Linear TPSA and Map

Yunhai Cai
FEL & Beam Physics Department
SLAC National Accelerator Laboratory

November 6-17, 2011

6th International Accelerator School for Linear Collider,

Pacific Grove, California, USA

Differential Algebra

Alex's example

$$f(x)=\frac{1}{x+\frac{1}{x}},\ f'(x)=-\frac{1-\frac{1}{x^2}}{(x+\frac{1}{x})^2},$$

$$x=2,$$

$$f(2)=\frac{2}{5},\ f'(2)=-\frac{3}{25},$$

$$f'(2)\approx\frac{f(2.1)-f(2)}{2.1-2}=\frac{0.38817-0.4}{0.1}=-0.1183$$

$$v=(2,1)$$

$$(a_1,a_2)+(b_1,b_2)=(a_1+b_1,a_2+b_2),$$

$$\frac{1}{(a_1,a_2)}=(\frac{1}{a_1},-\frac{a_2}{a_1^{\ 2}}),$$

$$f(v)=\frac{1}{(2,1)+\frac{1}{(2,1)}}=\frac{1}{(2,1)+(\frac{1}{2},-\frac{1}{4})}=\frac{1}{(\frac{5}{2},\frac{3}{4})}=(\frac{2}{5},-\frac{3}{25})$$

Definition of Linear TPSA

- Presentations:
  - $X=a_0+a_1\, x + a_2\, p_x + a_3\, y + a_4\, p_y + a_5\, \delta + a_6\ \ l_p = a_0 + X_1;$
  - $Y=b_0+b_1\, x + b_2\, p_x + b_3\, y + b_4\, p_y + b_5\, \delta + b_6\ \ l_p = b_0 + Y_1\, ;$
  - $Z=c_0+c_1\, x + c_2\, p_x + c_3\, y + c_4\, p_y + c_5\, \delta + c_6\ \ l_p = c_0 + Z_1\, ;$
- Rules
  - $Z = X + Y$; $Z=X-Y$; (plus, minus, like a linear polynomial)
  - $Z = d\, X$; (d multiply all terms, d is a "number")
  - $Z = X * Y=a_0b_0+a_0Y_1+b_0X_1$; (almost like polynomial but drop second order terms, why call TPSA)
  - $Z=f(X)=f(a_0)+f'(a_0)X_1$; (Taylor expansion around 0^{th}-order term)
  - $Z=X^{-1}=1/a_0 - X_1/a_0^2$; (Taylor expansion around 0^{th}-order term)
- 0th-order term is treated as the same "number"

Definition of Linear Map

- Linear Map is defined by six linear TPSA
  - $X = x_0 + m_{11}\, x + m_{12}\, p_x + m_{13}\, y + m_{14}\, p_y + m_{15}\, \delta + m_{16}\, l_p;$
  - $P_x = p_{x0} + m_{21}\, x + m_{22}\, p_x + m_{23}\, y + m_{24}\, py + m_{25}\, \delta + m_{26}\, l_p;$
  - $Y = y_0 + m_{31}\, x + m_{32}\, p_x + m_{33}\, y + m_{34}\, py + m_{35}\, \delta + m_{36}\, l_p;$
  - $P_y = y_0 + m_{41}\, x + m_{42}\, p_x + m_{43}\, y + m_{44}\, py + m_{45}\, \delta + m_{46}\, l_p;$
  - $\Delta = \delta_0 + m_{51}\, x + m_{52}\, p_x + m_{53}\, y + m_{54}\, py + m_{55}\, \delta + m_{56}\, l_p;$
  - $L = l_0 + m_{61}\, x + m_{62}\, p_x + m_{63}\, y + m_{64}\, py + m_{65}\, \delta + m_{66}\, l_p;$

column 3

where x_0, p_{x0}, y_0, p_{y0}, d_0, l_{p0}, and m_{ij} are coefficients of the linear polynomials. We note polynomials with capital letters. (x_0, p_{x0}, y_0, p_{y0}, d_0, l_{p0}) represents a reference orbit and matrix m for the linear perturbation relative to the reference orbit.

Track through a Matrix Element

$$\begin{pmatrix} X \\ P_x \\ Y \\ P_y \\ \Delta \\ L_p \end{pmatrix}_f = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} & t_{26} \\ t_{31} & t_{32} & t_{33} & t_{34} & t_{35} & t_{36} \\ t_{41} & t_{42} & t_{43} & t_{44} & t_{45} & t_{46} \\ t_{51} & t_{52} & t_{53} & t_{54} & t_{55} & t_{56} \\ t_{61} & t_{62} & t_{63} & t_{64} & t_{65} & t_{66} \end{pmatrix} \begin{pmatrix} X \\ P_x \\ Y \\ P_y \\ \Delta \\ L_p \end{pmatrix}_i$$

Results: orbit as a vector multiplication to matrix, $V_f = t * V_i$ and the 1th-order as a matrix multiplication, namely $m_f = t * m_i$.

