in terms of the Fourier transform of the field:

$$Z_z(\omega) = -\frac{1}{q_1}\tilde{E}_z(\kappa = \kappa_0, \omega) \quad \textbf{(1.43)}$$

For finite length cavities the delta function is replaced by the sinc function:

$$\frac{1}{2\pi}\int_{-L/2}^{L/2} e^{-jz(\kappa-\kappa_0)}dz = \frac{L}{2\pi}\frac{\sin\left[\left(\kappa-\kappa_0\right)\frac{L}{2}\right]}{\left(\kappa-\kappa_0\right)\frac{L}{2}} \qquad \textbf{(1.44)}$$

and the sinc becomes a delta function as the length L->infinity. For a finite a length, the sinc (=sin[x]/x) has a maximum at $\kappa=\kappa_0$ **and the zeros are located at** $\kappa=\kappa_0\pm 2\pi/L$**.**



- **For long wavelengths the fields tend to be confined to a given region in which they propagate and are non-propagating (evanescent) elsewhere.**

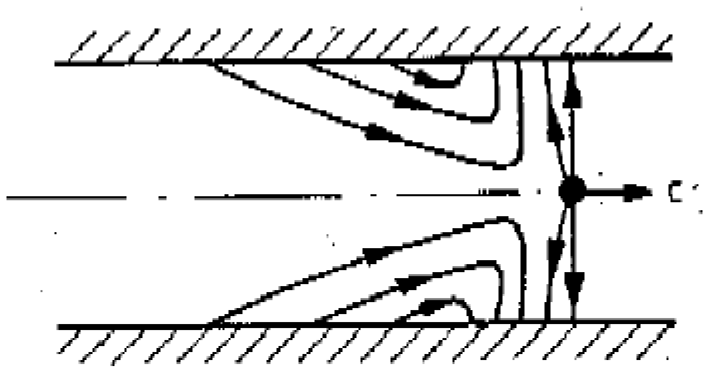
- **The integration path is confined to the propagating region.**

- **For short wavelengths fields propagate out of cavities into the beam tubes. However, the sinc function (1.63) shows that the contribution is small and hence the integration may be extended to infinite limits.**

Wakefield For Waveguide with Lossy Walls

- For a pipe with finite conductivity σ and if the skin depth is much smaller than the thickness of the pipe wall then all of the field is essentially contained within a skin depth or so.

- Thus, the pipe walls can be considered to be infinite.



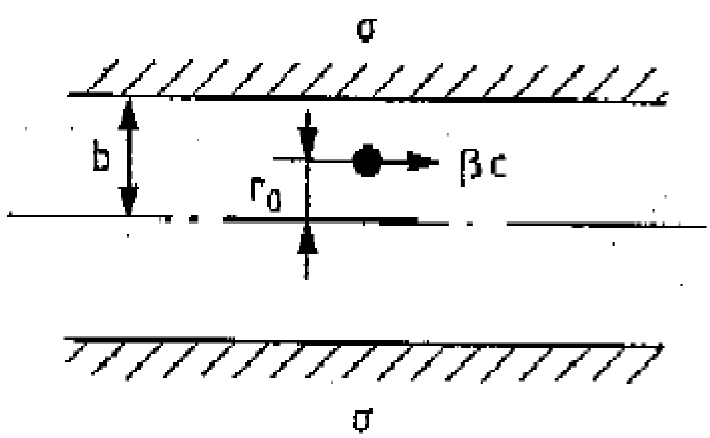
- During the motion of the charged particle bunch a non-zero wakefield will appear behind the charge.

- In general the wakefield that develops from reflections from waveguide discontinuities (obstacles, tuners and irises etc) is far larger than the resistive wall wake.
  - However, for a collimator or beam scraper the resistive wall wake can be dominant effect. The collimator is used to scrape any beam halo that will develop on accelerating a relativistic beam through several km.

Resistive Wall Wakefield

For a perfectly conducting matched waveguide with no obstacles to reflect back the field there is no overall wakefield. However, the presence of loss on the walls of the waveguide gives rise to a wakefield. For a cylindrical waveguide the E and H fields are given by:

$$\left.\begin{aligned} E_r(\omega) &= \frac{qZ_0}{2\pi r} e^{-jkz} \\ H_\phi(\omega) &= \frac{q}{2\pi r} e^{-jkz} \end{aligned}\right\} \qquad \textbf{(1.45)}$$



The continuity at the wall of the waveguide at r = b requires the magnetic field component inside the surface be the same as that outside. Inside the wall the field is sustained by a surface current flowing along the z direction (the waveguide is orientated along z). The electric field along the z-axis is given by:

$$E_z(\omega) = Z_c H_\phi(\omega) \qquad \textbf{(1.46)}$$

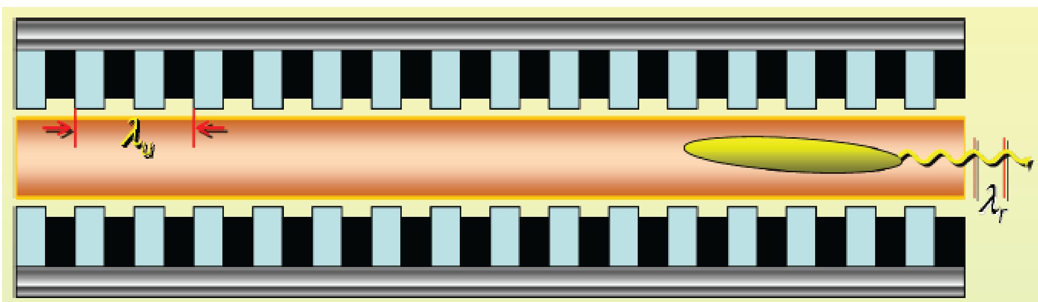
where the surface impedance is given by:

$$Z_c = \sqrt{\frac{j\omega Z_0}{\sigma c}} \qquad \textbf{(1.47)}$$

Z_0 (= 377) is the characteristic impedance of free space and c is the velocity of light. The flux of the Poynting vector on wall of the pipe gives the longitudinal impedance per unit length:

$$Z(\omega) = 2\pi b Z_c \left|H_\phi\right|^2 = \frac{1+j}{2\pi b}\sqrt{\frac{\omega Z_0}{2c\sigma}} \qquad \textbf{(1.48)}$$

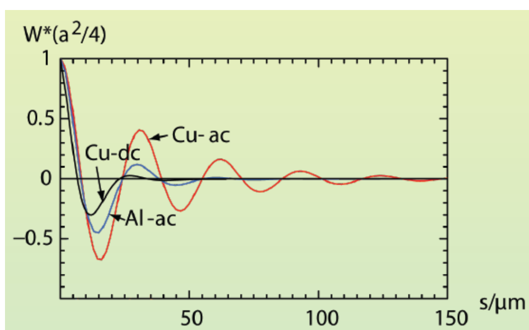
- **This is often a small effect compared to that encountered due to impedance of obstacles encountered in the accelerator structure. However, for short bunches with high charge considerable power dissipation may occur for non-superconducting cavities (e.g. LCLS vacuum chamber, LHC collimators and magnets).**



The radiation wavelength in an FEL is $\lambda_r = \dfrac{\lambda_u}{2\gamma^2\left(1+K^2/2\right)}$

The energy of the beam should be kept constant over the length of the undulator.
For LCLS with a 1 nC beam $\Delta\gamma/\gamma \approx 3-5\text{x}10^{-4}$
A uniform energy change, such as incoherent radiation of the beam can be compensated by tailoring K. However, wakefields generate a ΔE that varies along the bunch.
Clearly it is important to minimize these wakefields.

