Problems Lecture 1: Linac Basics

1) Calculate the relative longitudinal motion of two electrons with an energy of $9\,\mathrm{GeV}$ and a difference of 3% over a distance of $21\,\mathrm{km}$.

2) Calculate the solutions to Hill's equation for $K(s) = K_0 > 0$.

3) Calculate the solutions to Hill's equation for $K(s) = 0$ assuming $\beta(s = 0) = \beta_0$ and $\beta'(s = 0) = 0$. (Only for the accelerator experts)

4) How much energy is roughly stored in one ILC cavity at nominal gradient?

Solutions: Linac Basics

1) We calculate

$$\gamma = \frac{E_0}{mc^2} \approx \frac{9\,\mathrm{GeV}}{0.511\,\mathrm{MeV}} \approx 18000$$

then we use

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

to find an approximation for β

$$\beta \approx 1 - \frac{1}{2\gamma^2} \approx 1 - 1.5 \times 10^{-9}$$

over the length of a linac $21\,\mathrm{km}$ the longitudinal delay compared to light is $\approx 32\,\mu\mathrm{m}$ for two particles which have a energy difference of $\Delta\gamma$ the relative longitudinal motion would be

$$\beta_1 - \beta_2 \approx \frac{1}{2\gamma^2} - \frac{1}{2(\gamma+\Delta\gamma)^2} \approx \frac{\Delta\gamma}{\gamma^3}$$

for an example of 3% the motion is $\approx 1.94\,\mu\mathrm{m} \ll \sigma_z$

Note: Due to the acceleration the effect is even smaller

Solutions: Linac Basics

2) We use $K(s) = \mathrm{const} > 0$.

- We know the solution is a harmonic oszillation with a fixed amplitude

$$x = A\cos(\phi(s) + \phi_0)$$

  for the beta-function this should correspond to a constant value of beta, which we call β_0

- We now need to check that this fulfills the differential equation for β

- Ansatz: $\beta = \beta_0$, $\beta' = 0$ and $\beta'' = 0$:

$$\frac{\beta''\beta}{2} - \frac{\beta'^2}{4} + K\beta^2 = 1$$

$$\Rightarrow K\beta_0^2 = 1$$

Hence

$$\beta_0 = \frac{1}{\sqrt{K}}$$

Now one can plug this into the equation of motion to see that one recovers the known solution for a harmonic oszillator. ϵ is defined by the initial condition.

Solutions: Linac Basics

3) $K = 0$

$x = x_0 + x'(0)s$

Ansatz: β is a polynom of second order

- We use $\beta'(s=0) = 0$, $\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$

$$\frac{\beta''\beta}{2} - \frac{\beta'^2}{4} + K\beta^2 = 1$$

$$\Rightarrow \frac{\beta''\beta}{2} - \frac{\beta'^2}{4} = 1$$

$$\Rightarrow \frac{1}{2}\left(\beta_0 + \frac{s^2}{\beta_0}\right)\left(\frac{2}{\beta_0}\right) - \frac{1}{4}\left(\frac{2s}{\beta_0}\right)^2 = 1$$

$$\Rightarrow 1 + \frac{s^2}{\beta_0^2} - \frac{s^2}{\beta_0^2} = 1$$

4) Assuming $R/Q = 1\,\mathrm{k}\Omega$ we find approximately $120\,\mathrm{J}$

$$E = \frac{(GL)^2}{2\pi f_{RF} R/Q}$$