How to obtain the transfer matrix?

A Thin Quadrupole Magnet

Use s as "time" variable, Hamiltonian in paraxial approximation is given by

$$H_Q = \frac{K_1 L}{2}(x^2 - y^2)$$

Hamiltonian equation and its solution

$$x_f = x_i,$$
$$p_{xf} = p_{xi} - K_1 L x_i,$$
$$y_f = y_i,$$
$$p_{yf} = p_{yi} + K_1 L y_i$$
$$\delta_f = \delta_{i,}$$
$$l_{pf} = l_{pi}$$

This map is nonlinear but symplectic. It can be rewritten as Lie operator exp(-:H_QL:).
Its linear matrix (how to get it?) is also symplectic.

Track a Linear Map through a Thin Quadrupole Magnet

$$X_f = X_i = x_{i0} + x_{i1},$$

$$P_{xf} = P_{xi} - K_1 L X_i = p_{xi0} + p_{xi1} - K_1 L (x_{i0} + x_{i1}),$$

$$Y_f = Y_i = y_{i0} + y_{i1},$$

$$P_{yf} = P_{yi} + K_1 L Y_i = p_{yi0} + p_{yi1} + K_1 L (y_{i0} + y_{i1})$$

$$\Delta_f = \Delta_i = \delta_{i0} + \delta_{i1}$$

$$L_{pf} = L_{pi} = l_{pi0} + l_{pi1}$$

feed down
to dipole kicks

elements of transport matrix

What happen to the 0th-order term?
How to get the transport matrix here?
How to the linear part of linear map transported?
How this compares to a matrix code?

A Thin Sextupole Magnet

Use s as "time" variable, Hamiltonian in paraxial approximation is given by

$$H_S = \frac{K_2 L}{3}(x^3 - 3xy^2)$$

Hamiltonian equation and its solution

$$x_f = x_i,$$
$$p_{xf} = p_{xi} - K_2 L(x_i^2 - y_i^2),$$
$$y_f = y_i,$$
$$p_{yf} = p_{yi} + 2K_2 L x_i y_i$$
$$\delta_f = \delta_{i,}$$
$$l_{pf} = l_{pi}$$

This map is nonlinear but symplectic. It can be rewritten as Lie operator exp(-:H_SL:). Its linear matrix (how to get it?) is also symplectic.

Track a Linear Map through a Thin Sextupole Magnet

$$X_f = X_i = x_{i0} + x_{i1},$$

$$P_{xf} = P_{xi} - K_2 L(X_i^2 - Y_i^2) = p_{xi0} + p_{xi1} - K_2 L(x_{i0}^2 - y_{i0}^2 + 2x_{i0}x_{i1} - 2y_{i0}y_{i1}),$$

$$Y_f = Y_i = y_{i0} + y_{i1},$$

$$P_{yf} = P_{yi} + 2K_2 L X_i Y_i = p_{yi0} + p_{yi1} + 2K_2 L(x_{i0}y_{i0} + x_{i0}y_{i1} + y_{i0}x_{i1})$$

$$\Delta_f = \Delta_i = \delta_{i0} + \delta_{i1}$$

$$L_{pf} = L_{pi} = l_{pi0} + l_{pi1}$$

quadrupole

skew quadrupole

What happen to the 0th-order term?
How to get the transport matrix here?
How to the linear part of linear map transported?

Linear TPSA Approach

- Substitute the "ray" of orbit with linear map, namely $(x,p_x,y,p_y,\delta,l_p)\text{->}(X, P_x, Y, Py, \Delta, L_p)$
- Use the rules of linear TPSA to make the "tracking"
- Result is
  - 0th-order: same as the orbit vector
  - 1st-order: matrix concatenation
  - second-order and higher terms are dropped
- Advantage:
  - Automatic and universal (works for "quadrupole, sbend")
  - Allow to use so call "polymorphism" (operator overloading in C++ implementation)
- Disadvantage:
  - You may loss the understanding of physics

How to Use it in a Ring?

- Track a linear map
  - Search closed orbit
  - Use the one-term matrix for its derivative
- Once closed orbit is found
  - Perform eigen values and vector analysis to construct "Ascript" at a location of the ring as outlined in the first lecture
  - Propagate the "Ascript" around the ring as a linear map
    - 0^{th}-order is the closed orbit
    - 1^{st}-order is the initial "Ascript"
  - Calculate coupling, dispersions, C-S parameters using "Ascript" at every elements