Conductivity has frequency (ac) dependence $\sigma = \dfrac{\sigma_0}{1-i\omega\tau}$
Consider different walls of vacuum chamber to reduce wake (cu, al indicated)

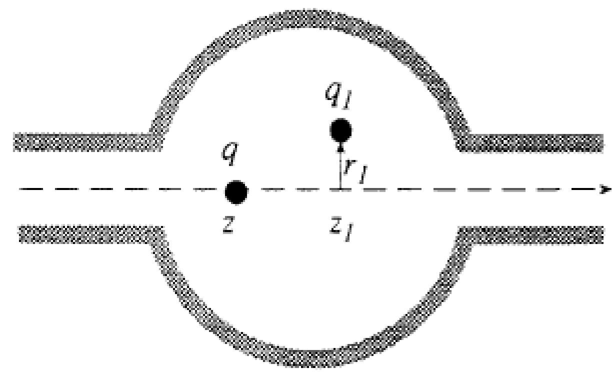


Ref: K.L.F. Bane, G. Stupakov 2004, SLAC PUB-10707

- **The collimators for the ILC and LHC have a significant resistive wall wakefield component as the energetic beam impinging on the walls of the collimator changes the conductivity. Experiments have been conducted in this area and means to mitigate for this are being employed –such as ceramic walls and breaking up the flow of currents in the conducting surfaces by creating variegated or slotted walls.**

Transverse Wake Function

- **Drive charge, q_l, is displaced with respect to the axis of the cavity**



- **Multipole components are exited in the transverse plane: dipole, quadrupole, sextupole etc**
- **Trailing charge q is subject to a Lorentz force which has both longitudinal and transverse components**
- **Transverse momentum kick imparted to trailing charges:**

$$M_{21}(r,r_l;\tau) = \int_{-\infty}^{\infty} F_{\perp}(\mathbf{r},z,\mathbf{r}_l,z_l;t)dz, \quad t = \frac{z_l}{v} + \tau \qquad \textbf{(1.49)}$$

- **The integration is assumed to be over an infinite distance.**
- **A transverse displacement can lead to both vertical and horizontal kicks**

- **Transverse kick measured in Volts/Coulomb defines the transverse wake function:**

$$w_{\perp}(r,r_l;\tau) = \frac{M_{21}(r,r_l;\tau)}{q_l q} \qquad \textbf{(1.50)}$$

- **The dipole transverse loss factor is defined as the amplitude of the transverse momentum kick given to the charge by its own wake per unit charge (V/C):**

$$k_{\perp}(r) = \frac{M_{11}(r_1)}{q_1^2} \qquad \textbf{(1.51)}$$

- **The dipole component of the transverse kick is the dominant term for ultra relativistic charges. The *transverse dipole wake function* is defined as the transverse wake per unit of transverse displacement (V/Cm):**

$$w'_{\perp}(r, r_1; \tau) = \frac{M_{21}(r, r_1; \tau)}{q_1 q r_1} \qquad \textbf{(1.52)}$$

- **and the transverse loss factor (V/Cm):**

$$k'_{\perp}(r) = \frac{M_{11}(r_1)}{q_1^2 r_1} \qquad \textbf{(1.53)}$$

Transverse Wake Function and Loss Factor of a Bunch

As in the case of the longitudinal wake we take the convolution of the bunch current with the point source transverse wake function to obtain the bunch dependent wake:

$$W_{\perp}(\tau)=\frac{1}{q_l}\int_{-\infty}^{\infty} i_b(\tau')w_{\perp}(\mathbf{r},\tau-\tau').d\tau' \qquad \textbf{(1.54)}$$

and the bunch transverse loss factor :

$$K_{\perp}(\tau)=\frac{1}{q_l}\int_{-\infty}^{\infty} i_b(\tau)W_{\perp}(r;\tau)d\tau \qquad \textbf{(1.55)}$$

The transverse wake and loss factor per unit displacement are:

$$W'_{\perp}(r;\tau)=\frac{W_{\perp}(r;\tau)}{r} \qquad \textbf{(1.56)}$$

$$K'_{\perp}(r;\tau)=\frac{K_{\perp}(r;\tau)}{r} \qquad \textbf{(1.57)}$$

both of which are measured in Volt/Coulomb/meter

Panofsky-Wenzel Theorem Relating Longitudinal and Transverse Wakes

- **Firstly, we consider both the driving charge and the witness charge both lying moving along the z-axis of the accelerator.**

- **Once the longitudinal component of the wakefield has been calculated the transverse wakefield can be derived from it in a straightforward manner. From Maxwell's equation:** $\nabla \mathrm{x} \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$ **then:**

$$\mathbf{e}_z \mathrm{x} \frac{\partial}{\partial t}\mathbf{B} = \frac{\partial}{\partial z}\mathbf{E}_\perp - \nabla_\perp E_z \qquad \textbf{(1.58)}$$

- **Using the relation for the total derivative:**

$$\frac{d}{dz}\mathbf{E}_\perp(r, r_1, (z+s)/c) = \left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathbf{E}_\perp(r, r_1, (z+s)/c) \qquad \textbf{(1.59)}$$

- **then the derivative of the transverse wake with respect to s is written in the form:**

$$\frac{\partial}{\partial s}\mathbf{W}_\perp(r, r_1, (z+s)/c) = \frac{1}{q_1}\int_{-\infty}^{\infty} dz \left[\frac{d}{dz}\mathbf{E}_\perp\left(r, r_1, (z+s)/c\right) - \nabla_\perp E_z\left(r, r_1, (z+s)/c\right)\right]$$

(1.60)

- **Performing the integrals as indicated above the first term vanishes provided $\mathbf{E}_\perp$ vanishes at the boundaries and we are left with (V/C/m):**

$$\frac{\partial}{\partial s}\mathbf{W}_\perp(r,r_1,s) = -\nabla_\perp W_{||}(r,r_1,s) \qquad \textbf{(1.61)}$$

- **This is the *Panofsky Wenzel* theorem (1956). A single integration provides the transverse wakefield once the longitudinal has been calculated:**

$$\mathbf{W}_\perp(r,r_1,s) = -\nabla_\perp \int_{-\infty}^{s} W_{||}(r,r_1,s')ds' \qquad \textbf{(1.62)}$$

- **In applying the above formula it has been assumed that $\lim_{s->\infty}\mathbf{W}_\perp(r,r_1,s)=0$. In practice one has finite limits and the lower limit is often taken at a point in which the field is zero (on the walls of a perfectly conductor for example).**

- **If the driving charge is slightly offset from the z-axis we expand the rhs of (1.61) retaining only the first order terms in the offset r_1:**

$$W_{||}(r,r_1,s) \sim W_{||}(r,0,s) + \left[\nabla_{\perp,r_1} W_{||}(r,r_1,s)\right]_{r_1=0} . r_1 + O(r_1^2) \qquad \textbf{(1.63)}$$

- **Thus (1.61) becomes:**

$$\frac{\partial}{\partial s}\mathbf{W}_{\perp}(r,r_1,s)=-\nabla_{\perp,r}\left\{W_{||}(r,0,s)+\left[\nabla_{\perp,r_1}W_{||}(r,r_1,s)\right]_{r_1=0}.r_1\right\} \qquad \mathbf{(1.64)}$$

- **The first term is in fact a monopole contribution to the transverse impedance and this often disappears according to the geometry (circular, rectangular elliptic). The remaining term is the dipole impedance:**

$$\frac{\partial}{\partial s}\mathbf{W}_{\perp}(r,r_1,s)=-\nabla_{\perp,r}\left[\nabla_{\perp,r_1}W_{||}(r,r_1,s)\right]_{r_1=0}.r_1 \qquad \mathbf{(1.65)}$$

Mode Expansion of Transverse Wake Function in Coordinates with Cylindrical Symmetry

- **As in the case of the longitudinal wake function, the transverse wake function is expressed as a superposition of multipole terms:**

$$w_{\perp}(r,r_l;\tau)=\sum_{m=0}^{\infty} w_{\perp,m}(r,r_l;\tau) \qquad \mathbf{(1.66)}$$

- **Applying Panofsky-Wenzel(1.80) and making use of the expansions of the longitudinal wake function(1.32), (1.35) we obtain:**

$$\frac{\partial}{\partial s} w_{\perp,m}(r,r_l,s)=-\overline{\overline{w}}_{z,m}(s)r^{m-1}r_l^m\left\{\cos(m\phi)\hat{r}-\sin(m\phi)\hat{\phi}\right\} \qquad \mathbf{(1.67)}$$

