Lecture 1:
Introduction to Damping Rings

Yunhai Cai
FEL & Beam Physics Department
SLAC National Accelerator Laboratory

November 6-17, 2011

6th International Accelerator School for Linear Collider, Pacific Grove, California, USA

Third-Generation Light Sources

APS

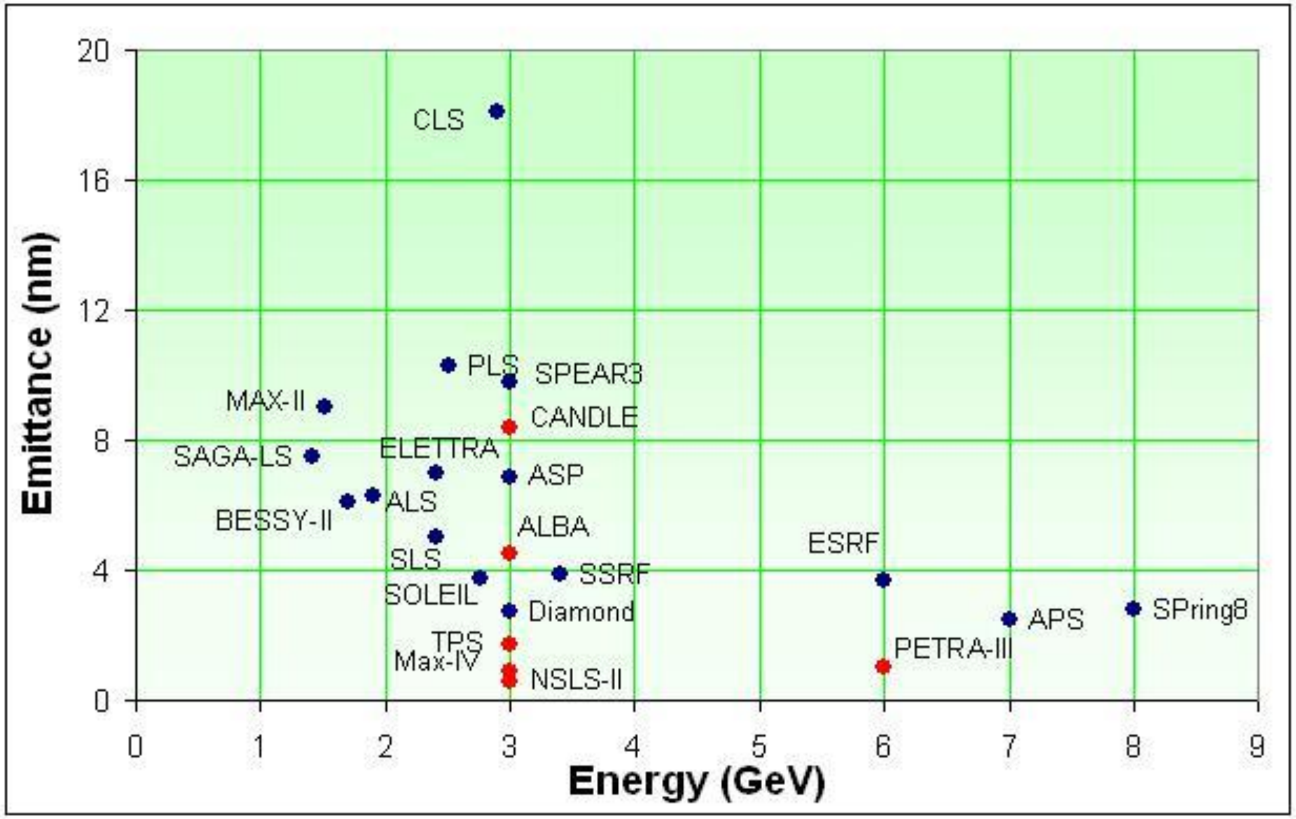
ALS

SSRF

ESRF

Worldwide Electron Storage Rings

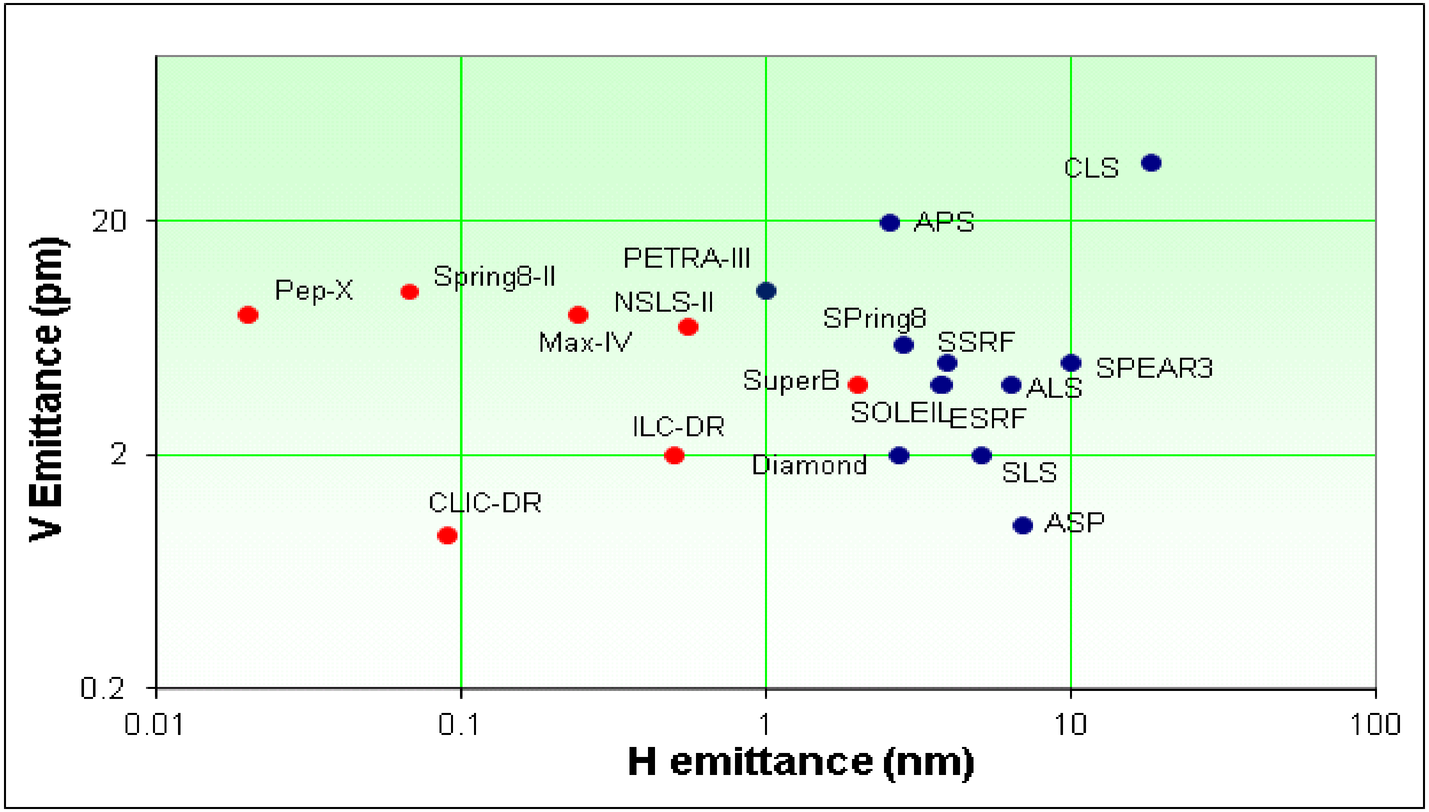
The electron beam emittance that defines the source size and divergence



