# Damping Rings and Ring Colliders

## *Radiation Damping and Equilibrium Emittance*

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# Outline

- A3.1 - DR Basics: Introduction to Damping Rings
- A3.2 - DR Basics: General Linear Beam Dynamics
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- A3.3 - LER Design: Radiation Damping and Equilibrium Emittance
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  - Quantum Excitation and Equilibrium Emittance
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These slides have been presented at the 2010 LC school by Mark Palmer

# Synchrotron Radiation and Radiation Damping

Up to this point, we have treated the transport of a relativistic electron (or positron) around a storage ring as a conservative process. In fact, the bending field results in the particles emitting synchrotron radiation.

The energy lost by an electron beam on each revolution is replaced by radiofrequency (RF) accelerating cavities. Because the synchrotron radiation photons are emitted in a narrow cone (of half-angle $1/\gamma$) around the direction of motion of a relativistic electron while the RF cavities are designed to restore the energy by providing momentum kicks in the $\hat{s}$ direction, this results in a gradual loss of energy in the transverse directions. This effect is known as *radiation damping*.

# Synchrotron Radiation

We will only concern ourselves with electron/positron rings. The instantaneous power radiated by a relativistic electron with energy $E$ in a magnetic field resulting in bending radius $\rho$ is:

$$P_\gamma = \frac{cC_\gamma E^4}{2\pi\rho^2} = \frac{e^2c^3}{2\pi}C_\gamma E^2B^2 \quad \text{where} \quad C_\gamma = 8.85\times10^{-5}\, m/(GeV)^3$$

We can integrate this expression over one revolution to obtain the energy loss per turn:

$$U_0 = \frac{C_\gamma E^4}{2\pi}\oint\frac{ds}{\rho^2} = \frac{C_\gamma E^4}{2\pi}I_2 \quad \text{where} \quad I_2 \text{ is the 2nd radiation integral}$$

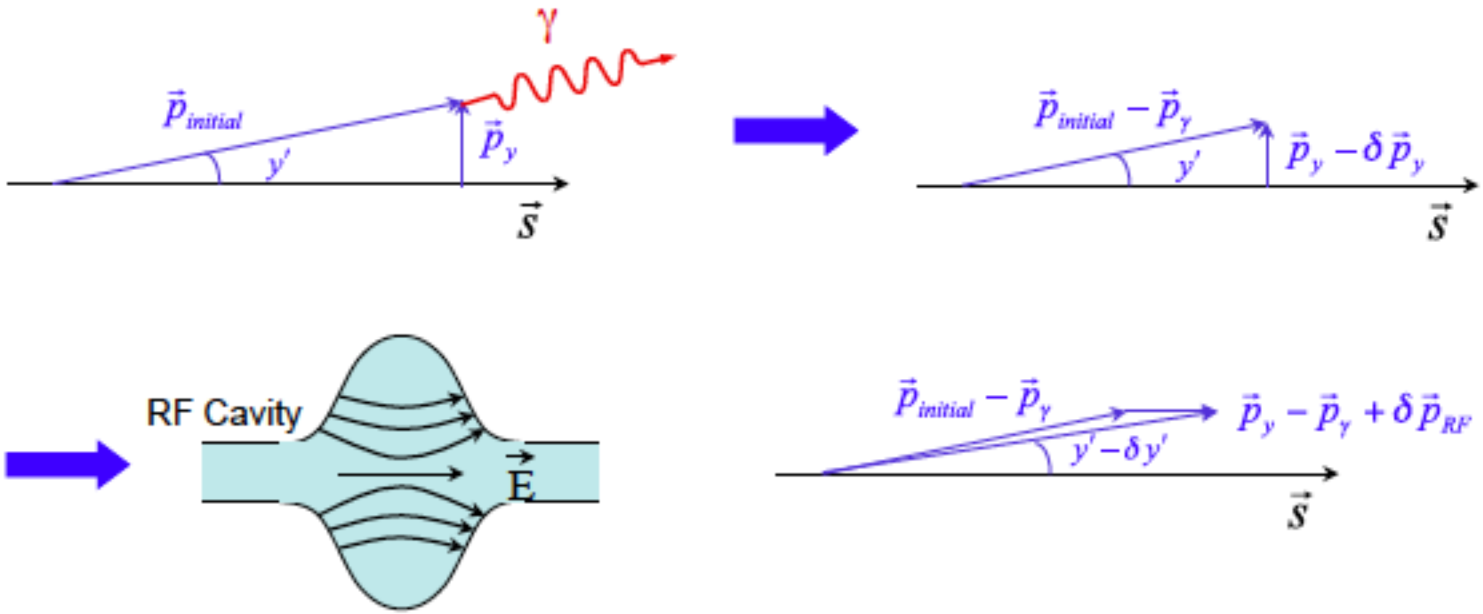
For a lattice with uniform bending radius (iso-magnetic) this yields:

