# Linear Normal Form and "Ascript"

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# "Asrcipt"

- Definition
  - "Ascript" is a symplectic transformation from the normal to physical coordinates
- Why ascript?
  - Only have to deal with real matrix and TPSA
  - Relate to a rotation
  - Closer to conventional treatment such as Courant-Synder parameters
  - Natural extension from one-dimensional case
  - Include coupling and effects of errors

# Courant-Snyder Parameters

One-turn matrix:

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Rotation matrix:

$$R = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}$$

We have:

$$M = ARA^{-1}$$

where $A^{-1}$ is a transformations from physical to normalized coordinates:

$$A^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}, A = \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{-\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

A is an "ascript" and is not unique. Since two-dimensional rotational group is commutative AR(q) is also an ascript. Courant and Synder choose to have $A_{12}$=0.

# Symplectic Matrix

M is a symplectic matrix if it has the property that

$$MJM^T = J,$$

where J is

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

# How to Construct "Ascript"

We use eigen vectors to construct a complex symplectic matrix

$$U = [E_I, iE_{-I}, E_{II}, iE_{-II}, E_{III}, iE_{-III}],$$

which is symplectic and has the property that

$$U^{-1}MU = \Lambda = diag(e^{i2\pi\nu_I}, e^{-i2\pi\nu_I}, e^{i2\pi\nu_{II}}, e^{-i2\pi\nu_{II}}, e^{i2\pi\nu_{III}}, e^{-i2\pi\nu_{III}})$$

"Ascript" is defined as A=UK has the property that

$$A^{-1}MA = R = K^{-1}\Lambda K$$

$$K = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i & 0 & 0 & 0 & 0 \\ -i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -i & 0 & 0 \\ 0 & 0 & -i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 & -i & 1 \end{pmatrix}$$

Further more A is symplectic and real.
Clearly, it is an extension of one dimension case.

# A Solution of Ascript

Explicitly, ascript can be written

$$A = \sqrt{2}[\mathrm{Re}\, E_I, \mathrm{Im}\, E_I, \mathrm{Re}\, E_{II}, \mathrm{Im}\, E_{II}, \mathrm{Re}\, E_{III}, \mathrm{Im}\, E_{III}],$$

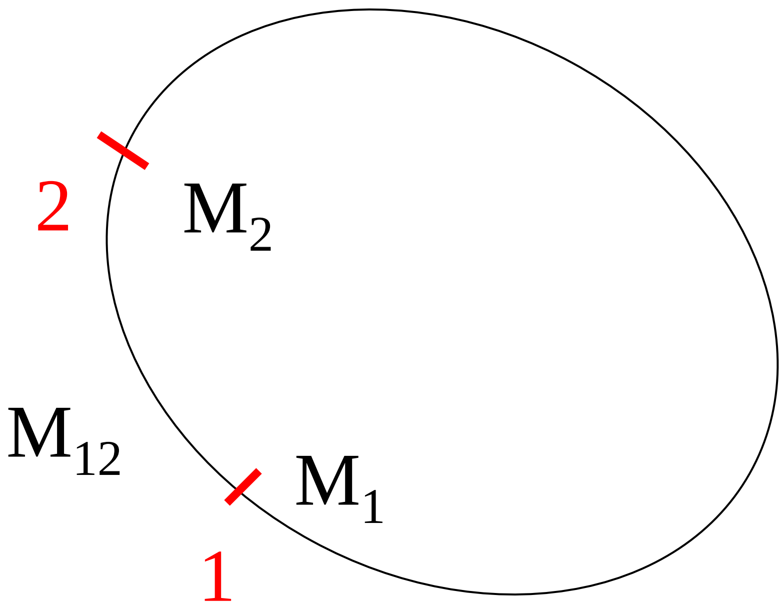
The eigen vectors are normalized as

$$E^{T^*}_{I,II,III} J E_{I,II,III} = i,$$

$$E^{T^*}_{-I,-II,-III} J E_{-I,-II,-III} = -i$$

How to get ascript directly from the one-turn matrix? Given ascript, we have $U=AK^{-1}$, which we should use in our map analysis. How about propagation of U? $A_2=T_{12}*A_1$ leads to $U_2=T_{12}*U_1$. But that implies we need to write force in complex, That is rather "dangerous". Therefore, we should use the complex coordinates only in the analysis.

# Propagation of "Ascript"

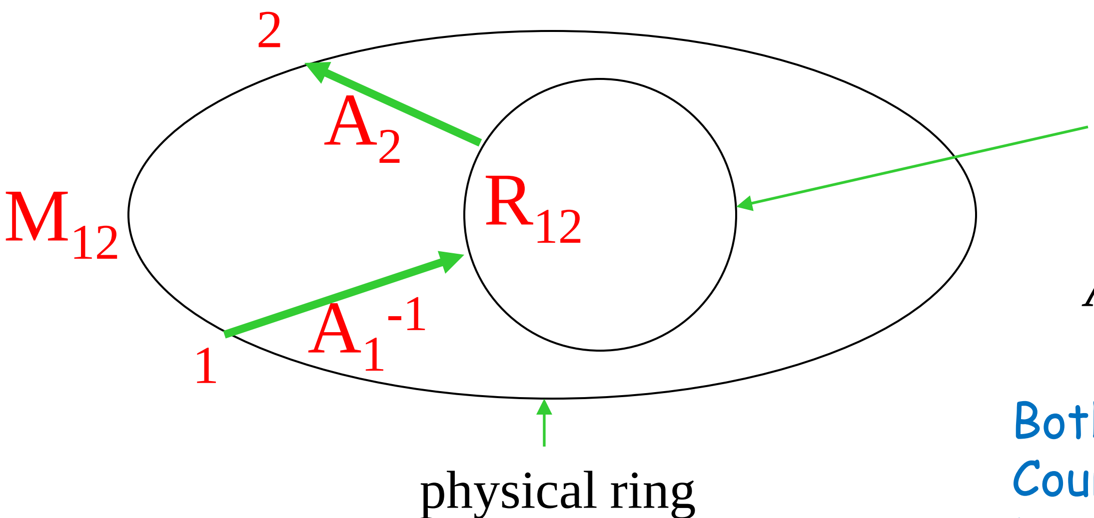



$M_1$ and $M_2$ are one-turn matrices at position 1 and respectively. $M_{12}$ is the transport matrix from 1 to 2. It is easy to show