- **It is interesting to note that the dipole term, m=1 is linearly proportional to the offset of the driving charge and it is independent of the witness charge. The dipole transverse force is directed along the offset of the leading charge:**

$$\frac{\partial}{\partial s} w_{\perp,1}(r,r_l,s)=-\overline{\overline{w}}_{z,m}(s)r_l \qquad \mathbf{(1.68)}$$

- **where $\overline{\overline{w}}_{z,m}(s)$ is the amplitude of the dipole longitudinal wake function (V/C/m^2)**

Transverse Coupling Impedance

- **The impedance is defined in terms of the Fourier transform of the transverse wake function with the additional imaginary factor (Ohms):**

$$Z_{\perp}(r,r_2;\omega)=j\int_{-\infty}^{\infty} w_{\perp,1}(r,r_2;\tau)\exp\left(-j\omega\tau\right)d\tau \qquad \textbf{(1.69)}$$

- **The imaginary constant was introduced in order to make the transverse impedance play the same role as the longitudinal one in beam stability theory.**
- **The dipole wake is usually the dominant one therefore it is natural to normalize with respect to the offset of the drive bunch (Ohms/m):**

$$Z'_{\perp}(r_1,r_2;\omega)=\frac{Z_{\perp}(r_1,r_2;\omega)}{r_1} \qquad \textbf{(1.70)}$$

The transverse wake is obtained via the inverse Fourier transform:

$$w_{\perp,1}(r,r_2;\tau)=-\frac{j}{2\pi}\int_{-\infty}^{\infty} Z_{\perp}(r,r_2;\omega)\exp\left(j\omega\tau\right)d\omega \qquad \textbf{(1.71)}$$

The Fourier transform of (1.61) gives the dipole transverse impedance in terms of the longitudinal one (Ohms):

$$FT\left\{\frac{\partial}{\partial s}\mathbf{W}_{\perp}(r,r_1,s)=-\nabla_{\perp}W_{||}(r,r_1,s)\right\}$$

$$\Rightarrow Z_{\perp}(r,r_1;\omega)=-\frac{c}{\omega}\nabla_{\perp}Z(r,r_1;\omega) \qquad \textbf{(1.72)}$$

For an arbitrary shape (1.84) gives:

$$\mathrm{FT}\left\{\frac{\partial}{\partial s}\mathbf{W}_{\perp}(r,r_1,s) = -\nabla_{\perp,r}\left[\nabla_{\perp,r_1}W_{||}(r,r_1,s)\right]_{r_1=0}.r_1\right\}$$

$$\Rightarrow Z_{\perp}(r,r_1;\omega) = \frac{c}{\omega}\nabla_{\perp,r}\left[\nabla_{\perp,r_1}Z(r,r_1,s)\right]_{r_1=0}.r_1 \qquad \textbf{(1.73)}$$

In cylindrical symmetry we obtain:

$$Z_{\perp,1}(r,r_1;\omega) = \frac{c}{\omega}\overline{\overline{Z_1}}(\omega)r_1 \qquad \textbf{(1.74)}$$

where (1.87) and (1.36) have been used.

Practical Wake Function Expansions

The longitudinal and transverse wake functions are written (justification for summations given in appendix):

$$W_{||}(\mathbf{r_1},\mathbf{r};s) = 2\theta(s)\sum_n \kappa_n(\mathbf{r_1},\mathbf{r})\cos(\omega_n s/c) \quad \textbf{(1.75)}$$

$$\mathbf{W}_{\perp}(\mathbf{r_1},\mathbf{r};s) = 2\theta(s)\sum_n \boldsymbol{\kappa}_{n\perp}(\mathbf{r_1},\mathbf{r})\sin(\omega_n s/c) \quad \textbf{(1.76)}$$

where the longitudinal and transverse loss factors are given by:

$$\kappa_n = \frac{V_n^*(\mathbf{r_1})V_n(\mathbf{r})}{4U_n}, \ \boldsymbol{\kappa}_{n\perp} = \frac{V_n^*(\mathbf{r_1})\nabla_{\perp}V_n(\mathbf{r})}{4U_n\omega_n/c} \quad \textbf{(1.77)}$$

and U_n is the energy stored in a particular mode n and the voltage evaluated from the integral of the axial electric field along L, the length of the cavity:

$$V_n(\mathbf{r}) = \int_0^L E_{zn}(r,z)\exp\left(\frac{i\omega_n z}{c}\right)dz \quad \textbf{(1.78)}$$

and a similar expression for $V_n(\mathbf{r_1})$**.**

The transverse wake function is zero at the bunch (s=0) and it increases linearly in close proximity behind the bunch. There is no wake in front of the bunch from causality considerations.

These wake functions are valid for v=c. For v<c additional correction terms of O(γ^{-2}) occur.

The m^{th} order multipole wake functions are:

$$W_{\|}^{m}(s) = 2\theta(s)\left(\frac{r_1}{a}\right)^{m}\left(\frac{r}{a}\right)^{m}\sum_{n}\kappa_{n}^{m}(\mathbf{r_1},\mathbf{r})\cos(\omega_{n}^{m}s/c) \qquad \textbf{(1.79)}$$

$$\mathbf{W}_{\perp}^{m}(s) = 2\theta(s)\left(\frac{r_1}{a}\right)^{m}\left(\frac{r}{a}\right)^{m-1}\left[\hat{r}\cos m\theta - \hat{\theta}\sin m\theta\right]\sum_{n}\frac{\kappa_n}{\omega_{n}^{m}a/c}(\mathbf{r_1},\mathbf{r})\sin(\omega_{n}^{m}s/c)$$

(1.80)

In particular for calculations on X-band structures for the NLC, the dipole (m=1) wake function is often computed in the form:

$$\mathbf{W}_{d}(s) = \frac{W_{\perp}(s)}{r_1 L} = 2\theta(s)\sum_{n}K_{n}(\mathbf{r_1})\sin(\omega_{n}^{m}s/c) \qquad \textbf{(1.81)}$$

where the "kick factor" is defined in terms of the loss factor:

$$K_n = \frac{c\kappa_n}{\omega_n a^2 L} \qquad \textbf{(1.82)}$$

Both the kick factor and specially defined dipole wake have units of V/C/m^2, and for X-band high energy linacs these units are often rewritten as: V/pC/mm/m.

Why use these strange units?

- **The millimeter factor arises from the offset of the drive bunch and this is usually of the order of mm (the iris has a radius of ~4mm or so).**
- **The per meter factor arises because each accelerating structure is of the order of 1m or less.**
- **The beam usually has a charge of ~ a few pC (approx 1.1 10^{10} particles are present in one bunch).**

To obtain the transverse momentum change of a trailing particle due to the wake left behind the drive particle:

- **multiply the specially defined dipole kick factor by the driving and witness charge, the length of the accelerating structure, and the offset of the drive charge and divide by the velocity of light ($\Delta p = Kqq_l La/c$).**
- **In terms of the normal kick factor one would compute $\Delta p = Kqq_l/c$.**

Wake Function in A Pill Box Cavity

- **A closed off, circular cylinder is taken as the cavity to analyze.**
- **In microwave parlance this is often known as a "pill box" cavity.**
- **This cavity permits a formally exact calculation of the wake functions. However, few of the summations can be evaluated in closed form.**

    - **Before proceeding, it is worth noting the general feature that the summation of the series of modes which describe the wake function converges rather slowly within the bunch itself.**
    - **Nonetheless, for positions well behind the bunch, the series converges much faster and hence the energy loss can be accurately evaluated.**

To evaluate the wake function we will use:

$$G(s) = 2\sum_{\mu} k_{\mu} \cos\left(\omega_{\mu}\frac{s}{c}\right), \quad k_{\mu} = \frac{V_{\mu}V_{\mu}}{4U_{\mu}} \qquad \textbf{(1.83)}$$

Multiplying by the charge of the drive particle gives in the potential seen by the trailing particle.

