CAVITY FUNDAMENTALS

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RF Cavity

- Mode transformer (TEM→TM)
- Impedance transformer (Low Z→High Z)
- Space enclosed by conducting walls that can sustain an infinite number of resonant electromagnetic modes
- Shape is selected so that a particular mode can efficiently transfer its energy to a charged particle
- An isolated mode can be modeled by an LRC circuit

RF Cavity

Lorentz force $$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

An accelerating cavity needs to provide an electric field E longitudinal with the velocity of the particle

Magnetic fields provide deflection but no acceleration

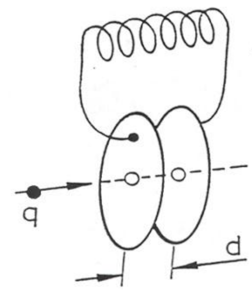
DC electric fields can provide energies of only a few MeV

Higher energies can be obtained only by transfer of energy from traveling waves →resonant circuits

Transfer of energy from a wave to a particle is efficient only is both propagate at the same velocity

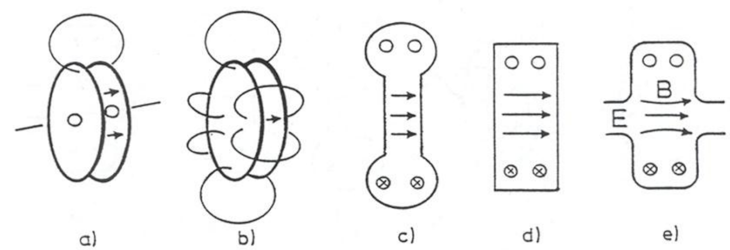
Equivalent Circuit for an rf Cavity

Simple LC circuit representing an accelerating resonator



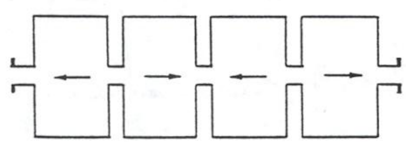
Simple lumped L-C circuit repesenting an accelerating resonator.
$\omega_o^2 = 1/LC$

Metamorphosis of the LC circuit into an accelerating cavity



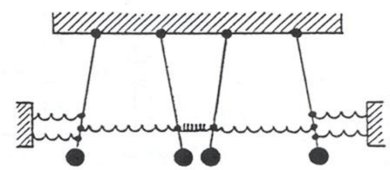
Metamorphosis of the L-C circuit of Fig. 1 into an accelerating cavity (after R.P.Feynman[33]). Fig. 5d shows the cylindrical "pillbox cavity" and Fig. 5e a slightly modified pillbox cavity with beam holes (typical β between 0.5 and 1.0). Fig. 5c resembles a low β version of the pillbox variety (0.2<β<0.5).

Chain of weakly coupled pillbox cavities representing an accelerating module



Chain of weakly-coupled pillbox cavities representing an accele-rating module

Chain of coupled pendula as its mechanical analogue



Chain of coupled pendula as a mechanical analogue to Fig. 6a

Electromagnetic Modes

Electromagnetic modes satisfy Maxwell equations

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix}\vec{E}\\ \vec{H}\end{Bmatrix} = 0$$

With the boundary conditions (assuming the walls are made of a material of low surface resistance)

no tangential electric field $\vec{n} \times \vec{E} = 0$

no normal magnetic field $\vec{n} \bullet \vec{H} = 0$

Electromagnetic Modes

Assume everything $\sim e^{-i\omega t}$

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\begin{Bmatrix}\vec{E}\\ \vec{H}\end{Bmatrix} = 0$$

For a given cavity geometry, Maxwell equations have an infinite number of solutions with a sinusoidal time dependence

For efficient acceleration, choose a cavity geometry and a mode where:

Electric field is along particle trajectory

Magnetic field is 0 along particle trajectory

Velocity of the electromagnetic field is matched to particle velocity

Accelerating Field (gradient)