$$U_0[eV] = 8.85\times10^4\,\frac{E^4[GeV]}{\rho[m]}$$

If this energy were not replaced, the particles would lose energy and gradually spiral inward until they would be lost by striking the vacuum chamber wall. The RF cavities replace this lost energy by providing momentum kicks to the beam in the longitudinal direction.

# Radiation Damping of Vertical Betatron Motion

We look first at the vertical dimension where, for an ideal machine, we do not need to consider effects of vertical dispersion.




The change in $y'$ after the RF cavity can be written as:

$$\delta y' = -y' \frac{\delta p_{RF}}{p} = -y' \frac{\delta E}{E}$$

# Radiation Damping (Vertical)

Recall that an oscillation with amplitude $A$ is described by:

$$A^2 = \gamma y^2 + 2\alpha y y' + \beta y'^2$$

If we assume that the β-function is slowly varying, so that

$\alpha = -\beta'/2 \sim 0$, we can write:

$$\delta\left(A^2\right) \approx \underbrace{\delta\left(\gamma y^2\right) + \delta\left(\beta y'^2\right)}_{=0}$$

$$\Rightarrow A\delta A = \beta y'^2 \frac{\delta y'}{y'} = -\beta y'^2 \frac{\delta E}{E}$$

and (using the solution to Hill's equation we obtained previously):

$$y'(s) \approx -\frac{A}{\sqrt{\beta_y(s)}} \sin\left[\psi_y(s) + \phi_0\right]$$

Substituting and averaging then gives:

$$\frac{\delta A}{A} = -\frac{1}{2}\frac{\delta E}{E_0}$$

# Radiation Damping (Vertical)

Thus the damping decrement, ie, the fractional decrease in amplitude in one revolution, is:

$$\alpha_y = \frac{\langle \delta A \rangle}{A T_0} = \frac{U_0}{2 E_0 T_0}$$

We can re-write this in exponential decay form as:

$$A(t) = A(0)\exp(-\alpha_y t)$$

or equivalently, the damping of the vertical emittance is given by:

$$\boxed{\varepsilon(t) = \varepsilon(0)\exp(-2\alpha_y t)}$$

# Radiation Damping (Transverse)

The situation for horizontal radiation damping is somewhat more complicated than the vertical case because of the presence of dispersion generated by the bending magnets. A similar procedure to that followed for the vertical case yields the result:

$$\alpha_x = \frac{U_0}{2E_0T_0}(1-\mathcal{D})$$

$$\mathcal{D} = \frac{I_4}{I_2} \quad \text{with} \quad I_2 = \oint \frac{ds}{\rho^2} \quad \text{and} \quad I_4 = \oint \frac{D}{\rho}\left(\frac{1}{\rho^2}+2k\right)ds$$

It is usual to write the transverse damping decrements as:

$$\alpha_i = \frac{U_0}{2E_0T_0}J_i \quad \text{with} \quad J_x = 1-\mathcal{D} \quad \text{and} \quad J_y = 1$$

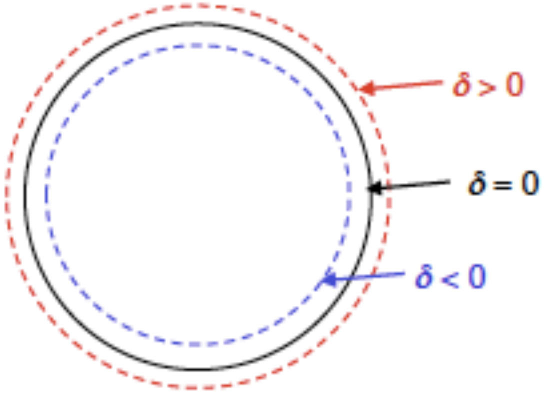
The transverse emittances will damp as:

$$\frac{d\varepsilon_i}{dt} = -2\alpha_i\varepsilon_i$$

# Synchrotron Motion

As particles circulate in a ring, the phase of their passage through the RF accelerating cavities must stay synchronized with respect to the RF frequency in order for their orbits to be stable. This stability is provided by the principle of phase focusing. In the relativistic limit we take:

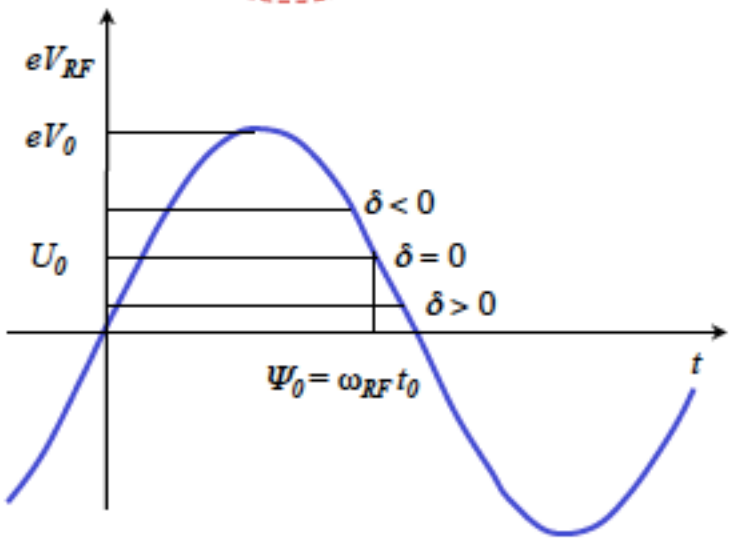
$$\delta = \frac{\Delta p}{p} = \frac{\Delta E}{E}$$




The arrival time for each particle is given by:

$$\frac{\Delta t}{T_0} = \frac{\Delta C}{C} = \alpha_c \delta$$

where $\alpha_c$ is the momentum compaction factor. Thus particles with $\delta>0$ will be delayed and will receive a smaller kick from the RF while particles with $\delta<0$ will arrive early and receive a larger kick as long as the default arrival time in the RF cavity is as shown on the right. This leads to synchrotron oscillations around a stable point.



# Synchrotron Equations of Motion

For our description of the longitudinal motion, we will use the variables:

$$\delta = \frac{\Delta E}{E_0} \quad \text{and} \quad \tau = t - t_0$$

where the *0* subscripts are for the synchronous particle.

Thus we can write:

$$\frac{d\tau}{dt} = -\alpha_c \delta$$

and

$$\frac{d\delta}{dt} = \frac{eV_{RF}(\tau) - U(E)}{E_0 T_0}$$

Note that we write the energy loss term as a function of $E$

where we have assumed that any synchrotron oscillations are far slower than the revolution time (a good assumption in practice) so that using the average energy loss per turn is valid. For small values of $\tau$ the RF voltage can be linearized as:

$$V_{RF}(\tau) = \frac{U_0}{e} + \tau \left.\frac{dV}{dt}\right|_{t=t_0} = \frac{U_0}{e} + \tau \omega_{RF} V_0 \cos\Psi_s$$

where $\sin\Psi_s = \frac{U_0}{eV_0}$

# Synchrotron Equation of Motion

We can now write:

$$\frac{d^2\delta}{dt^2} + 2\alpha_E \frac{d\delta}{dt} + \omega_s^2 \delta = 0$$

Synchrotron EOM

where:

$$\alpha_E = \frac{1}{2T_0} \left. \frac{dU(E)}{dE} \right|_{E=E_0}$$

$$\omega_s^2 = \frac{e\alpha_c \omega_{RF} V_0 \cos\Psi_s}{E_0 T_0}$$

The solutions to the synchrotron EOM can be written as:

$$\delta(t) = A_E e^{-\alpha_E t} \cos(\omega_s t - \Psi_s)$$

with

$$\tau(t) = \frac{-\alpha_c A_E}{E_0 \omega_s} e^{-\alpha_E t} \sin(\omega_s t - \Psi_s)$$

which describes the oscillation in energy and time of a particle with respect to the ideal synchronous particle.