The longitudinal modes in the cavity are given by:

$$\omega_{np}^2/c^2 = \left(j_n/R\right)^2 + \left(\pi p/g\right)^2 = \nu_{np}^2 \qquad \textbf{(1.84)}$$

where is the nth solution of $J_0(j_n)$=0, g is the length of the cavity and R is the radius. The cavity fields are:

$$\begin{aligned} E_z^{n,p} &= \frac{j_n}{R} J_0\left(j_n \frac{r}{R}\right)\cos\left(\frac{\pi p z}{g}\right)\exp\left(i\omega_{np}t\right) \\ E_r^{n,p} &= \frac{\pi p}{g} J_1\left(j_n \frac{r}{R}\right)\sin\left(\frac{\pi p z}{g}\right)\exp\left(i\omega_{np}t\right) \\ H_\theta^{n,p} &= i\omega_{np}\varepsilon_0 J_1\left(j_n \frac{r}{R}\right)\cos\left(\frac{\pi p z}{g}\right)\exp\left(i\omega_{np}t\right) \end{aligned} \qquad \textbf{(1.85)}$$

The voltage is evaluated on the axis (r=0):

$$V_{np} = \int_0^g E_z(r=0, z, t=z/c) = \frac{i\nu_{np}R}{j_n}\left[1-\left(-1\right)^p \exp\left(i\nu_{np}g\right)\right] \qquad \textbf{(1.86)}$$

and thus:

$$V_{np}V_{np}^* = 2\left(\frac{\nu_{np}R}{j_n}\right)^2\left[1-\left(-1\right)^p \cos\left(\nu_{np}g\right)\right] \qquad \textbf{(1.87)}$$

The energy stored in the cavity is given by:

$$U_{np} = \int_0^R dr \int_0^{2\pi} d\theta \int_0^g dz\, H_\theta^{np} H_\theta^{*np} = \frac{\pi\varepsilon_0}{4}\nu_{np}^2 g R^2 J_1^2(j_n) \qquad \textbf{(1.88)}$$

Thus, the loss factor is evaluated as:

$$k_{np} = \frac{1}{\pi\varepsilon_0 g}\frac{2}{1+\delta_p^0}\frac{1-(-1)^p\cos(\nu_{np}g)}{j_n^2 J_1^2(j_n)} \qquad \textbf{(1.89)}$$

where δ_p^0 is the Kronecker delta function. The point charge longitudinal wake function is then given by the double summation:

$$\pi\varepsilon_0 gG(s) = 2\theta(s)\sum_{n=1}^{\infty}\sum_{p=-\infty}^{\infty}\frac{1-(-1)^p\cos(\nu_{np}g)}{j_n^2 J_1^2(j_n)}\cos(\nu_{np}s) \qquad \textbf{(1.90)}$$

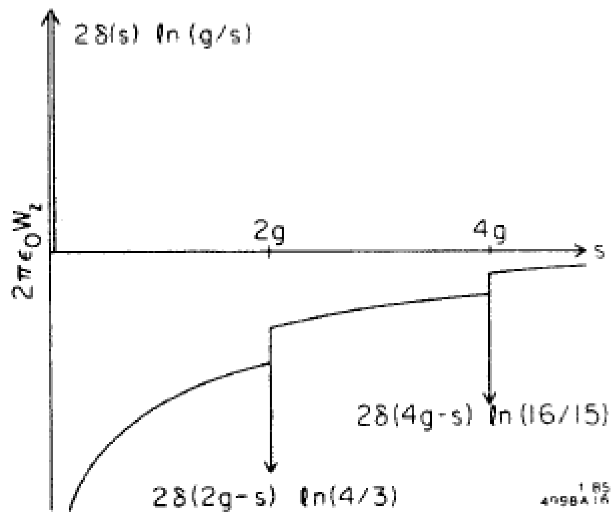
It is not possible, in general, to obtain a closed from to the above sum. However, for the special case of $s < s_0 \equiv \sqrt{4R^2+g^2} - g$ the sums can partially be evaluated to a series of delta functions:

$$2\pi\varepsilon_0 gG(s) = 2\delta(s)\ln\left[\frac{g}{s}\right] -$$

$$2\sum_{n=1}^{\infty}\delta(2ng-s)\ln\left[\frac{s^2}{s^2-g^2}\right] - \frac{1}{g}\left\{\left(\left[\frac{s}{2g}\right]_{IP} + \frac{s}{2g}\right)^{-1} - \left(\left[\frac{s}{2g}\right]_{IP} + \frac{s}{2g}+1\right)^{-1}\right\} \qquad \textbf{(1.91)}$$

- **Applying causality, this wake must be the same as that produced in between two parallel plates. That is, no signal is able to propagate from the point where the driving charge enters the cavity, be reflected from the outer wall, and return to the path followed by the driving charge with a distance s_0 behind it.**

- **The wake is accelerating at all points accept at s = 0. The driving charge itself experiences an infinite retarding potential at the moment it exits the second plane of two parallel plates.. Spherical wavefronts, which expand with the velocity of light, are generated when the charge enters through the first plane and again when it leaves through the second plane. On the axis two of these wavefronts join in the double cusp geometry shown at position X. When a trailing particle meets and passes through this singularity or a later reflection of it, it will experience a finite accelerating potential given by the third term on (1.91). For small s this accelerating potential diverges as 1 /s. If s is a multiple of 2g the test particle will travel with the singularity across the cavity and experience an infinite accelerating potential.**



The convolution of the point source wake in (1.90) with a Gaussian current:

$$I(s) = \frac{\exp\left(-\frac{s^2}{2\sigma^2}\right)}{\sqrt{2\pi}\sigma} \qquad \mathbf{(1.92)}$$

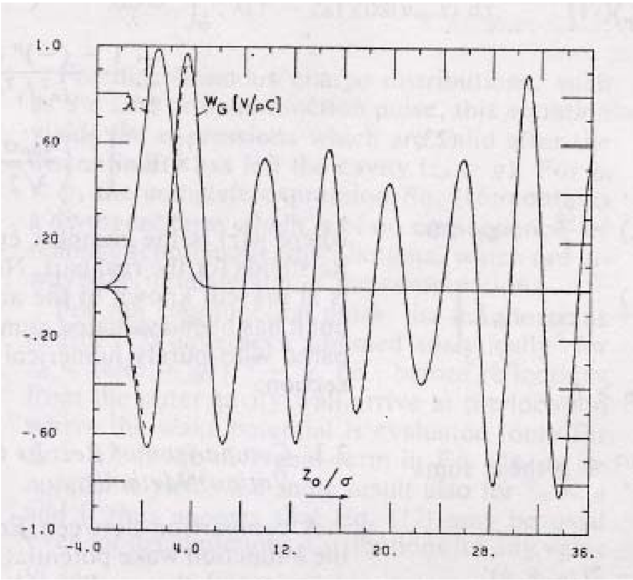
gives the Gaussian bunch wake function:

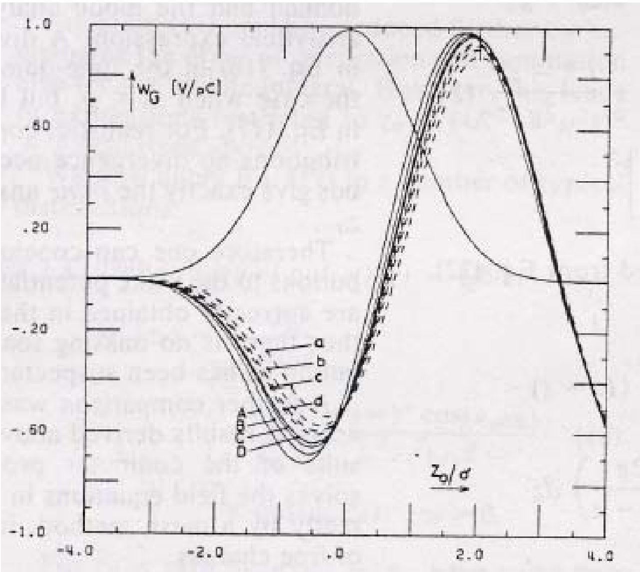
$$\pi\varepsilon_0 g G(s) =$$

$$\theta(s)\exp\left(-\frac{s^2}{2\sigma^2}\right)\sum_{n=1}^{\infty}\sum_{p=-\infty}^{\infty}\frac{1-\left(-1\right)^p\cos\left(\nu_{np}g\right)}{j_n^2 J_1^2(j_n)}\mathrm{Re}\left\{w\left(\frac{\nu_{np}\sigma}{\sqrt{2}}-\frac{is}{\sqrt{2}}\right)\right\} \qquad \mathbf{(1.93)}$$

where w(z) is the complex error function:

$$\int_0^{\infty} dt\exp\left(-a^2t^2+izt\right) = \frac{\sqrt{\pi}}{2a}w\left(\frac{z}{2a}\right) \qquad \mathbf{(1.94)}$$





Computed Pill-Box Wake Function for Gaussian Bunch

Mode results are shown with solid lines. Dashed lines correspond to a time domain simulation. The fig on the right is shows the effect of changing the number of modes: a,b,c,d =10, 40, 160, and 64 modes, respectively.