Voltage gained by a particle divided by a reference length

$$E = \frac{1}{L}\int E_z(z)\cos(\omega z / \beta c)dz$$

For velocity-of-light particles $L = \frac{N\lambda}{2}$

For less-than-velocity-of-light cavities, there is no universally adopted definition of the reference length

Design Considerations

$\frac{H_{s,\max}}{E_{acc}}$	minimum	critical field
$\frac{E_{s,\max}}{E_{acc}}$	minimum	field emission
$\frac{< H_s^2 >}{E_{acc}^2}$	minimum	shunt impedance, current losses
$\frac{< E_s^2 >}{E_{acc}^2}$	minimum	dielectric losses
$\frac{U}{E_{acc}^2}$	minimum maximum	control of microphonics voltage drop for high charge per bunch

Energy Content

Energy density in electromagnetic field:

$$u = \frac{1}{2}\left(\varepsilon_0 \mathbf{E}^2 + \mu_0 \mathbf{H}^2\right)$$

Because of the sinusoidal time dependence and the 90º phase shift, he energy oscillates back and forth between the electric and magnetic field

Total energy content in the cavity:

$$U = \frac{\varepsilon_0}{2}\int_V dV\,|\mathbf{E}|^2 = \frac{\mu_0}{2}\int_V dV\,|\mathbf{H}|^2$$

Power Dissipation

Power dissipation per unit area

$$\frac{dP}{da} = \frac{\mu_0 \omega \delta}{4} \left| \mathbf{H}_{||} \right|^2 = \frac{R_s}{2} \left| \mathbf{H}_{||} \right|^2$$

Total power dissipation in the cavity walls

$$P = \frac{R_s}{2} \int_A da \left| \mathbf{H}_{||} \right|^2$$

Quality Factor

Quality Factor Q_0:

$$Q_0 \equiv \frac{\text{Energy stored in cavity}}{\text{Energy dissipated in cavity walls per radian}} = \frac{\omega_0 U}{P_{diss}}$$

$$= \omega_0 \tau_0 = \frac{\omega_0}{\Delta\omega_0}$$

$$Q_0 = \frac{\omega\mu_0}{R_s} \frac{\int_V dV \left|\mathbf{H}\right|^2}{\int_A da \left|\mathbf{H}_{||}\right|^2}$$

Geometrical Factor

Geometrical Factor QRs (Ω)

Product of the Quality Factor and the surface resistance

Independent of size and material

Depends only on shape of cavity and electromagnetic mode

$$G = QR_s = \omega\mu_0 \frac{\int_V dV \left|\mathbf{H}\right|^2}{\int_A da \left|\mathbf{H}_{||}\right|^2} = 2\pi\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{\lambda}\frac{\int_V dV \left|\mathbf{H}\right|^2}{\int_A da \left|\mathbf{H}_{||}\right|^2} = \frac{2\pi\eta}{\lambda}\frac{\int_V dV \left|\mathbf{H}\right|^2}{\int_A da \left|\mathbf{H}_{||}\right|^2}$$

$\eta \approx 377\Omega$ Impedance of vacuum

Shunt Impedance, R/Q

Shunt impedance R_{sh}:

$$R_{sh} \equiv \frac{V_c^2}{P_{diss}} \quad \text{in } \Omega$$

V_c = accelerating voltage

Note: Sometimes the shunt impedance is defined as $\frac{V_c^2}{2P_{diss}}$ or quoted as impedance per unit length (ohm/m)

R/Q (in Ω)

$$\frac{R}{Q} = \frac{V^2}{P}\frac{P}{\omega U} = \frac{E^2}{U}\frac{L^2}{\omega}$$

Q – Geometrical Factor (Q R_s)

Q: $$\frac{\text{Energy content}}{\text{Energy disspated during one radian}} = \omega\frac{U}{P} = \omega\tau = \frac{\omega}{\Delta\omega}$$