Courtesy of R. Bartolini, Low Emittance Rings Workshop, 2010, CERN

Worldwide Electron Storage Rings

The electron beam emittance that defines the source size and divergence



Courtesy of R. Bartolini, Low Emittance Rings Workshop, 2011, Greece

SLC Damping Rings (SLAC)

Radiation Damping

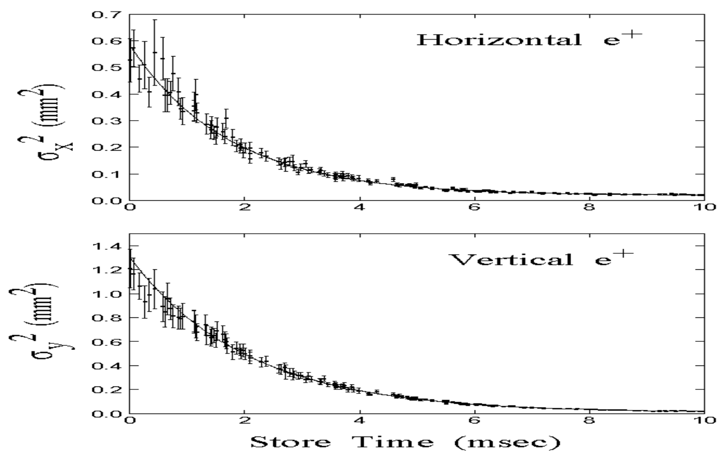


Figure 2. Sample of data for the positron damping ring. The vertical scale represents the real size of the bunch.

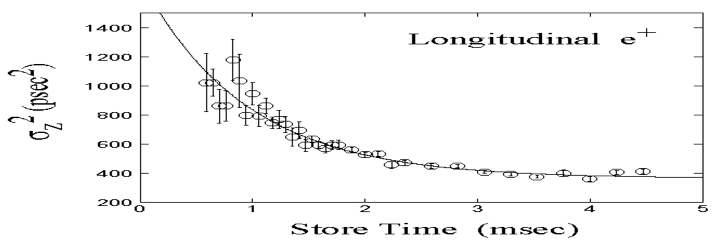
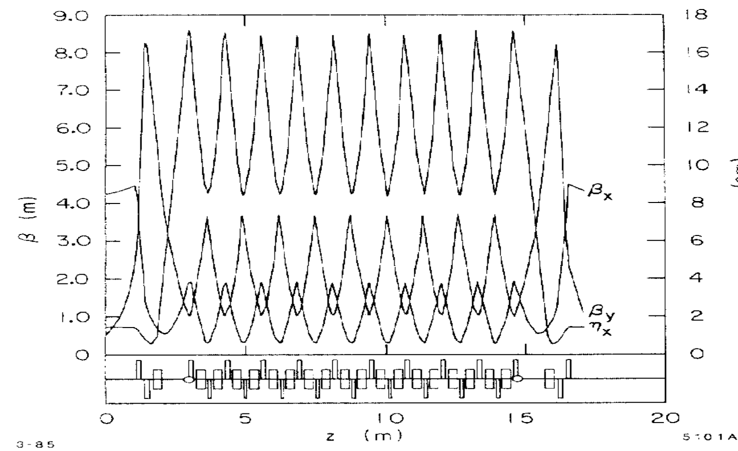


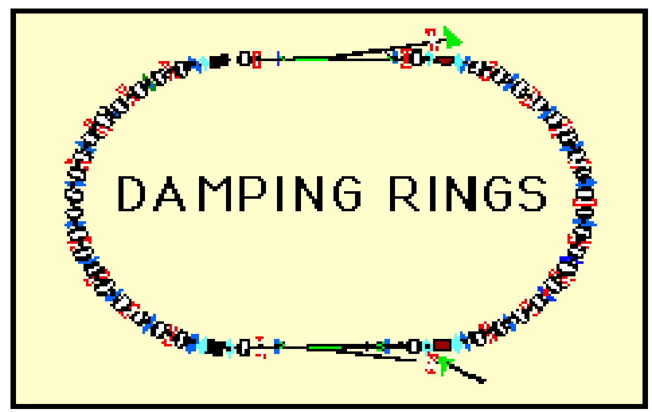
Figure 3. Longitudinal damping time data. The origin of the horizontal axis represents injection time.