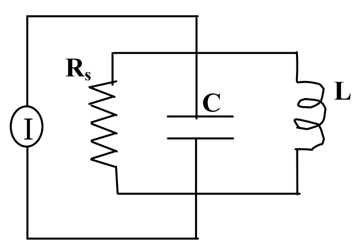
- **"Outside" the realm of the bunch, i.e. $|s|>4\sigma$ the wake function is accurately computed with a limited number of modes –the time domain and modal analysis agree very well! Ten modes is sufficient in this particular example for an accurate computation of the wake.**

- **Inside the region the bunch, i.e. $|s|<4\sigma$ many modes are necessary in order to accurately compute the wake function. As the number of modes is increased and the number of mesh points in the time domain method is increases both methods converge towards the same value -they converge from opposite ends.**

- **The wake function is decelerating within the bunch. Is this physically correct? Yes, otherwise the bunch would be continuously accelerated by its own field. Although, the tail of the bunch is in an accelerating region and this does permit acceleration of the tail by the head of the bunch.**

- **There is no damping in this system. Hence the oscillation in the wake is allowed to rise up again at some point.**

R-L-C Circuit Model of Single Mode and Impedance-Wake Relations



Each cell of the accelerating structure is represented by an R-L-C circuit. The circuit has a shunt impedance R_s, an inductance L and a capacitance C. In practice this represents the fields present in the structure and they cannot readily be measured. However, related quantities can be measured for a so simple R-L-C circuit, namely, the cavity resonance frequency, ω_r, the quality factor Q and the damping factor α:

$$\omega_r = (LC)^{-1/2}, \;\; Q = R_s(C/L)^{1/2}, \;\; \alpha = \omega_r/(2Q) \quad \textbf{(1.95)}$$

The circuit is driven by a current I and the voltages across each element are identical:

$$V = I_R R_s = \frac{1}{C}\int dt\, I_c = L\frac{dI_L}{dt} \qquad \textbf{(1.96)}$$

Differentiating with respect to time t give the total current as:

$$\frac{dI}{dt} = \left(\frac{1}{R_s}\frac{d}{dt} + C\frac{d^2}{dt^2} + \frac{1}{L}\right)V \qquad \textbf{(1.97)}$$

and this is readily rewritten as:

$$\left(\frac{d^2}{dt^2} + \frac{\omega_r}{Q}\frac{d}{dt} + \omega_r^2\right)V = \frac{\omega_r R}{Q}\frac{dI}{dt} \qquad \textbf{(1.98)}$$

The solution is a damped oscillation:

$$V(t) = e^{-\alpha t} A\cos\omega_r^{'} t + e^{-\alpha t} B\sin\omega_r^{'} t,\ \ \omega_r^{'} = \omega_r\left(1-(4Q^2)^{-1}\right)^{1/2} \qquad \textbf{(1.99)}$$

The Wake Potential is calculated by enforcing a delta function driving current:

$$I(t) = q\delta(t) \qquad \textbf{(1.100)}$$

This instantaneously induces a voltage across the capacitor:

$$V(0^+) = q/C = (\omega_r R_s/Q)q \qquad \textbf{(1.101)}$$

The energy stored in the capacitor is equal to the energy lost by the point charge:

$$U = \frac{q^2}{C} = \frac{\omega_r R_s}{2Q} q^2 = \frac{V(0^+)}{2} q = \kappa q^2 \qquad \textbf{(1.102)}$$

where κ is the mode loss factor:

$$\kappa = \frac{U}{q^2} = \frac{\omega_r R_s}{2Q} \qquad \textbf{(1.103)}$$

This capacitor then discharges through the resistor and through the inductance:

$$\left.\frac{dV}{dt}\right|_{t=0^+} = -\frac{1}{C}\frac{dq}{dt} = -\frac{I_R}{C} = -\frac{1}{C}\frac{V(0^+)}{R_s} = -\frac{\omega_r^2 R_s}{Q^2} q = -\frac{2\omega_r \kappa}{Q} q \qquad \textbf{(1.104)}$$

Thus we have the initial conditions:

$$V(0^+) = 2\kappa q \;\; \textbf{and} \;\; \dot{V}(0^+) = -\frac{2\omega_r \kappa}{Q} q \qquad \textbf{(1.105)}$$

Thus we now enforce these initial conditions in order to solve for the constants A and B. The differential of the voltage is given by:

$$\frac{dV}{dt} = e^{-\alpha t}[(-A\alpha + B\omega_r^{'})\cos\omega_r^{'} t - (B\alpha + A\omega_r^{'})\sin\omega_r^{'} t] \qquad \textbf{(1.106)}$$

Thus we obtain:

$$A = 2\kappa q \;\; \textbf{and} \;\; -A\alpha + B\omega_r^{'} = -\frac{2\omega_r \kappa q}{Q} \qquad \textbf{(1.107)}$$

and this allows the voltage response to a delta function current excitation to be obtained as:

$$V(t) = 2q\kappa e^{-\alpha t}\left[\cos\omega_r' t - \frac{\sin\omega_r' t}{2Q(\omega_r'/\omega_r)}\right], \quad \omega_r' = \omega_r(1-(2Q)^{-2})^{1/2} \quad \textbf{(1.108)}$$

This voltage is induced by a point charge going through a cavity at t =0. A second charge q' will at a time t gain or loose energy U = q'V(t). This energy loss or gain per unit source and probe charge is the wake or Green function G(t). For this cavity resonance we have:

$$G(t) = 2\kappa e^{-\alpha t}\left[\cos\omega_r' t - \frac{\sin\omega_r' t}{2Q(\omega_r'/\omega_r)}\right] \quad \textbf{(1.109)}$$

Typically the quality factor is very high and thus:

$$G(t) = 2\kappa e^{-\alpha t}\cos\omega_r' t \quad \textbf{(1.110)}$$

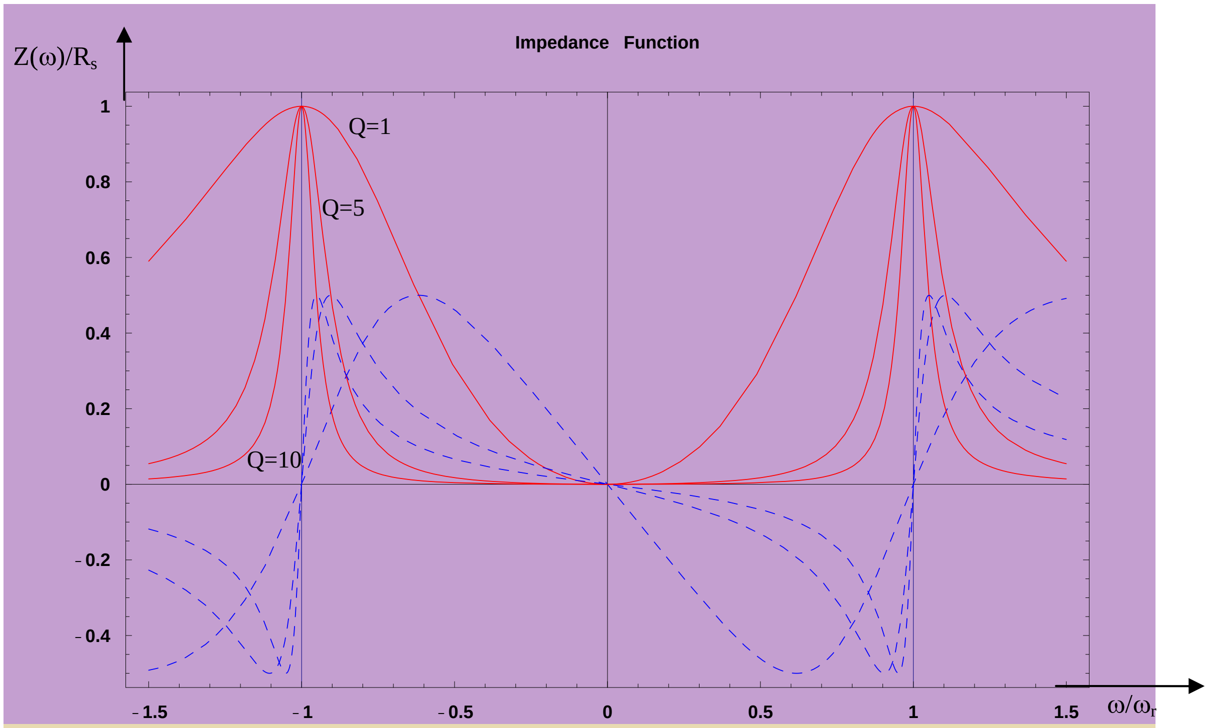
To evaluate the impedance we switch to a complex phasor notation:

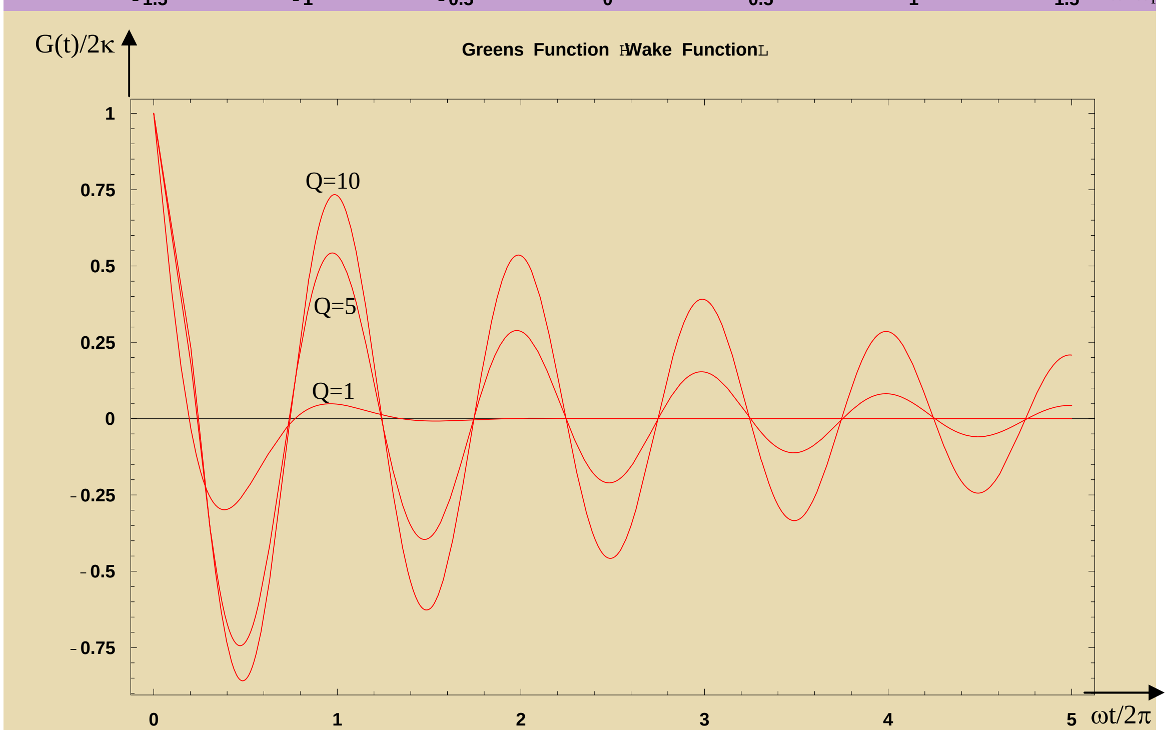
$$I(t) = \hat{I}\exp(j\omega t), \quad V(t) = V_0\exp(j\omega t) \quad \textbf{(1.111)}$$

and thus the differential equation for the R-L-C circuit becomes:

$$\left(-\omega^2 + j\frac{\omega_r\omega}{Q} + \omega_r^2\right)V_0\exp(j\omega t) = j\frac{\omega_r\omega R_s}{Q}\hat{I}\exp(j\omega t) \quad \textbf{(1.112)}$$

The impedance is the ratio of the voltage to the current:


Impedance Function
Z(ω)/Rs
Q=1
Q=5
Q=10
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5
ω/ωr


Greens Function HWake FunctionL
G(t)/2κ
Q=10
Q=5
Q=1
1
0.75
0.5
0.25
0
-0.25
-0.5
-0.75
0
1
2
3
4
5
ωt/2π

$$Z(\omega)=V/I=V_0/\hat{I}=R_s\frac{j\omega_r\omega Q^{-1}}{\omega_r^2-\omega^2+j\omega_r\omega Q^{-1}}=R_s\frac{1-jQ\left(\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)}{1+Q^2\left(\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2}=\mathrm{Re}\{Z\}+j\mathrm{Im}\{Z\}$$

(1.113)

For relatively loss-less systems with very high quality factors then the impedance is large in the vicinity of $\omega\approx\omega_r$ **or** $\Delta\omega/\omega_r(=|\omega-\omega_r|/\omega_r)\ll 1$ **and this allows the impedance to be simplified to:**

$$Z(\omega)=R_s\frac{1-j2Q\Delta\omega/\omega_r}{1+4Q^2(\Delta\omega/\omega_r)^2} \qquad \textbf{(1.114)}$$

Resonator has the following properties, which are used in the coupled circuit design described in the following lectures:

$\omega=\omega_r\rightarrow Z_r(\omega_r)$ **has a maximum, and** $Z(\omega_r)=0$

$|\omega|<\omega_r\rightarrow Z_i(\omega)>0$ **the impedance is inductive**

$|\omega|>\omega_r\rightarrow Z_i(\omega)<0$ **the impedance is capacitive**

Further, for any impedance or potential it can readily be shown that:

$$Z_r(\omega)=Z_r(-\omega),\quad Z_i(\omega)=-Z_i(\omega) \qquad \textbf{(1.115)}$$

$Z(\omega)=\int_{-\infty}^{\infty}G(t)\exp(-j\omega t)dt$, **Z(ω) = the Fourier transform of G(t), the Wake function**

Methods of Wakefield HOM Calculation

1. ***Finite difference + finite element codes => MAFIA, Omega3 (3-d frequency domain), Tau3(3-d time domain) GdfidL (3-D freq/time domain), ABCI (time domain), HFSS (finite difference freq. domain)***

2. Mode-matching (frequency domain) => Smart2D(2-d, match modes transversely), Transvrs (single periodic iris, match modes longitudinally), Cascade(2-d, match modes transversely)

3. Circuit models: Single and dual mode manifold-damped, frequency domain models

Lecture #1 Homework

1. Measurements made on an accelerator cavity indicate that the impedance of the cavity is almost entirely inductive in nature ($Z = j\omega L$)

(a) By taking the Fourier transform calculate the longitudinal wake function, W(s) for this case. (You may find it helpful, to use the following FT pairs $W(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega)\exp(j\omega\tau)d\omega$, $Z(\omega) = \int_{-\infty}^{\infty} W(t')\exp(-j\omega t')dt$)

(b) Take the convolution with a general time dependent current, I(t) and hence obtain the voltage.

(c) Obtain the bunch wakefield for a charge with a Gaussian distribution by taking the convolution with a Gaussian line density ($\lambda(s) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{s^2}{2\sigma_z{}^2}\right)$) with the wake of part a.