Rough estimate (factor of 2) for fundamental mode

$$\omega = \frac{2\pi c}{\lambda} \simeq \frac{2\pi}{\sqrt{\varepsilon_0\mu_0}}\frac{1}{2L} \qquad U = \frac{\mu_0}{2}\int H^2 dv \simeq \frac{\mu_0}{2}\frac{1}{2}H_0^2\frac{\pi L^3}{6}$$

$$P = \frac{1}{2}R_s\int H^2 dA = \frac{1}{2}R_s\frac{1}{2}H_0^2\,\pi L^2$$

$$QR_s \sim \frac{\pi}{6}\sqrt{\frac{\mu_0}{\varepsilon_0}} = 200\Omega$$

$G = QR_s$ is size (frequency) and material independent.

It depends only on the mode geometry

It is independent of number of cells

For superconducting elliptical cavities $QR_s \sim 275\Omega$

Shunt Impedance (R_{sh}), R_{sh} R_s, R/Q

$$R_{sh} = \frac{V^2}{P} \simeq \frac{E_z^2 L^2}{\frac{1}{2} R_s H_0^2 \pi L^2 \frac{1}{2}}$$

In practice for elliptical cavities

$R_{sh} R_s \simeq 33{,}000 \ \left(\Omega^2\right)$ per cell

$R_{sh} / Q \simeq 100\Omega$ per cell

$R_{sh}\, R_s$ and R_{sh} / Q

Independent of size (frequency) and material

Depends on mode geometry

Proportional to number of cells

Power Dissipated per Unit Length or Unit Area

$$\frac{P}{L} \propto \frac{1}{\frac{R}{Q} QR_S} \frac{E^2 R_S}{\omega}$$

For normal conductors $R_S \propto \omega^{1/2}$

$$\frac{P}{L} \propto \omega^{-\frac{1}{2}}$$

$$\frac{P}{A} \propto \omega^{\frac{1}{2}}$$

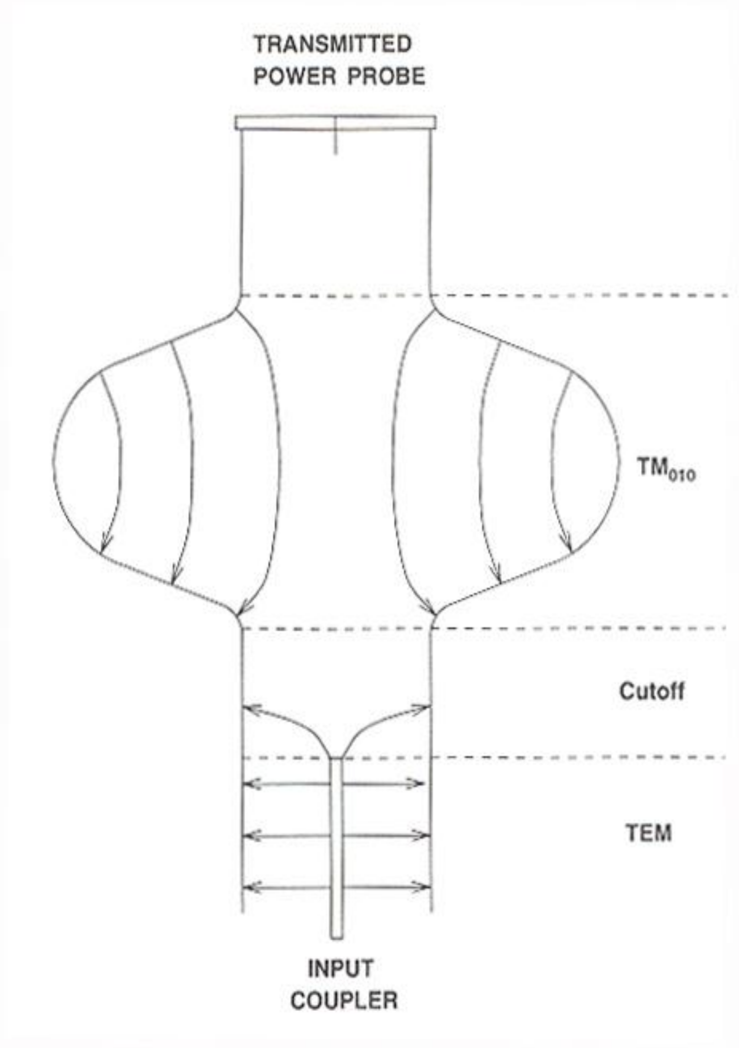
For superconductors $R_S \propto \omega^2$