C. Simopoulos and R.L. Holtzaple (1996)

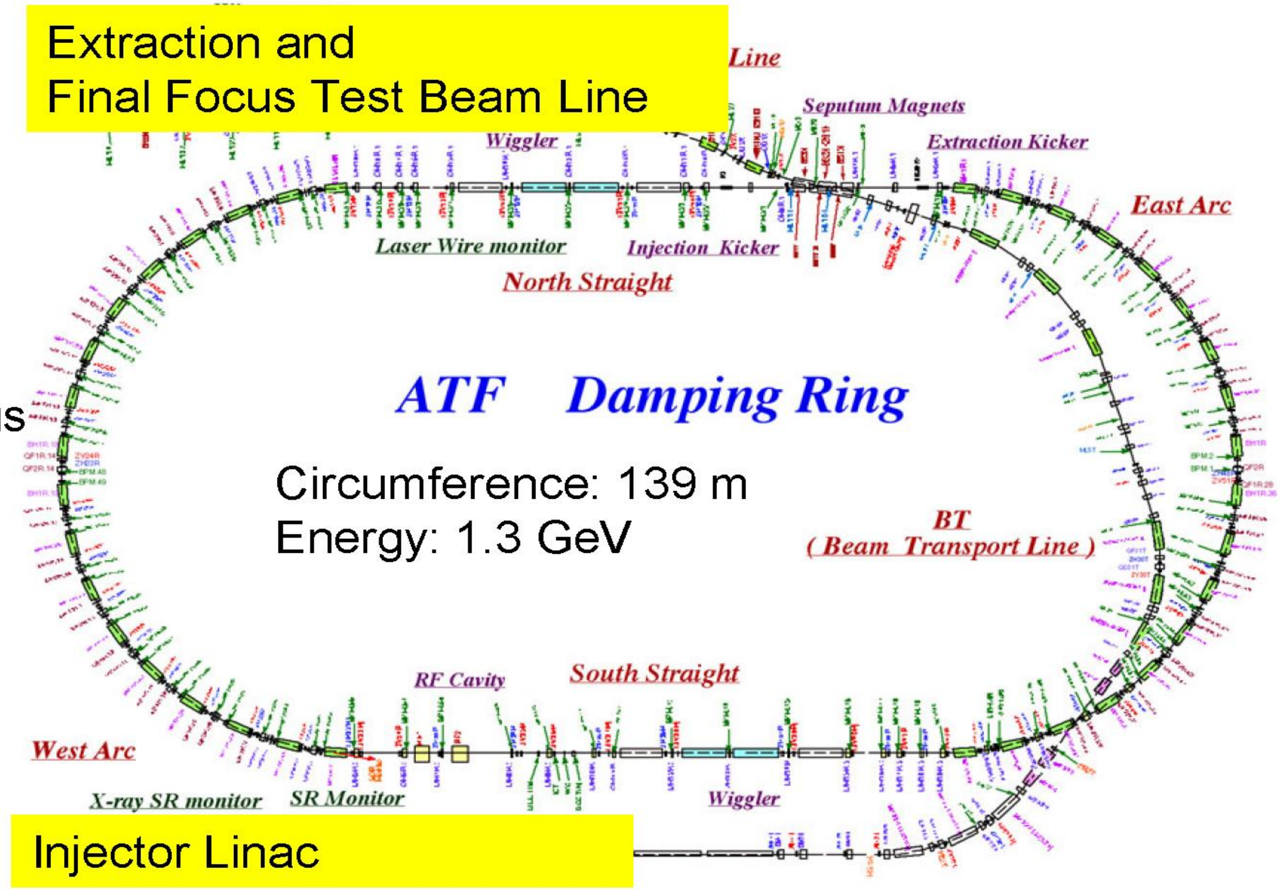
FODO Lattice (1985)



Layout



ATF Damping Ring (KEK)



Test Facility for LC
- Test of Low emittance beam tuning
- Deliver low emittance beam, e.g. for final focus test (ATF2)
- R & D of instrumentations, etc.

Required or target of low vertical emittance
For ATF2(Final Focus test): 12 pm
ILC damping ring dewsign: 2 pm

ILC Damping Ring

S. Guiducci and M.E. Biagini (2010)

Table 1: Parameter list for the TILC08 version compared to the SB2009.

	TILC08	SB2009
Energy (GeV)	5	5
Circumference (m)	6476	3238
Number of bunches	2610	1305
N particles/bunch	$2x10^{10}$	$2x10^{10}$
Damping time τ_x (ms)	21	24
Emittance ε_x (nm)	0.48	0.66
Emittance ε_y (pm)	2	2
Momentum compaction	$1.7x10^{-4}$	$1.5x10^{-4}$
Energy loss/turn (MeV)	10.3	4.5
Energy spread	$1.3x10^{-3}$	$1.2x10^{-3}$
Bunch length (mm)	6	6
RF voltage (MV)	21	7.5
RF frequency (MHz)	650	650
B wiggler (T)	1.6	1.6
Total wiggler length (m)	216	78
Number of wigglers	88	32

Phase space plots

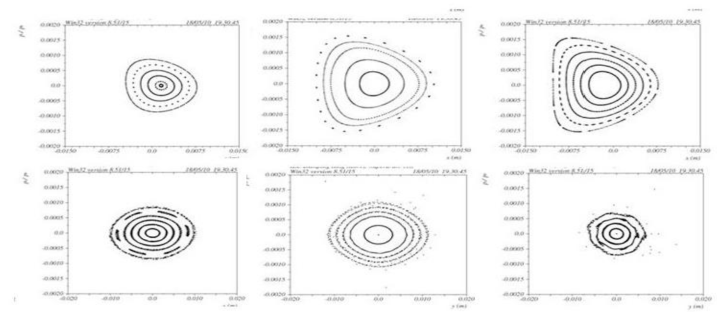


Figure 7: Phase space plots: x (top) and y (bottom) for $\Delta p/p=0$ (centre), 1% (left), -1% (right)

Layout

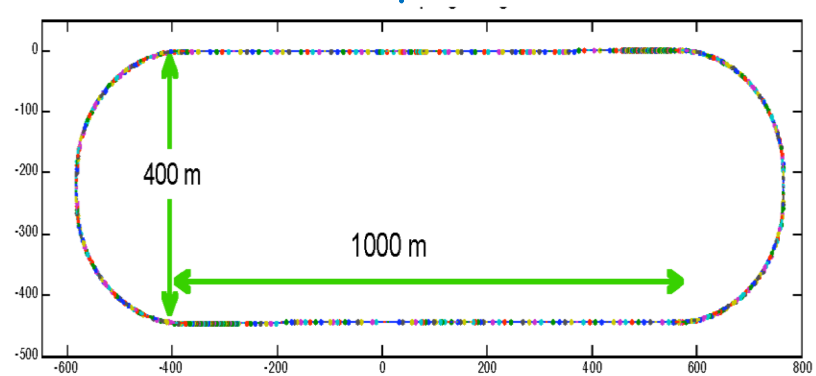


Figure 1: Layout of the 3.2km damping rings.