2. Given the fact that the wakefield $W(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega)\exp(j\omega\tau)d\omega$, is a real quantity and that for a beam traveling at the velocity of light there can be no wake ahead of the beam (causality), prove that:

(a) Re{Z(ω)} = Re{Z(-ω)} and Im{Z(ω)} = Im{Z(-ω)}

(b) the self-wake seen by the driving particle itself is half that of the total wakefield that a trailing particle will see (hint: expand the exp(i ωt) into sin and cos functions and take the real and imaginary parts of the wake given in terms of the inverse Fourier transform of impedance, then derive an expression for the wake that does not take into account causality and compare it to one that does include it) . This is known as the *fundamental theorem of beam loading*

3. Given the longitudinal impedance (per unit length) of a lossy circular waveguide of radius, b and, conductivity, σ: Z = $(1+j)\frac{1}{2\pi b}\sqrt{\frac{\omega Z_0}{2c\sigma}}$, calculate the associated wake function. Here, Z_0 =377 Ohms is the impedance of free space and c is the velocity of light (hint: refer to the table 2.1 of transforms in chapter 2 of A. Chao's book or consider a branch cut in the contour integral).

4. (a)Derive the expression for the longitudinal loss factor of a bunch in terms of the λ(k), the Fourier transform of the line density of the bunch and the impedance of the wake Z(k) (answer: $k_{loss} = \frac{c}{\pi}\int_0^{\infty} Z(k)\,|\,\lambda(k)\,|^2 dk$).

(b) Now do the same for the transverse loss factor (often called the kick factor) for $\lambda(s) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{s^2}{2\sigma_z{}^2}\right)$ and the transform $\lambda(\omega) = \exp\left(-\frac{\omega^2}{2\sigma_z{}^2 c^2}\right)$. In this case the impedance is defined in terms of the transverse wake by: $Z_t(\omega) = -\frac{i}{c}\int_0^\infty W_t(s)e^{i\omega s/c}ds$ (hint: take real and imaginary parts of the wake and realize that the imaginary part of the impedance now plays the same role as the real part did in part a. Answer: $k_{t,loss} = \frac{2}{\pi^{3/2}}\int_0^\infty Z_r(w)D(\omega\sigma_z / c)d\omega$ where Dawson's integral is given by: $D(x) = e^{-x^2}\int_0^x e^{y^2}dy$).

In all questions, pay attention to the units of your answers. For example, you will always expect to see longitudinal impedance in units of Ohms/m and wakefields in V/C/m (or V/pC/m etc).

APPENDIX A: Modal Sum Representation of Wakefield via Field Function Analysis

It will be shown that for any cavity the wakefield may be expanded in a modal sum:

$$W_{||}(s) = \sum_n 2\kappa_n \cos(\omega_n s / c) \ \forall s > 0 \qquad \textbf{(1.116)}$$

Where s>0 is refers to the distance behind the driving bunch (by causality for s<0 then W =0). The κ_n are the characteristic loss factors of the structure and ω_n are the cavity resonance frequencies; both of which are readily calculated with computer codes, HFSS, KN7C, MAFIA or OMEGA3 (a finite element computer code code developed at SLAC), for example.

In order to prove the above general expansion we resort to a vector and scalar potential representation of the fields:

$$\mathbf{E} = \frac{\partial}{\partial t}\mathbf{A} - \nabla\Phi, \quad \mathbf{B} = \nabla \mathrm{x} \mathbf{A} \qquad \textbf{(1.117)}$$

Substituting these relations into Maxwell's equations readily yields:

$$\nabla^2 \mathbf{A} - c^{-2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0 \mathbf{j} + c^{-2}\frac{\partial}{\partial t}\nabla\Phi \qquad \textbf{(1.118)}$$

$$\nabla^2\Phi = -\varepsilon_0^{-1}\rho \qquad \textbf{(1.119)}$$

Where a Coulomb gauge has been used for the potentials:

$$\nabla.\mathbf{A}=0 \qquad \textbf{(1.120)}$$

The vector potential itself can be expanded into the modes a_n of the closed resonator structure:

$$\mathbf{A}(r,t)=\sum_n q_n(t)\mathbf{a}_n(r) \qquad \textbf{(1.121)}$$

where the a_n are required to satisfy the equation:

$$\left[\nabla^2+(\omega_n/c)^2\right]\mathbf{a}_n(r)=0 \qquad \textbf{(1.122)}$$

where the ω_n are the structure eigenfrequencies. Also, the a_n form a complete orthogonal set of basis vectors and we choose:

$$\frac{\varepsilon_0}{2}\int d^3r\,\mathbf{a}_\mathbf{n}^*(\mathbf{r}).\mathbf{a}_\mathbf{m}(\mathbf{r})=U_n\delta_{nm} \qquad \textbf{(1.123)}$$

where U_n is a normalizing factor and δ is the usual kronecker delta function.
This allows the wave equation to be rewritten in the form:

$$U_n\left[\omega_n^2+\frac{d^2}{dt^2}\right]q_n(t)=\frac{1}{2}\int d^3r\,\mathbf{a}_\mathbf{n}^*(r,t).\mathbf{j}(r,t)-\frac{\varepsilon_0}{2}\frac{\partial}{\partial t}\int d^3r\,\mathbf{a}_\mathbf{n}^*(r,t).\nabla\Phi(r,t) \qquad \textbf{(1.124)}$$

The second volume integral is transformed into a surface integral by Gauss' theorem and this vanishes

due to the boundary conditions imposed on Φ. Here, $\mathbf{a}_n^*.\nabla\Phi = \nabla(\mathbf{a}_n^*\Phi) - \Phi\nabla.\mathbf{a}_n^* = \nabla(\mathbf{a}_n^*\Phi)$ **has been used and the Coulomb gauge has been applied once more. Thus, the Fourier transform of the expansion coefficient is obtained as:**

$$\tilde{q}_n(\omega) = \int_{-\infty}^{\infty} dt\, q_n(t)\exp(-i\omega t) = \frac{1}{2U_n}\frac{1}{\omega-\omega_n^2}\int d^3r\,\mathbf{a}_n^*.\tilde{\mathbf{j}}(r,\omega) \tag{1.125}$$

The Fourier transform of the vector potential is:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \sum_{\mathbf{n}} \tilde{q}_n(\omega)\mathbf{a}_n(\mathbf{r}) \tag{1.126}$$

The scalar potential is also expanded into a complete orthonormal system:

$$\Phi(\mathbf{r},\omega) = \sum_{\mathbf{n}} r_n(\omega)\phi_n(\mathbf{r}) \tag{1.127}$$

and ϕ satisfy the equation:

$$\left[\nabla^2 + (\Omega_n/c)^2\right]\phi_n(r) = 0 \tag{1.128}$$

with boundary conditions that ϕ is zero on the surface surrounding the volume of the cavity. The orthogonalisation condition is chosen to define T such that:

$$\frac{\varepsilon_0}{2}\left(\Omega_n/c\right)^2\int d^3r\,\phi_n^*(r)\phi_m(r) = T_n\delta_{nm} \tag{1.129}$$

This allows the expansion coefficients of Φ to be obtained as:

$$r_n(t)=\left(2T_n\right)^{-1}\int d^3r\phi_n^*(r)\rho(r,t) \qquad \textbf{(1.130)}$$

Evaluation of Impedance:

The longitudinal impedance is defined as:

$$Z(x,y,s)=\frac{1}{q}\int_{-\infty}^{\infty} dz\,\tilde{E}_z(\mathbf{r},\omega)\exp(i\omega z/c) \qquad \textbf{(1.131)}$$

The electric field $\tilde{\mathbf{E}}(\mathbf{r},\omega)=i\omega\tilde{\mathbf{A}}(\mathbf{r},\omega)-\nabla\tilde{\Phi}(\mathbf{r},\omega)$ is excited by a charge q and here we consider a point charge moving parallel to the z-axis:

$$\begin{aligned}\rho(\mathbf{r},t)&=q\delta(z-ct)\delta(x-x_0)\delta(y-y_0)\\ \mathbf{j}(\mathbf{r},t)&=c\mathbf{e}_z\rho(\mathbf{r},t)\end{aligned} \qquad \textbf{(1.132)}$$