# Energy Oscillation Damping

There are a couple points to note about the synchrotron EOM.

- First, we note that the synchrotron motion is intrinsically damped towards the motion of the synchronous particle. In the $\delta$-$\tau$ plane, an off-energy particle will exponentially spiral towards the origin – the synchronous particle's parameters
- Second, the damping coefficient, $\alpha_E$, is dependent on the energy of the particle. This happens in two ways. First the power radiated depends on energy. Secondly, the time it takes an electron to complete a revolution around the ring depends on the circumference of the orbit which also depends on the energy. Thus we still have some work to do to understand the rate of damping.

We start by writing the energy lost in one turn as:

$$U = \int_0^T P_\gamma dt$$

# Radiation Damping of Synchrotron Motion

We want to convert the integral over time to an integral over s. For a particle that is not on the closed orbit, the path length that it traverses can be written as:

$$d\ell = \left(1 + \frac{x}{\rho}\right) ds \qquad \Rightarrow \qquad dt = \frac{d\ell}{c} = \frac{1}{c}\left(1 + \frac{x}{\rho}\right) ds$$

where $x$ represents the orbit displacement due to the energy deviation. We can thus write the time differential as:

$$dt = \left(1 + \frac{D\delta}{\rho}\right) ds$$

and the energy loss per turn becomes:

$$U = \frac{1}{c}\oint P_\gamma \left(1 + \frac{D\delta}{\rho}\right) ds$$

# Radiation Damping

Evaluating $\left.\frac{dU}{dE}\right|_{E=E_0}$ yields (after a bit of work):

$$\alpha_E = \frac{1}{2T_0}\frac{dU}{dE} = \frac{U_0}{2T_0 E_0} J_E$$

where $$J_E = 2 + \mathcal{D} = 2 + \frac{I_4}{I_2}$$

and $$I_2 = \oint \frac{1}{\rho^2} ds \qquad I_4 = \oint \frac{D}{\rho}\left(\frac{1}{\rho^2} + 2k\right) ds \qquad k = \frac{1}{B\rho}\frac{dB_y}{dx}$$

Thus an energy deviation will damp with a time constant

$$\tau_E = \frac{2T_0 E_0}{J_E U_0}$$

# Summary of Radiation Damping

We can now summarize the radiation damping rates for each of the beam degrees of freedom:

$$\alpha_E = \frac{U_0}{2T_0E_0}J_E \qquad J_E = 2 + \mathcal{D} \qquad \mathcal{D} = 1 + \frac{I_4}{I_2}$$

$$\alpha_x = \frac{U_0}{2T_0E_0}J_x \qquad J_x = 1 - \mathcal{D}$$

$$\alpha_y = \frac{U_0}{2T_0E_0}J_y \qquad J_y = 1$$

and we can immediately write:

$$J_E + J_x + J_y = 4$$

Robinson's Theorem

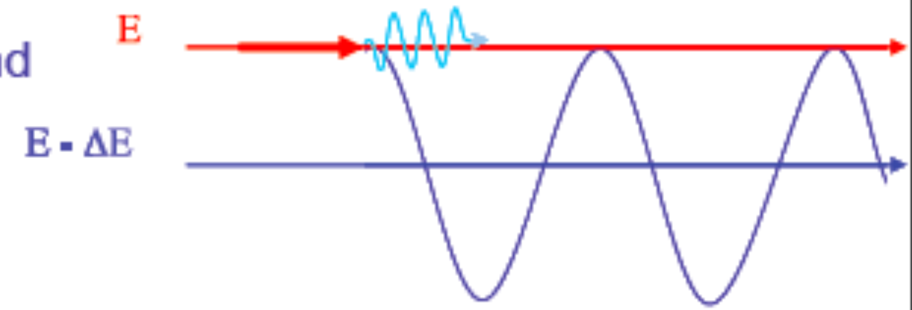
For separated function lattices, $\mathcal{D} \ll 1$ and the longitudinal damping occurs at roughly twice the rate of the damping in the two transverse dimensions.

Radiation damping plays a very special role in electron/positron rings because it provides a direct mechanism to take *hot* injected beams and reduce the equilibrium parameters to a regime useful for high luminosity colliders and high brightness light sources. At the same time, the radiated power plays a dominant role in the design of the technical systems – we will discuss some aspects of this further in tomorrow's lecture.