Physical Constants and CGS units

For electron:

Rest energy $mc^2 = 0.51$ MeV

Classic radius $r_e = \frac{e^2}{mc^2} = 2.82 \times 10^{-15}$ meter

Compton wavelength/2π $\lambda\!\!\!^{-}_e = \frac{\hbar}{mc} = r_e / \alpha = 3.86 \times 10^{-13}$ meter

Impedance of free space $Z_0 = \frac{4\pi}{c} = 120\pi$ Ω

Alfren current $I_A = \frac{ec}{r_e} = 17045$ A

For 1 GeV electron:

$$\gamma = \frac{1000}{0.51} \approx 2000$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = 0.999999869$$

Dynamics of Relativistic Particles

Relative velocity $\beta = v/c,$

Lorentz factor $\gamma = \dfrac{1}{\sqrt{1-\beta^2}},$

Momentum $\vec{p} = \gamma m \vec{v},$

Energy $E = \gamma mc^2 = \sqrt{c^2 p^2 + m^2 c^4} = cp/\beta.$

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$

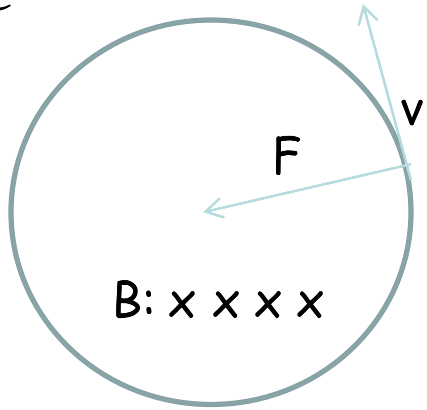
Equation of motion $\dfrac{d\vec{p}}{dt} = e(\vec{E} + \dfrac{\vec{v}}{c} \times \vec{B}),$ Lorentz force

Energy gain $\dfrac{dE}{dt} = e\vec{v} \cdot \vec{E}.$

in Uniform Magnetic Field

Equation of motion
$$\frac{d(\gamma m\vec{v})}{dt} = m\gamma\frac{d\vec{v}}{dt} = e\frac{\vec{v}}{c}\times\vec{B},$$

Assuming no velocity component in direction of B,



$$m\gamma\dot{v} = m\gamma\frac{v^2}{\rho} = evB/c$$

$$\Rightarrow \frac{pc}{e} = B\rho,$$

where ρ is the radius of the circular motion of the charged particle. This is the zeroth-order equation of circular accelerators. $B\rho$ is called the magnetic rigidity.

1. Energy E=pc, so the higher energy the larger the ring.
2. Conversion: 1 GeV => 10/2.998 T-m.

1.0 GeV => ρ=6.67 m and 42 m circumference, if B=0.5 T (e)
7 TeV => ρ=2.6 km and 16.3 km circumference, if B=9 T (p)

Radiation Damping

Instantaneous synchrotron radiated power is given by (Lienard 1898)

$$P_\gamma = \frac{2}{3} r_e mc^2 \frac{c\beta^4\gamma^4}{\rho^2},$$

Energy loss per turn is

$$U_0 = \frac{2\pi\rho}{c\beta} P_\gamma = \frac{4\pi}{3} \frac{r_e mc^2}{\rho} \beta^3\gamma^4.$$

or

$$\frac{U_0}{E} = \frac{4\pi}{3} \frac{r_e}{\rho} (\beta\gamma)^3.$$ (1.33×10^{-5} for our 1 GeV ring)

which is at order of the damping increments. Therefore the damping time $\tau \sim T_0 E/U_0$ (10 ms) The damping of the emittance is

$$\varepsilon_{ext} = \varepsilon_{inj} e^{-2t/\tau} + \varepsilon_{equ} (1 - e^{-2t/\tau})$$

Hamiltonian of a Charged Particle in Electromagnetic field

The Hamiltonian is given by

$$H = e\phi + [m^2c^4 + c^2(\vec{p} - e\vec{A}/c)^2]^{1/2},$$

where p is the canonical momentum,

the one in the equation of motion

$$\vec{p} = \vec{P} + \frac{e}{c}\vec{A}.$$

The equation of motion is given by the Hamiltonian equation,

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

Here we have $(q_1, q_2, q_3) = (x, y, s)$ and $(p_1, p_2, p_3) = (p_x, p_y, p_s)$. They are a set of the first ordinary differential equations.

Hamiltonian Equation

Time t is the independent variable:

$$\frac{dx}{dt}=\frac{\partial H}{\partial p_x},\frac{dp_x}{dt}=-\frac{\partial H}{\partial x},$$

$$\frac{dy}{dt}=\frac{\partial H}{\partial p_y},\frac{dp_y}{dt}=-\frac{\partial H}{\partial y},$$

$$\frac{ds}{dt}=\frac{\partial H}{\partial p_s},\frac{dp_s}{dt}=-\frac{\partial H}{\partial s}.$$

Path length s is the independent variable:

$$\frac{dx}{ds}=\frac{\partial \mathcal{H}}{\partial p_x},\frac{dp_x}{ds}=-\frac{\partial \mathcal{H}}{\partial x},$$

$$\frac{dy}{ds}=\frac{\partial \mathcal{H}}{\partial p_y},\frac{dp_y}{ds}=-\frac{\partial \mathcal{H}}{\partial y},$$

$$\frac{dt}{ds}=\frac{\partial \mathcal{H}}{\partial(-H)},\frac{d(-H)}{dt}=-\frac{\partial \mathcal{H}}{\partial t}.$$

with

$$\mathcal{H}=-p_s(x,p_x,y,p_y,t,-H)$$

Derivation:

$$\frac{dx}{ds}=\frac{dx}{dt}\frac{dt}{ds}=\frac{\partial H}{\partial p_x}/\frac{\partial H}{\partial p_s}=-\frac{\partial p_s}{\partial p_x}=\frac{\partial \mathcal{H}}{\partial p_x}$$

Scaled with Design Momentum

$$\frac{dx}{ds} = \frac{\partial \mathcal{H}/p_0}{\partial p_x / p_0}, \frac{dp_x / p_0}{ds} = -\frac{\partial \mathcal{H}/p_0}{\partial x},$$

$$\frac{dy}{ds} = \frac{\partial \mathcal{H}/p_0}{\partial p_y / p_0}, \frac{dp_y / p_0}{ds} = -\frac{\partial \mathcal{H}/p_0}{\partial y},$$

$$\frac{dt}{ds} = \frac{\partial \mathcal{H}/p_0}{\partial (-H / p_0)}, \frac{d(-H / p_0)}{dt} = -\frac{\partial \mathcal{H}/p_0}{\partial t}.$$

with new scaled Hamiltonian

$$\mathcal{H} = -p_s(x, p_x, y, p_y, t, -H) / p_0$$

The form of the Hamiltonian equation is preserved.

