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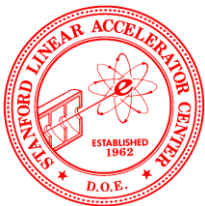
# Linear TPSA and Map

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**November 6-17, 2011**

*6<sup>th</sup> International Accelerator School for  
Linear Collider,  
Pacific Grove, California, USA*



# Differential Algebra

Alex's example

$$f(x) = \frac{1}{x + \frac{1}{x}}, f'(x) = -\frac{1 - \frac{1}{x^2}}{(x + \frac{1}{x})^2},$$

$$x = 2,$$

$$f(2) = \frac{2}{5}, f'(2) = -\frac{3}{25},$$

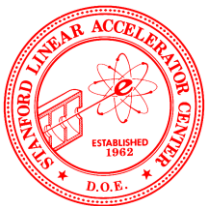
$$f'(2) \approx \frac{f(2.1) - f(2)}{2.1 - 2} = \frac{0.38817 - 0.4}{0.1} = -0.1183$$

$$v = (2, 1)$$

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2),$$

$$\frac{1}{(a_1, a_2)} = \left( \frac{1}{a_1}, -\frac{a_2}{a_1^2} \right),$$

$$f(v) = \frac{1}{(2, 1) + \frac{1}{(2, 1)}} = \frac{1}{(2, 1) + \left( \frac{1}{2}, -\frac{1}{4} \right)} = \frac{1}{\left( \frac{5}{2}, \frac{3}{4} \right)} = \left( \frac{2}{5}, -\frac{3}{25} \right)$$



# Definition of Linear TPSA

- Presentations:

- $X = a_0 + a_1 x + a_2 p_x + a_3 y + a_4 p_y + a_5 \delta + a_6 I_p = a_0 + X_1;$
- $Y = b_0 + b_1 x + b_2 p_x + b_3 y + b_4 p_y + b_5 \delta + b_6 I_p = b_0 + Y_1;$
- $Z = c_0 + c_1 x + c_2 p_x + c_3 y + c_4 p_y + c_5 \delta + c_6 I_p = c_0 + Z_1;$

- Rules

- $Z = X + Y; Z = X - Y;$  (plus, minus, like a linear polynomial)
- $Z = d X;$  (d multiply all terms, d is a "number")
- $Z = X * Y = a_0 b_0 + a_0 Y_1 + b_0 X_1;$  (almost like polynomial but drop second order terms, why call TPSA)
- $Z = f(X) = f(a_0) + f'(a_0) X_1;$  (Taylor expansion around 0<sup>th</sup>-order term)
- $Z = X^{-1} = 1/a_0 - X_1/a_0^2;$  (Taylor expansion around 0<sup>th</sup>-order term)

- 0<sup>th</sup>-order term is treated as the same "number"

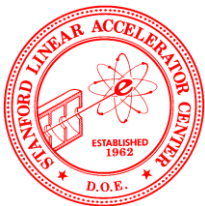


# Definition of Linear Map

- Linear Map is defined by six linear TPSA

- $X = x_0 + m_{11} x + m_{12} p_x + m_{13} y + m_{14} p_y + m_{15} \delta + m_{16} l_p$ ; column 3
- $P_x = p_{x0} + m_{21} x + m_{22} p_x + m_{23} y + m_{24} p_y + m_{25} \delta + m_{26} l_p$
- $Y = y_0 + m_{31} x + m_{32} p_x + m_{33} y + m_{34} p_y + m_{35} \delta + m_{36} l_p$
- $P_y = y_0 + m_{41} x + m_{42} p_x + m_{43} y + m_{44} p_y + m_{45} \delta + m_{46} l_p$
- $\Delta = \delta_0 + m_{51} x + m_{52} p_x + m_{53} y + m_{54} p_y + m_{55} \delta + m_{56} l_p$
- $L = l_0 + m_{61} x + m_{62} p_x + m_{63} y + m_{64} p_y + m_{65} \delta + m_{66} l_p$

where  $x_0, p_{x0}, y_0, p_{y0}, d_0, l_{p0}$ , and  $m_{ij}$  are coefficients of the linear polynomials. We note polynomials with capital letters.  $(x_0, p_{x0}, y_0, p_{y0}, d_0, l_{p0})$  represents a reference orbit and matrix  $m$  for the linear perturbation relative to the reference orbit.

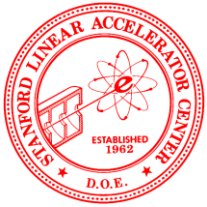


# Track through a Matrix Element

$$\begin{pmatrix} X \\ P_x \\ Y \\ P_y \\ \Delta \\ L_p \end{pmatrix}_f = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} & t_{26} \\ t_{31} & t_{32} & t_{33} & t_{34} & t_{35} & t_{36} \\ t_{41} & t_{42} & t_{43} & t_{44} & t_{45} & t_{46} \\ t_{51} & t_{52} & t_{53} & t_{54} & t_{55} & t_{56} \\ t_{61} & t_{62} & t_{63} & t_{64} & t_{65} & t_{66} \end{pmatrix} \begin{pmatrix} X \\ P_x \\ Y \\ P_y \\ \Delta \\ L_p \end{pmatrix}_i$$

Results: orbit as a vector multiplication to matrix,  $V_f = t * V_i$  and the 1th-order as a matrix multiplication, namely  $m_f = t * m_i$ .