# Equilibrium Beam Properties

Now that we have determined the radiation damping rates, we can explore the equilibrium properties of the beam

- The emission of photons by the beam is a random process around the ring
- Photons are emitted within a cone around the direction of the beam particle with a characteristic angle $1/\gamma$




- This quantized process excites oscillations in each dimension
- In the absence of resonance or collective effects, which also serve to *heat* the beam, the balance between quantum excitation and radiation damping results in the equilibrium beam properties that are characteristic of a given lattice

# Quantum Excitation - Longitudinal

We will first look at the impact of quantum excitation in the longitudinal dimension.

For the very short timescales corresponding to photon emission, we can take the equations of motion we previously obtained for synchrotron motion and write:

$$\delta_E^2(t) + \frac{E_0^2 \omega_s^2}{\alpha_c^2} \tau^2(t) = A_E^2$$

where $A_E$ is a constant of the motion.

We want to consider the change in $A_E$ due to the emission of individual photons. The emission of an individual photon will not affect the time variable, however, it will cause an instantaneous change in the value of $\delta_E$.

# Quantum Excitation - Longitudinal

Thus we can write:

$$\Delta\delta = A_0 \cos\omega_s\left(t - t_0\right) - \frac{u}{E_0}\cos\omega_s\left(t - t_1\right) = A_1 \cos\omega_s\left(t - t_1\right)$$

where $u$ is the energy radiated at time $t_1$. Thus

$$A_1^2 = A_0^2 + \left(\frac{u}{E_0}\right)^2 - \frac{2A_0 u}{E_0}\cos\omega_s\left(t_1 - t_0\right)$$

and

$$\Delta A^2 = \left\langle A^2 - A_0^2 \right\rangle = \frac{u^2}{E_0^2}$$

We can thus write the average change in synchrotron amplitude due to photon emission as:

$$\frac{d\left\langle A^2\right\rangle}{dt} = \mathcal{N}\left(\frac{u}{E_0}\right)^2$$

where $\mathcal{N}$ is the rate of photon emission and $u$ is the photon energy.

# Quantum Excitation - Longitudinal

If we now include the radiation damping term, the net change in the synchrotron amplitude can be written as:

$$\frac{d\left\langle A^2\right\rangle}{dt} = -2\alpha_E\left\langle A^2\right\rangle + \mathcal{N}\frac{u^2}{E_0^2}$$

The equilibrium properties of a bunch are obtained when the rate of growth from quantum excitation and the rate of damping from radiation damping are equal. For an ensemble of particles where we identify the RMS energy amplitude with the energy spread, we can then write the equilibrium condition as:

$$\sigma_\delta^2 = \left(\frac{\sigma_E}{E_0}\right)^2 = \frac{\left\langle A^2\right\rangle}{2} = \frac{\left\langle \mathcal{N}\left\langle u^2\right\rangle\right\rangle_s}{4\alpha_E E_0^2}$$

# Photon Emission

$\left\langle \mathcal{N}\left\langle u^2\right\rangle\right\rangle_s$ is the ring-wide average of the photon emission rate, $\mathcal{N}$, times the mean square energy loss associated with each emission. In other words:

$$\mathcal{N} = \int_0^\infty n(u)du \quad \text{and} \quad \mathcal{N}\left\langle u^2\right\rangle = \int_0^\infty u^2 n(u)du$$

where $n(u)$ is the photon emission rate at energy $u$, and

$$\left\langle \mathcal{N}\left\langle u^2\right\rangle\right\rangle_s = \frac{1}{C}\oint \mathcal{N}\left\langle u^2\right\rangle ds$$

where $C$ is the ring circumference. Derivations of the photon spectrum emitted in a magnetic field are available in many texts and we will simply quote the result:

$$\mathcal{N}\left\langle u^2\right\rangle = 2C_q\gamma^2\frac{E_0 P_\gamma}{\rho} \quad \text{where} \quad C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc} = 3.84\times 10^{-13}\,m$$

# Energy Spread and Bunch Length

Integrating around the ring then yields the beam energy spread:

$$\sigma_\delta^2 = \left(\frac{\sigma_E}{E_0}\right)^2 = C_q \gamma^2 \frac{I_3}{J_E I_2} \quad \text{where} \quad I_3 = \oint \frac{ds}{|\rho|^3}$$

Using our solution to the synchrotron equations of motion, the bunch length is related to the energy spread by:

$$\sigma_\ell = \frac{c\alpha_c}{\omega_s E_0} \quad \text{where} \quad \omega_s^2 = \frac{e\alpha_c \omega_{RF} V_0 \cos\Psi_s}{E_0 T_0}$$

We note that the bunch length scales inversely with the square root of the RF voltage.