Hamiltonian Using the Path Length *s* as Independent Variable

The scaled Hamiltonian is suitable of quadrupole, sextupole, octopole, and skew quadrupole magnets is given by

$$H = -\frac{eA_s}{cp_0} - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2},$$

where $\delta=(p-p_0)/p_0$ and p_0 is the reference momentum and A_s the component of the vector potential along the direction of propagation. For a storage ring, we choose $cp_0=eB\rho$ as shown previously. Under the paraxial approximation, namely $p_x \ll 1$ and $p_y \ll 1$, the Hamiltonian can be simplified to

$$H = -\frac{A_s}{B\rho} + \frac{p_x^2 + p_y^2}{2(1+\delta)}.$$

For quadrupole magnets, the dependence of δ leads to chromaticity. That is the reason to introduce the sextupole magnets into the storage rings.

Paraxial Approximation

$$H = -\frac{eA_s}{cp_0} - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$$

$$= -\frac{eA_s}{cp_0} - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$$

$$= \frac{eA_s}{cp_0} - (1+\delta)[1 - \frac{p_x^2}{(1+\delta)^2} - \frac{p_y^2}{(1+\delta)^2}]^{1/2}$$

$$= \frac{eA_s}{cp_0} - (1+\delta)[1 - \frac{p_x^2}{(1+\delta)^2} - \frac{p_y^2}{(1+\delta)^2}]^{1/2}$$

paraxial approximation $$\approx \frac{eA_s}{cp_0} - (1+\delta)[1 - \frac{p_x^2}{2(1+\delta)^2} - \frac{p_y^2}{2(1+\delta)^2}]$$

$$= \frac{eA_s}{cp_0} - (1+\delta) + \frac{p_x^2}{2(1+\delta)} + \frac{p_y^2}{2(1+\delta)}$$

drop out if we only need the relative (ct)

Hamiltonian and Transfer Map for a Drift

Use s as the independent variable, Hamiltonian in the paraxial approximation is given by

$$H_D = \frac{1}{2(1+\delta)}(p_x^2 + p_y^2).$$

Solving the Hamiltonian equation, we obtain the transfer map of the drift:

$$\begin{aligned}
x_f &= x_i + \frac{p_{xi}}{1+\delta}\Delta s, \\
p_{xf} &= p_{xi}, \\
y_f &= y_i + \frac{p_{yi}}{1+\delta}\Delta s, \\
p_{yf} &= p_{yi}, \\
\delta_f &= \delta_{i}, \\
\ell_f &= \ell_i + \frac{\Delta s}{2(1+\delta_i)^2}(p_{xi}^2 + p_{yi}^2),
\end{aligned}$$

where Δs is the length of the draft, subscript "i'" for the initial canonical coordinates and "f" for the final ones. One can show that it is indeed a symplectic map.

Third Pair: Canonical Coordinate

After scaling by p_0, Ruth's choice of the third pair of canonical coordinate is given by

$$\frac{dt}{ds} = \frac{\partial H}{\partial(-E/p_0)}, \qquad \frac{d(-E/p_0)}{ds} = -\frac{\partial H}{\partial t}. \quad \xrightarrow{E=cp} \quad \frac{d(ct)}{ds} = \frac{\partial H}{\partial(-p/p_0)}, \qquad \frac{d(-p/p_0)}{ds} = -\frac{\partial H}{\partial(ct)}.$$

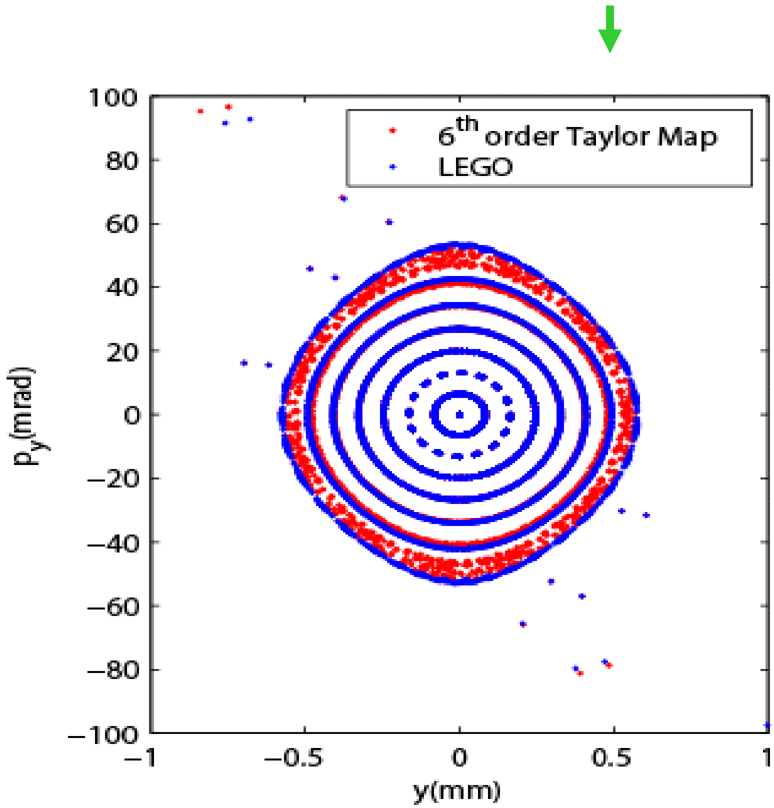
The third pair of canonical coordinate can be derived from Ruth's