How to obtain the transfer matrix?



# A Thin Quadrupole Magnet

Use  $s$  as "time" variable, Hamiltonian in paraxial approximation is given by

$$H_Q = \frac{K_1 L}{2} (x^2 - y^2)$$

Hamiltonian equation and its solution

$$x_f = x_i,$$

$$p_{xf} = p_{xi} - K_1 L x_i,$$

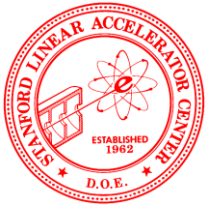
$$y_f = y_i,$$

$$p_{yf} = p_{yi} + K_1 L y_i$$

$$\delta_f = \delta_i,$$

$$l_{pf} = l_{pi}$$

This map is nonlinear but symplectic. It can be rewritten as Lie operator  $\exp(-:H_Q L:)$ . Its linear matrix (how to get it?) is also symplectic.



# Track a Linear Map through a Thin Quadrupole Magnet

$$X_f = X_i = x_{i0} + x_{i1},$$

$$P_{xf} = P_{xi} - K_1 L X_i = p_{xi0} + p_{xi1} - K_1 L (x_{i0} + x_{i1}),$$

$$Y_f = Y_i = y_{i0} + y_{i1},$$

$$P_{yf} = P_{yi} + K_1 L Y_i = p_{yi0} + p_{yi1} + K_1 L (y_{i0} + y_{i1})$$

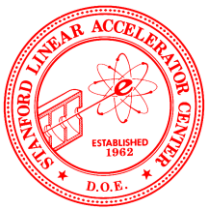
$$\Delta_f = \Delta_i = \delta_{i0} + \delta_{i1}$$

$$L_{pf} = L_{pi} = l_{pi0} + l_{pi1}$$

feed down  
to dipole kicks

elements of transport matrix

- What happen to the  $O^{\text{th}}$ -order term?
- How to get the transport matrix here?
- How to the linear part of linear map transported?
- How this compares to a matrix code?



# A Thin Sextupole Magnet

Use  $s$  as "time" variable, Hamiltonian in paraxial approximation is given by

$$H_s = \frac{K_2 L}{3} (x^3 - 3xy^2)$$

Hamiltonian equation and its solution

$$x_f = x_i,$$

$$p_{xf} = p_{xi} - K_2 L (x_i^2 - y_i^2),$$

$$y_f = y_i,$$

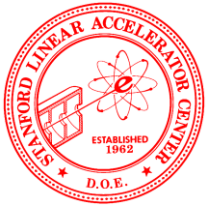
$$p_{yf} = p_{yi} + 2K_2 L x_i y_i$$

$$\delta_f = \delta_i,$$

$$l_{pf} = l_{pi}$$

This map is nonlinear but symplectic. It can be rewritten as Lie operator  $\exp(-:H_sL:)$ . Its linear matrix (how to get it?) is also symplectic.





# Track a Linear Map through a Thin Sextupole Magnet

$$X_f = X_i = x_{i0} + x_{i1},$$

$$P_{xf} = P_{xi} - K_2 L (X_i^2 - Y_i^2) = p_{xi0} + p_{xi1} - K_2 L (x_{i0}^2 - y_{i0}^2 + 2x_{i0}x_{i1} - 2y_{i0}y_{i1}),$$

$$Y_f = Y_i = y_{i0} + y_{i1},$$

$$P_{yf} = P_{yi} + 2K_2 L X_i Y_i = p_{yi0} + p_{yi1} + 2K_2 L (x_{i0}y_{i0} + x_{i0}y_{i1} + y_{i0}x_{i1})$$

$$\Delta_f = \Delta_i = \delta_{i0} + \delta_{i1}$$

$$L_{pf} = L_{pi} = l_{pi0} + l_{pi1}$$

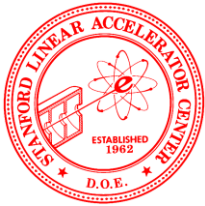
quadrupole

skew  
quadrupole

What happen to the 0<sup>th</sup>-order term?

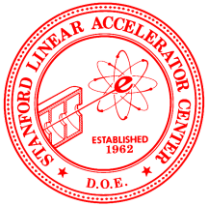
How to get the transport matrix here?

How to the linear part of linear map transported?



# Linear TPSA Approach

- Substitute the "ray" of orbit with linear map, namely  $(x, p_x, y, p_y, \delta, l_p) \rightarrow (X, P_x, Y, P_y, \Delta, L_p)$
- Use the rules of linear TPSA to make the "tracking"
- Result is
  - 0th-order: same as the orbit vector
  - 1st-order: matrix concatenation
  - second-order and higher terms are dropped
- Advantage:
  - Automatic and universal (works for "quadrupole, sbend")
  - Allow to use so call "polymorphism" (operator overloading in C++ implementation)
- Disadvantage:
  - You may loss the understanding of physics



# How to Use it in a Ring?

- Track a linear map
  - Search closed orbit
  - Use the one-term matrix for its derivative
- Once closed orbit is found
  - Perform eigen values and vector analysis to construct "Ascript" at a location of the ring as outlined in the first lecture
  - Propagate the "Ascript" around the ring as a linear map
    - 0<sup>th</sup>-order is the closed orbit
    - 1<sup>st</sup>-order is the initial "Ascript"
  - Calculate coupling, dispersions, C-S parameters using "Ascript" at every elements