The Fourier transform of the charge density and current are given by:

$$\begin{aligned}\tilde{\rho}(\mathbf{r},\omega)&=\frac{q}{c}\exp(-i\omega z/c)\delta(x-x_0)\delta(y-y_0)\\ \tilde{j}_z(\mathbf{r},\omega)&=q\exp(-i\omega z/c)\delta(x-x_0)\delta(y-y_0)\end{aligned} \qquad \textbf{(1.133)}$$

Making use of the relations for r_n and q_n, derived above then the electric field is obtained as:

$$\tilde{\mathbf{E}}(r,\omega)=q\sum_n \frac{-i\omega}{\omega^2-\omega_n^2}\mathbf{a}_n(r)\frac{1}{2U_n}\int_{-\infty}^{\infty} dz' a_{nz}^*(x_0,y_0,z')\exp(-i\omega z'/c) -\frac{q}{c}\sum_n \nabla\phi_n^*(x_0,y_0,z')\exp(-i\omega z'/c) \quad \textbf{(1.134)}$$

The complex voltages are defined:

$$V_n(x,y,\omega)=\int_{-\infty}^{\infty} dz\, a_{zn}(x,y,z)\exp(i\omega z/c)$$
$$v_n(x,y,\omega)=\int_{-\infty}^{\infty} dz\left(\frac{\partial}{\partial z}\phi_n(x,y,z)\right)\exp(i\omega z/c) \quad \textbf{(1.135)}$$

Thus the longitudinal impedance is written as:

$$Z_0(x,y,\omega)=\sum_n \frac{-i\omega}{\omega^2-\omega_n^2}\frac{1}{2U_n}V_n(x,y,\omega)V_n^*(x_0,y_0,\omega) +\sum_n \frac{i}{\omega}\frac{1}{2T_n}v_n(x,y,\omega)v_n^*(x_0,y_0,\omega) \quad \textbf{(1.136)}$$

where the following relation, derived by integrating by parts, has been used:

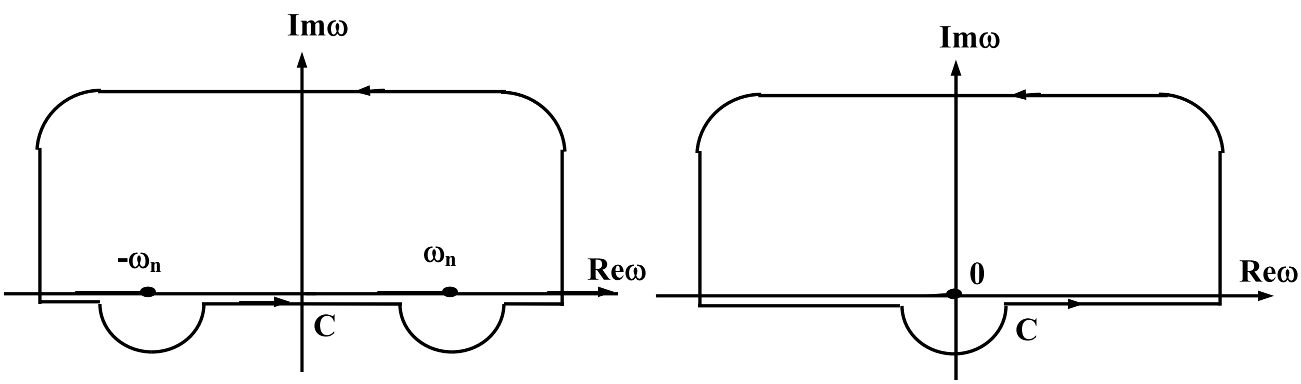
$$\frac{1}{c}\int_{-\infty}^{\infty} dz\,\phi_n^*(x_0,y_0,z)\exp(-i\omega z/c)=-\frac{i}{\omega}\int_{-\infty}^{\infty} dz\left(\frac{\partial}{\partial z}\phi(x_0,y_0,z)\right)\exp(-i\omega z/c)$$

$$\textbf{(1.137)}$$

and the condition ϕ_n =0 at the boundary has been invoked.

The longitudinal Wake Potential is obtained by the inverse transform of the impedance:

$$W_{\parallel}(x,y,s)=\frac{1}{2\pi}\oint_C d\omega Z(x,y,\omega)\exp(i\omega s/c) \qquad \textbf{(1.138)}$$



We integrate around the closed contour indicated and for s>0 we close the contour in the upper half plane and for s<0 we close the contour in the lower half plane. The second term of the impedance (which occurs due to the scalar potential) has a pole at ω=0 but it does not contribute to the wake because:

$$v_n(x,y,0)=\int_{-\infty}^{\infty} dz\left(\frac{\partial}{\partial z}\phi_n(x,y,z)\right)=\left[\phi_n(x,y,z)\right]_{\text{boundary}} \qquad \textbf{(1.139)}$$

Thus the wake function is given as:

$$\begin{aligned} W_{\parallel}(x,y,s)=\sum_n \frac{1}{4U_n}[&V_n(x,y,\omega_n)V_n^*(x_0,y_0,\omega_n)\exp(i\omega_n s/c) \\ &+V_n(x,y,-\omega_n)V_n^*(x_0,y_0,-\omega_n)\exp(-i\omega_n s/c)] \end{aligned} \qquad \textbf{(1.140)}$$

We are at liberty to choose real eigenvectors a_n and this makes:

$$V_n^*(x,y,-\omega)=V_n(x,y,\omega) \qquad \textbf{(1.141)}$$

and we will specialize to the case {x,y} = {x_0,y_0} to give:

$$W_{||}(x,y,s)=\sum_n 2\frac{|V_n(x,y,\omega_n)|^2}{4U_n}\cos(\omega_n s/c) \quad \forall s>0 \qquad \textbf{(1.142)}$$

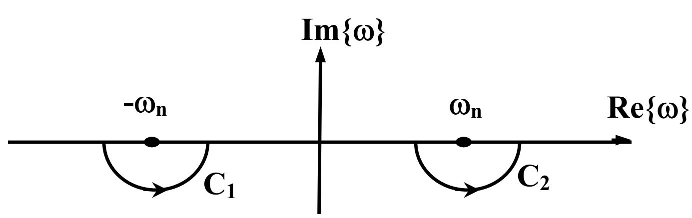
The loss parameter, of the transverse mode is given by:

$$\kappa_n = |V_n(x,y,\omega_n)|^2/(4U_n) \qquad \textbf{(1.143)}$$

It remains to calculate the wakefield at s = 0:

$$W_0(x,y,0)=\frac{1}{2\pi}\oint_C d\omega Z_0(x,y,\omega) \qquad \textbf{(1.144)}$$

The impedance function is an odd function ($Z_0(x,y,\omega)=-Z_0(x,y,-\omega)$) and thus it must be evaluated for the two contours C_1 and C_2 which consist of semi-circles with radius ε (where the limit ε->0 will be taken).



The contour C_1 gives:

$$\begin{aligned}\frac{1}{2\pi}\int_{C_1} d\omega \frac{-\omega}{\omega^2-\omega_n^2}\frac{|V_n(x,y,\omega)|^2}{2U_n} &= \frac{1}{2}\frac{1}{i\pi}\int_{C_1} d\omega \frac{1}{\omega+\omega_n}\frac{|V_n(x,y,\omega)|^2}{2U_n} \\ &= \frac{1}{2}\frac{1}{i\pi}\int_0^{\pi} d\phi \frac{i|V_n(x,y,\omega_n-\varepsilon e^{i\phi})|^2}{4U_n} \\ &= \frac{1}{2}\kappa_n\end{aligned} \qquad \textbf{(1.145)}$$

Similarly for the contour C_2 and thus in general the longitudinal wakefield is given by:

$$W_{||}(s) = \theta(s)\sum_n 2\kappa_n \cos(\omega_n s/c) \qquad \textbf{(1.146)}$$

$$\theta(s) = \begin{cases} 0 & s<0 \\ 1/2 & s=0 \\ 1 & s>0 \end{cases}$$

Causality is expressed in this equation by the fact that the wakefield is zero ahead of the driving bunch. Further, what is sometimes called the *fundamental theorem of beam loading* is expressed by the factor of ½ which describes the wakefield felt by the driving bunch itself.