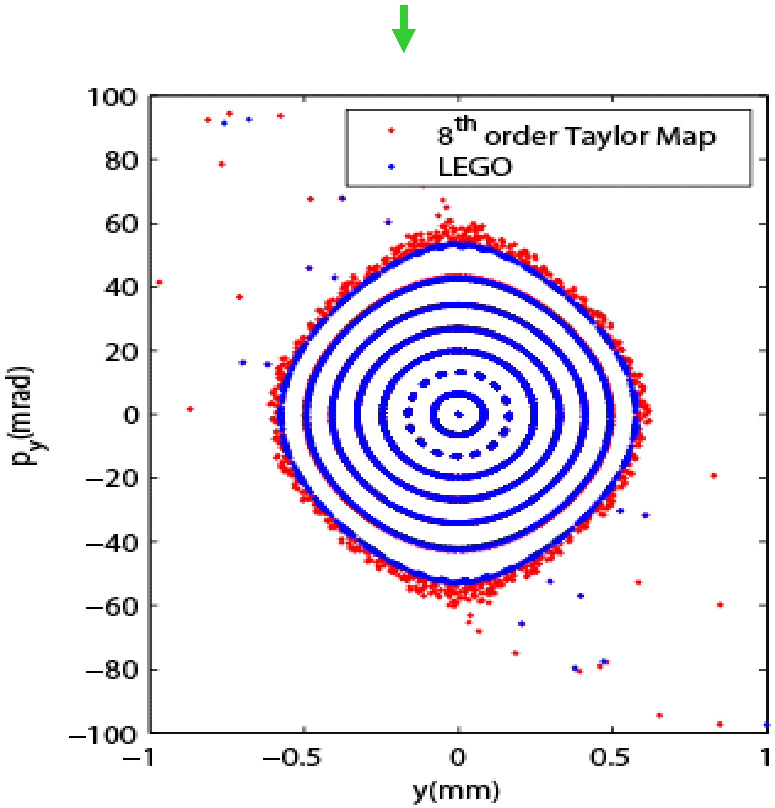
$$\frac{d\delta}{ds} = \frac{\partial H}{\partial \ell}, \qquad \frac{d\ell}{ds} = -\frac{\partial H}{\partial \delta}, \quad \xrightarrow{z=-\ell} \quad \frac{dz}{ds} = \frac{\partial H}{\partial \delta}, \qquad \frac{d\delta}{ds} = -\frac{\partial H}{\partial z},$$

where ℓ=ct and $\delta=(p-p_0)/p_0$. Obviously, we have used the ultra-relativistic approximation. For most electron rings, it is very good approximation.

Importance of Symplecticity

artificial damping or growth





Vector Potential of Magnets

A_x=A_y=0 and the component of vector potential along the propagating axis a

$$A_s = -Re[\sum_{n=1} \frac{1}{n}(b_n + ia_n)(x+iy)^n].$$

b_n and a_n for normal and skew components respectively. For a quadrupole magnet, we have

$$V_Q(x,y) = -\frac{A_s}{B\rho} = \frac{b_2}{2B\rho}(x^2 - y^2) = \frac{K_1}{2}(x^2 - y^2).$$

$K_1 > 0$, it focuses in x and defocuses in y. For a sextupole magnet, we have

$$V_S(x,y) = -\frac{A_s}{B\rho} = \frac{b_3}{3B\rho}(x^3 - 3xy^2) = \frac{K_2}{6}(x^3 - 3xy^2).$$

K_1, K_2 are the standard strengths for quadrupole and sextupole used in the program MAD.

Magnets for NSLS-II (BNL)

courtesy of Weiming Guo

Dipole magnet

Quadrupole and Sextupole

Hamiltonian and Transfer Map for a Focusing Quadrupole Magnet

Use s as the independent variable, Hamiltonian in the paraxial approximation is given by

$$H_Q = \frac{1}{2(1+\delta)}(p_x^2 + p_y^2) + \frac{K_1}{2}(x^2 - y^2).$$

Solving the Hamiltonian equation, we obtain the transfer map of a focusing quadrupole:

$$\begin{aligned}
x_f &= x_i \cos(\kappa\Delta s) + \frac{p_{xi}}{\kappa(1+\delta)}\sin(\kappa\Delta s),\\
p_{xf} &= -\kappa(1+\delta)x_i \sin(\kappa\Delta s) + p_{xi}\cos(\kappa\Delta s),\\
y_f &= y_i \cosh(\kappa\Delta s) + \frac{p_{yi}}{\kappa(1+\delta)}\sinh(\kappa\Delta s),\\
p_{yf} &= \kappa(1+\delta)y_i \sinh(\kappa\Delta s) + p_{yi}\cosh(\kappa\Delta s),\\
\delta_f &= \delta_{i,}\\
\ell_f &= \ell_i + \Delta_Q(x_i, p_{xi}, y_i, p_{yi}, \delta_i, \Delta s),
\end{aligned}$$

where Δs is the length of the quadrupole, $\kappa = \sqrt{K_1/(1+\delta)}$, the function Δ_Q in the path length can be found in ref. *Nucl. Inst. Meth. A645:168-174, 2011.*

Hill's Equation and its Solution

Hamiltonian for an one-dimensional quadrupole is given by

$$H = \frac{1}{2} p_x^2 + \frac{1}{2} K(s) x^2,$$

Its Hamiltonian equation leads to the Hill's equation

$$\frac{d^2 x}{ds^2} + K(s)x = 0,$$

where K(s+L)=K(s) and L is the periodicity of lattice. Its solution

$$x(s) = \sqrt{2J_x \beta(s)} \cos[\psi(s) + \phi],$$

where $\beta(s)$, $\alpha(s)=-\beta'(s)/2$, $\gamma(s)=(1+\alpha(s)^2)/\beta(s)$, are called the Courant-Snyder parameters. Its invariant is given by

$$\gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 = 2J_x$$

Transporting Matrices

Given x, x' at position s_1, the value of x, x' at position s_2 can be written as

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} (\frac{\beta_2}{\beta_1})^{1/2}(\cos\psi_{12}+\alpha_1\sin\psi_{12}) & (\beta_1\beta_2)^{1/2}\sin\psi_{12} \\ -\frac{1+\alpha_1\alpha_2}{(\beta_1\beta_2)^{1/2}}\sin\psi_{12}+\frac{\alpha_1-\alpha_2}{(\beta_1\beta_2)^{1/2}}\cos\psi_{12} & (\frac{\beta_1}{\beta_2})^{1/2}(\cos\psi_{12}-\alpha_2\sin\psi_{12}) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

This formula can be derived from the explicit form of the solution in the previous slide. If there is a periodicity from s_1 to s_2, it reduces to

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s+C} = \begin{pmatrix} \cos\mu+\alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s},$$

where $\mu = \Psi_{12}$. This is the Courant-Snyder parameterization of the one-turn matrix. The betatron tune is defined by $\nu=\mu/2\pi$. All these matrices are symplectic, $MJM^T = J$, and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Normalized Coordinates

One-turn matrix:

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Rotation matrix:

$$R = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}$$

We have:

$$M = ARA^{-1}$$

where A^{-1} is a transformations from the physical to the normalized coordinates:

$$A^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}, A = \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{-\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

All these matrices are symplectic. However, the transformation matrix A is not quite unique because of the commuting property of the rotational matrices.