$$M_2 = M_{12} M_1 M_{12}^{-1}$$

As a result of this identity, $\tilde{A}_2 = M_{12} A_1$ is an "ascript" At position 2 if $A_1$ is an "ascript" at position 1. We do not need to solve eigen vectors at every position in the ring.

# Propagation of "Lattice Functions"




$$\tilde{A}_2 = A_2 R(\psi_{12}) = M_{12} A_1$$

Both $A_1$ and $A_2$ are in Courant-Synder form, namely its 12 element is zero.

lattice functions at location 2:

$$\beta = \tilde{A}_{11}^2 + \tilde{A}_{12}^2, \alpha = -(\tilde{A}_{11}\tilde{A}_{21} + \tilde{A}_{12}\tilde{A}_{22}), \gamma = \tilde{A}_{21}^2 + \tilde{A}_{22}^2$$

phase advance:

$$\psi_{12} = \tan^{-1} \tilde{A}_{12} / \tilde{A}_{11}$$

# Edwards-Teng Coupling Parameters

Given an one-turn matrix M, we can decouple it with a symplectic transformation:

$C_{ET}$

$$M = \begin{pmatrix} gI & \bar{w} \\ -w & gI \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} gI & -\bar{w} \\ w & gI \end{pmatrix},$$

where $u_1$ and $u_2$ can be parameterized as if no coupling case and w is a symplectic matrix:

$$u_1 = \begin{pmatrix} \cos\mu_1 + \alpha_1 \sin\mu_1 & \beta_1 \sin\mu_1 \\ -\gamma_1 \sin\mu_1 & \cos\mu_1 - \alpha_1 \sin\mu_1 \end{pmatrix},$$

$$u_2 = \begin{pmatrix} \cos\mu_2 + \alpha_2 \sin\mu_2 & \beta_2 \sin\mu_2 \\ -\gamma_2 \sin\mu_2 & \cos\mu_2 - \alpha_2 \sin\mu_2 \end{pmatrix},$$

$$w = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}.$$

There are ten independent parameters. Bar notes symplectic conjugate. $g^2=1-\det(w)$.

# "Ascript" for Coupled Lattices

$$A = C_{ET} A_{CS} = \begin{pmatrix} g\sqrt{\beta_1} & 0 & \frac{w_{12}\alpha_2 + w_{22}\beta_2}{\sqrt{\beta_2}} & -\frac{w_{12}}{\sqrt{\beta_2}} \\ -\frac{g\alpha_1}{\sqrt{\beta_1}} & \frac{g}{\sqrt{\beta_1}} & -\frac{w_{11}\alpha_2 + w_{21}\beta_2}{\sqrt{\beta_2}} & \frac{w_{11}}{\sqrt{\beta_2}} \\ \frac{w_{12}\alpha_1 - w_{11}\beta_1}{\sqrt{\beta_1}} & -\frac{w_{12}}{\sqrt{\beta_1}} & g\sqrt{\beta_2} & 0 \\ \frac{w_{22}\alpha_1 - w_{21}\beta_1}{\sqrt{\beta_1}} & -\frac{w_{22}}{\sqrt{\beta_1}} & -\frac{g\alpha_2}{\sqrt{\beta_2}} & \frac{g}{\sqrt{\beta_2}} \end{pmatrix}$$

$$g = \sqrt{1-(w_{11}w_{22} - w_{12}w_{21})}, A^{-1} = -JA^T J$$

A is sympletic and its presentation is far from unique. In fact, there are two independent angles. There are eight independent parameters.

# " Symplectic Dispersion Matrix"
## by Ohmi, Hirata, and Oide

$$H = \begin{pmatrix} (1-\frac{|h_x|}{1+a})I & -\frac{h_x \bar{h}_y}{1+a} & h_x \\ -\frac{h_y \bar{h}_x}{1+a} & (1-\frac{|h_y|}{1+a})I & h_y \\ -\bar{h}_x & -\bar{h}_y & aI \end{pmatrix}$$

$h_x$ and $h_y$ are 2x2 matrices and parameter a is related to their determinates by

$$a^2 + |h_x| + |h_y| = 1$$

H has 8 independent parameters. Four parameters describe dispersions and the other fours for "crab dispersions"

# A Symplectic Factorization of "Ascript"

$$A = H_{OHO} C_{ET} A_{CS} R(\psi_1, \psi_2, \psi_3)$$

- $H_{OHO}$ is a dispersion matrix by Ohmi, Hirata, and Oide (8 independent parameters)
- $C_{ET}$ is coupling matrix by Edwards and Teng (4 independent parameters)
- $A_{CS}$ is "three two-dimensional ascripts" in Courant-Synder form (6 independent parameters)
- $R(\psi_1,\psi_2,\psi_3)$ are "three rotation matrix" for phase advances (3 independent parameters)
- $A$ has 21 independent parameters, which is the dimensionality of 6x6 symplectic matrix