# Quantum Excitation - Horizontal

In order to evaluate the impact of the radiated photon on the motion of the emitting electron, we recall

$$A^2 = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s)$$

The change in closed orbit due to losing a unit of energy, $u$, is given by:

$$\delta x = -D(s)\frac{u}{E_0}$$

$$\delta x' = -D'(s)\frac{u}{E_0}$$

and we can then write:

$$\delta A^2 = \left(\gamma D^2 + 2\alpha D D' + \beta D'^2\right)\frac{u^2}{E_0^2} = \mathcal{H}(s)\frac{u^2}{E_0^2}$$

where $\mathcal{H}(s)$ is the *curly-H* function.

# Horizontal Emittance

We can then write an excitation term for the rms emittance as:

$$\left.\frac{d\varepsilon_x}{dt}\right|_{QE} = \frac{1}{2}\frac{d\langle A^2\rangle}{dt} = \frac{\left\langle \mathcal{N}\mathcal{H}\langle u^2\rangle\right\rangle_s}{2E_0^2}$$

Equating this expression to the damping rate yields (after some calculation) the equilibrium horizontal emittance:

$$\varepsilon_x = C_q \frac{\gamma^2 \left\langle \frac{\mathcal{H}}{\rho^3} \right\rangle}{J_x \left\langle \frac{1}{\rho^2} \right\rangle} = C_q \frac{\gamma^2 I_5}{J_x I_2}$$

where we have defined the next synchrotron radiation integral:

$$I_5 = \oint \frac{\mathcal{H}}{\rho^3} ds$$

# Quantum Excitation - Vertical

In the vertical dimension, where we assume the ideal case of no vertical dispersion, the quantum excitation of the emittance is determined by the opening angle of the emitted photons. The resulting perturbation to the vertical motion can be described as:

$$\delta y = 0 \qquad \delta y' = \frac{u}{E_0}\theta_\gamma$$

and we can write:

$$\delta\left\langle A^2\right\rangle = \left(\frac{u\theta_\gamma}{E_0}\right)^2 \beta_y$$

Thus, proceeding as we have on the preceding pages, we can write the expression for the equilibrium emittance as:

$$\varepsilon_y = \frac{\left\langle \mathcal{N}\left\langle u^2\right\rangle \beta_y\right\rangle_s \left\langle \theta_\gamma^2\right\rangle}{4E_0^2} = \frac{\left\langle \mathcal{N}\left\langle u^2\right\rangle \beta_y\right\rangle_s}{4\gamma^2 E_0^2}$$

$$\boxed{\varepsilon_y \approx \frac{C_q}{2J_y I_2}\oint \frac{\beta_y}{\rho^3} ds}$$

# Vertical Emittance & Emittance Coupling

For typical storage ring parameters, the vertical emittance due to quantum excitation is negligible. Assuming a typical $\beta_y$ values of a few 10's of meters and bending radius of ~100m, we can estimate $\varepsilon_y \leq 0.1 \text{ pm}$. The observed sources of vertical emittance are:

- ***emittance coupling*** whose source is ring errors which couple the vertical and horizontal betatron motion
- ***vertical dispersion*** due to vertical misalignment of the quadrupoles and sextupoles and angular errors in the dipoles

The vertical and horizontal emittances in the presence of a collection of such errors around a storage ring is commonly described as:

$$\varepsilon_y = \frac{\kappa}{1+\kappa}\varepsilon_0; \quad \varepsilon_x = \frac{1}{1+\kappa}\varepsilon_0 \quad \text{for } 0 < \kappa < 1$$

$\varepsilon_0$ is the natural emittance.