Propagating Optical Functions

Using the transformation matrix A and A^{-1}, the transporting matrix M_{12} can be rewritten as

$$M_{12} = A_2 R_{12} A_1^{-1}.$$

This leads to

$$\tilde{A}_2 = A_2 R_{12} = M_{12} A_1.$$

Since M_{12} is determined by the components in beamline, we can use this formula to compute the optical function at position s_2 if their initial values at s_1 is known. For the position s_2, it is easy to show that

$$\beta = \tilde{A}_{11}^2 + \tilde{A}_{12}^2, \alpha = -(\tilde{A}_{11}\tilde{A}_{21} + \tilde{A}_{12}\tilde{A}_{22}), \gamma = \tilde{A}_{21}^2 + \tilde{A}_{22}^2.$$

The phase advance is

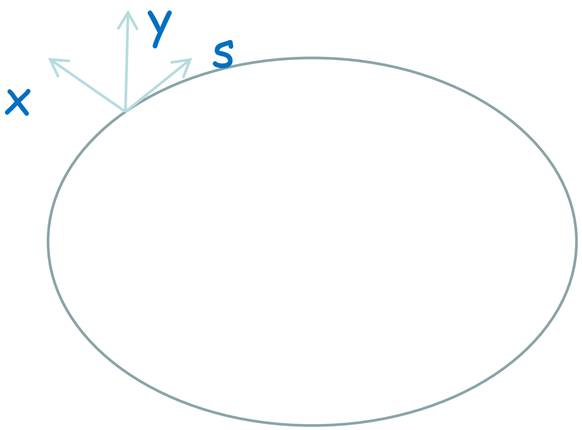
$$\psi_{12} = \tan^{-1}(\tilde{A}_{12} / \tilde{A}_{11})$$

Hamiltonian of Sector Bending Magnet

Similarly, the scaled Hamiltonian of a sector bending magnet can be derived using a curved coordinate system. Under the paraxial approximation, it is given by

$$H_D = -\frac{x}{\rho}\delta + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}.$$

Here we have assumed that the magnetic field B matches with the bending radius ρ, namely $cp_0=eB\rho$. The first term generates the dispersion and the second gives little focusing in the horizontal plane.



Sign convention:

s is the parrticle moving direction.
For a positive charge e, B_y is also positive.

Hamiltonian and Transfer Map for a Sector Bending Magnet

Use s as the independent variable, Hamiltonian in the paraxial approximation is given by

$$H_D = -\frac{x}{\rho}\delta + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}.$$

Solving the Hamiltonian equation, we obtain the transfer map of a sector bend:

$$x_f = x_i \cos(\kappa\Delta s) + \frac{p_{xi}}{\kappa(1+\delta)}\sin(\kappa\Delta s) + \rho\delta_i(1-\cos(\kappa\Delta s)),$$

$$p_{xf} = -\kappa(1+\delta)x_i \sin(\kappa\Delta s) + p_{xi}\cos(\kappa\Delta s) + \kappa(1+\delta_i)\rho\delta_i \sin(\kappa\Delta s),$$

dispersion

$$y_f = y_i + \frac{\Delta s p_{yi}}{(1+\delta_i)},$$

$$p_{yf} = p_{yi}$$

$$\delta_f = \delta_{i,}$$

$$\ell_f = \ell_i + \Delta_D(x_i, p_{xi}, y_i, p_{yi}, \delta_i, \rho, \Delta s),$$

where Δs is the length of the quadrupole, $\kappa = 1/(\rho\sqrt{(1+\delta)})$, the function Δ_D in the path length can be found ref. *Nucl. Inst. Meth. A645:168-174, 2011.*

Transporting Matrices

1. Drift with length L:

$$\begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2. Focusing quadrupole with length L and strength K

$$\begin{pmatrix} \cos(L\sqrt{K}) & \frac{1}{\sqrt{K}}\sin(L\sqrt{K}) & 0 & 0 & 0 & 0 \\ -\sqrt{K}\sin(L\sqrt{K}) & \cos(L\sqrt{K}) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(L\sqrt{K}) & \frac{1}{\sqrt{K}}\sinh(L\sqrt{K}) & 0 & 0 \\ 0 & 0 & \sqrt{K}\sinh(L\sqrt{K}) & \cosh(L\sqrt{K}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

3. Sector bend with radius ρ and length L

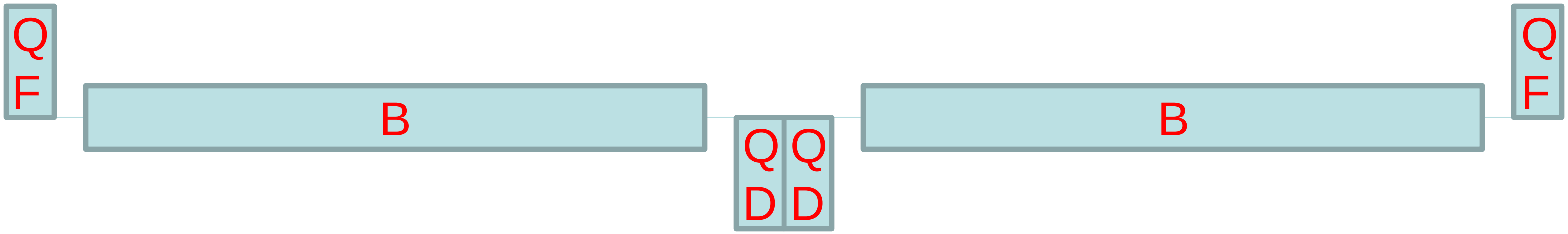
$$\begin{pmatrix} \cos\frac{L}{\rho} & \rho\sin\frac{L}{\rho} & 0 & 0 & \rho(1-\cos\frac{L}{\rho}) & 0 \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} & 0 & 0 & \sin\frac{L}{\rho} & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \sin\frac{L}{\rho} & \rho(1-\cos\frac{L}{\rho}) & 0 & 0 & L-\rho\sin\frac{L}{\rho} & 1 \end{pmatrix}$$

Canonical coordinates used:

$z=(x, p_x, y, p_y, \delta, \ell)$
and $\delta=(p-p_0)/p_0$.

path length

Simplest Periodic Cell: FODO



How to compute the Courant-Synder parameters and dispersions? For simplicity, we can use thin lens approximation for quadrupoles, and short length approximation for dipoles, and no gaps between any magnets.

What's the problem if we use these FODO cells to build entire ring? Why do we need to introduce sextupole magnets? How they work? Any unintended consequences of sextupoles?

What determines the beam sizes or beam distribution?

Chromaticity and its Correction

Transporting matrix for a quadupole magnet is given by

$$\begin{pmatrix} \cos(L\sqrt{K_1}) & \frac{1}{\sqrt{K_1}}\sin(L\sqrt{K_1}) & 0 & 0 & 0 & 0 \\ -\sqrt{K_1}\sin(L\sqrt{K_1}) & \cos(L\sqrt{K_1}) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(L\sqrt{K_1}) & \frac{1}{\sqrt{K_1}}\sinh(L\sqrt{K_1}) & 0 & 0 \\ 0 & 0 & \sqrt{K_1}\sinh(L\sqrt{K_1}) & \cosh(L\sqrt{K_1}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Actually, $K_1 \to K_1/(1+\delta)$ in the exact solution. Or equivalently, we can make the potential of quadupole:

$$\tilde{V}_Q = \frac{K_1}{2(1+\delta)}(x^2 - y^2) \approx \frac{K_1}{2}(x^2 - y^2) - \frac{K_1}{2}(x^2 - y^2)\delta.$$

On the other hand, the sextupole potential relative to a dispersive orbit is given by

$$V_S(x,y) = \frac{K_2}{6}[(x+\eta_x\delta)^3 - 3(x+\eta_x\delta)y^2].$$

To make a local compensation of δ term, we set $K_2 = K_1/\eta_x$.

Energy Gain in RF Cavity

From
$$\frac{dE}{dt} = e\vec{v}\cdot\vec{E},$$
$$\Rightarrow dE = eE_z dz,$$
$$\Rightarrow \Delta E = \int eE_z dz' = eV_{RF}(z).$$

With a proper choice of the RF cavity, we obtain

$$f_0 = c/C,$$
$$f_{RF} = hf_{0,}$$
$$\omega = 2\pi f$$
$$k = \frac{\omega}{c},$$
$$z = -\ell,$$

Energy loss ↓

Energy gain: $$\delta_f = \delta_i + \frac{eV_{RF}}{E_0}\sin\left(\frac{2\pi f_{RF}}{c} z_i + \varphi_s\right) - \frac{U_0}{E_0}.$$

RF Cavity and Synchrotron Oscillation

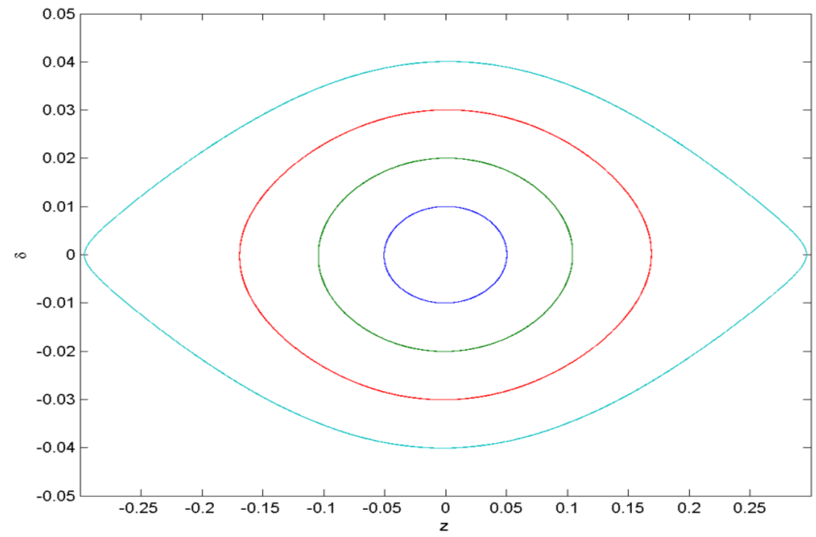
For a single RF in a ring, every turn we have

$$\begin{cases} \delta_{n+1} = \delta_n + \dfrac{eV_{RF}}{E_0}\sin(k_{RF} z_n + \phi_s) - \dfrac{U_0}{E_0} \\ z_{n+1} = z_n - \alpha C \delta_{n+1} \end{cases}$$

α momentum compaction factor.
Expand small z,

$$\Rightarrow \begin{cases} \dot{\delta} = \dfrac{eV_{RF}k_{RF}}{T_0 E_0}\cos\phi_s z \\ \dot{z} = -\dfrac{\alpha C}{T_0}\delta \end{cases}$$

RF Bucket



Synchrotron tune is given by

$$\nu_s = \sqrt{\frac{h\alpha}{2\pi}\frac{eV_{RF}}{E_0}\cos\phi_s},$$

where $\omega_s = \nu_s \omega_0$.

Summary

1. Hamiltonian is fundamental for the beam dynamics in storage rings, including the linear optics.
2. To make the particle motion stable, we use harmonic oscillators in all three dimensions. In the longitudinal plane, the RF bucket makes its stability extremely robust. That why we can focus on the transverse dynamics.

References

1. E.D. Courant and H.S. Synder, "Theory of the Alternating-Gradient Synchrotron," Annals of Physics: 3 1-148 (1958).
2. R.D. Ruth, "Single-particle dynamics in circular accelerators," AIP proceedings, volume 153, pages 150-235, 1987.
3. D.A. Edwards and M.J. Syphers, "An introduction to the physics of high energy accelerators," John Wiley & Sons, inc. 1993.
4. K.L. Brown and R.V. Servranckx, "First- and second-order charged particle optics," SLAC-PUB-3381, July 1984.