# Radiation Integrals and Equilibrium Quantities

Summary of Radiation Integrals:

$$I_1 = \oint \frac{D(s)}{\rho} ds$$

$$I_2 = \oint \frac{1}{\rho^2} ds$$

$$I_3 = \oint \frac{1}{|\rho|^3} ds$$

$$I_4 = \oint \frac{D(s)}{\rho}\left(\frac{1}{\rho^2} + 2k\right) ds$$

$$I_5 = \oint \frac{\mathcal{H}}{|\rho|^3} ds$$

$$\mathcal{H} = \gamma D^2 + 2\alpha D D' + \gamma D'^2$$

Summary of Equilibrium Beam Properties:

$$\alpha_c = \frac{I_1}{C}$$

$$U_0 = \frac{C_\gamma E^4}{2\pi} I_2 \quad \text{where} \quad C_\gamma = 8.85\times 10^{-5} m/(GeV)^3$$

$$\alpha_i = \frac{U_0}{2E_0 T_0} J_i, \quad i = x, y, E$$

$$J_x = 1 - \mathcal{D}; \quad J_y = 1; \quad J_E = 2 + \mathcal{D}; \quad \mathcal{D} = \frac{I_4}{I_2}$$

$$\left(\frac{\sigma_E}{E}\right)^2 = \frac{C_q \gamma^2 I_3}{J_E I_2} \quad \text{where} \quad C_q = 3.84\times 10^{-13} m$$

$$\sigma_\ell = \frac{c\alpha_c}{\omega_s E_0} \quad \text{where} \quad \omega_s^2 = \frac{e\alpha_c \omega_{RF} V_0 \cos\Psi_s}{E_0 T_0}; \quad \sin\Psi_s = \frac{U_0}{eV_0}$$

$$\varepsilon_x = \frac{C_q \gamma^2 I_5}{J_x I_2}; \quad \varepsilon_y \approx \frac{C_q}{2 J_y I_2} \oint \frac{\beta_y}{\rho^3} ds \quad \text{(quantum excitation)}$$

# Emittance Scaling in Lattices

The natural emittance of a lattice is given by:

$$\varepsilon_0 = \frac{C_q \gamma^2}{J_x} \frac{I_5}{I_2}$$

The ratio $\frac{I_5}{I_2}$ can be tailored to provide very low emittance. It can be shown that the natural emittance scales approximately as:

$$\varepsilon_0 \approx F \frac{C_q \gamma^2}{J_x} \theta^3$$

where $F$ is a function of the lattice design and $\theta$ is the bending angle from the dipoles in each lattice cell. The natural emittance can be made small by having small bending angles in the dipoles of each lattice cell and by optimizing $F$. The theoretical minimum emittance (TME) lattice has

$$F \approx \frac{1}{12\sqrt{15}}$$

Unfortunately, designing a very low emittance lattice in this way may have serious impact on the cost and/or performance of a low emittance ring.

# Achieving Ultra-Low Emittance

The path to low emittance that is pursued in a damping ring, is to provide insertion devices, wigglers, which dominate the radiation damping of the machine. For a sinusoidal wiggler, we can write the energy loss around the ring as:

$$U_0 = \frac{C_\gamma E^4}{2\pi}\left(\oint_{dipoles}\frac{1}{\rho^2}ds + \int_0^{L_{wiggler}}\frac{1}{\rho_{wig}^2}ds\right) = U_{dip} + U_{wig}$$

The overall length of the wiggler section, along with the wiggler period and peak field, can be adjust to make the second term dominate the radiation losses in the ring and hence the damping rate. The expressions

$$\varepsilon_{dip} = C_q\gamma^2\frac{I_{5dip}}{I_{2dip}} \quad \text{and} \quad \varepsilon_{wig} = C_q\gamma^2\frac{I_{5wig}}{I_{2wig}}$$

give the emittance contributions of the dipole and wiggler regions, respectively. We can then write the natural emittance of the ring as:

$$\varepsilon_0 = \frac{\varepsilon_{dip}}{1+F} + \frac{\varepsilon_{wig}F}{1+F} \quad \text{where} \quad F = \frac{U_{wig}}{U_{dip}}$$

Thus, if the wiggler radiation dominates, the emittance contribution due to the dipoles is reduced by a factor of $F$ and the ring emittance is dominated by the intrinsic wiggler emittance. In fact, the wiggler emittance can be quite small by placing the wigglers in zero dispersion regions with small $\beta_x$.