$$\frac{P}{L} \propto \omega$$

$$\frac{P}{A} \propto \omega^2$$

External Coupling

- **Consider a cavity connected to an rf source**
- **A coaxial cable carries power from an rf source to the cavity**
- **The strength of the input coupler is adjusted by changing the penetration of the center conductor**
- **There is a fixed output coupler, the transmitted power probe, which picks up power transmitted through the cavity. This is usually very weakly coupled**



Cavity with External Coupling

Consider the rf cavity after the rf is turned off.
Stored energy U satisfies the equation:

$$\frac{dU}{dt} = -P_{tot}$$

Total power being lost, P_{tot}, is:

$$P_{tot} = P_{diss} + P_e + P_t$$

P_e is the power leaking back out the input coupler.
P_t is the power coming out the transmitted power coupler.
Typically P_t is very small $\Rightarrow P_{tot} \approx P_{diss} + P_e$

Recall

$$Q_0 \equiv \frac{\omega_0 U}{P_{diss}}$$

Similarly define a "loaded" quality factor Q_L:

$$Q_L \equiv \frac{\omega_0 U}{P_{tot}}$$

Now

$$\frac{dU}{dt} = -\frac{\omega_0 U}{Q_L} \Rightarrow U = U_0 e^{-\frac{\omega_0 t}{Q_L}}$$

$\therefore$ energy in the cavity decays exponentially with time constant: $\tau_L = \frac{Q_L}{\omega_0}$

Cavity with External Coupling

Equation
$$\frac{P_{tot}}{\omega_0 U} = \frac{P_{diss} + P_e}{\omega_0 U}$$

suggests that we can assign a quality factor to each loss mechanism, such that

$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_e}$$

where, by definition,
$$Q_e \equiv \frac{\omega_0 U}{P_e}$$

Typical values for CEBAF 7-cell cavities: Q_0=1x10^{10}, $Q_e \approx Q_L$=2x10^{7}.

Cavity with External Coupling

- Define "coupling parameter": $\beta \equiv \dfrac{Q_0}{Q_e}$

therefore $$\frac{1}{Q_L} = \frac{(1+\beta)}{Q_0}$$

β is equal to: $$\beta = \frac{P_e}{P_{diss}}$$

- It tells us how strongly the couplers interact with the cavity. Large β implies that the power leaking out of the coupler is large compared to the power dissipated in the cavity walls.

Several Loss Mechanisms

$P = \sum P_i$ -wall losses

-power absorbed by beam

-coupling to outside world

Associate Q will each loss mechanism

$$Q_i = \omega \frac{U}{P_i}$$

(index 0 is reserved for wall losses)

Loaded Q: Q_L

$$\frac{1}{Q_L} = \frac{\sum P_i}{\omega U} = \sum \frac{1}{Q_i}$$

Coupling coefficient:

$$\beta_i = \frac{Q_0}{Q_i} = \frac{P_i}{P_0}$$

$$Q_L = \frac{Q_0}{1 + \sum \beta_i}$$

1300 MHz 9-cell

Pill Box Cavity

Hollow right cylindrical enclosure

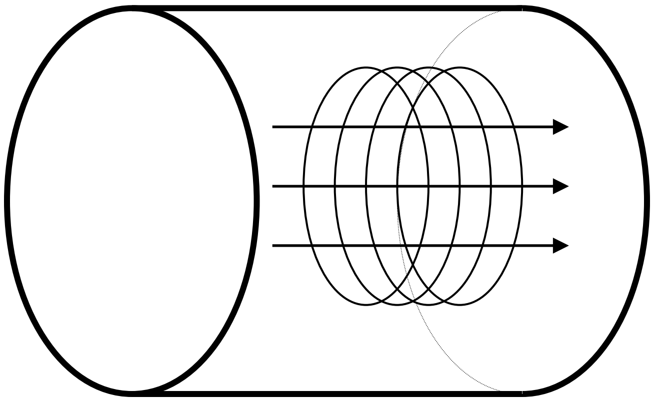
Operated in the TM_{010} mode $H_z = 0$

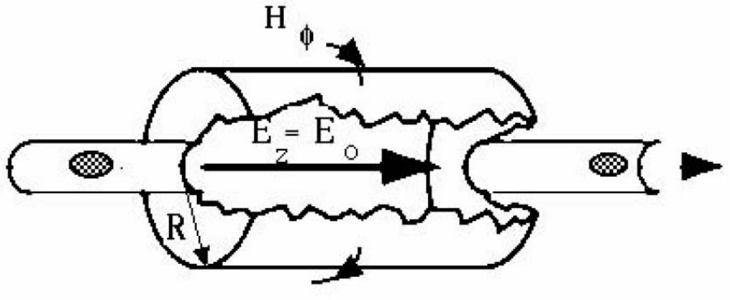
$$\frac{\partial^2 E_z}{\partial^2 r} + \frac{1}{r}\frac{\partial E_z}{\partial r} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial^2 t} \qquad \omega_0 = \frac{2.405c}{R}$$

$$E_z(r,z,t) = E_0 J_0\left(2.405\frac{r}{R}\right)e^{-i\omega_0 t}$$

$$H_\varphi(r,z,t) = -i\frac{E_0}{\mu_0 c}J_1\left(2.405\frac{r}{R}\right)e^{-i\omega_0 t}$$

TM_{010} mode





Modes in Pill Box Cavity

- TM_{010}
  - Electric field is purely longitudinal
  - Electric and magnetic fields have no angular dependence
  - Frequency depends only on radius, independent on length
- TM_{0mn}
  - Monopoles modes that can couple to the beam and exchange energy
- TM_{1mn}
  - Dipole modes that can deflect the beam
- TE modes
  - No longitudinal E field
  - Cannot couple to the beam

TM Modes in a Pill Box Cavity

$$\frac{E_r}{E_0} = -\frac{n\pi}{x_{lm}}\frac{R}{L} J_l'\left(x_{lm}\frac{r}{R}\right)\sin\left(n\pi\frac{z}{L}\right)\cos l\varphi$$

$$\frac{E_\varphi}{E_0} = \frac{ln\pi}{x_{lm}^2}\frac{R^2}{rL} J_l\left(x_{lm}\frac{r}{R}\right)\sin\left(n\pi\frac{z}{L}\right)\sin l\varphi$$

$$\frac{E_z}{E_0} = J_l\left(x_{lm}\frac{r}{R}\right)\sin\left(n\pi\frac{z}{L}\right)\cos l\varphi$$

$$\omega_{lmn} = c\sqrt{\left(\frac{x_{lm}}{R}\right)^2 + \left(\frac{\pi n}{L}\right)^2}$$

$$\frac{H_r}{E_0} = -i\omega\varepsilon\frac{l}{x_{lm}^2}\frac{R^2}{r} J_l\left(x_{lm}\frac{r}{R}\right)\cos\left(n\pi\frac{z}{L}\right)\sin l\varphi$$

x_{lm} is the mth root of $J_l(x)$

$$\frac{H_\varphi}{E_0} = -i\omega\varepsilon\frac{R}{x_{lm}} J_l'\left(x_{lm}\frac{r}{R}\right)\cos\left(n\pi\frac{z}{L}\right)\cos l\varphi$$

$$\frac{H_z}{E_0} = 0$$

TM_{010} Mode in a Pill Box Cavity

$$E_r = E_\varphi = 0 \qquad E_z = E_0 J_0\left(x_{01}\frac{r}{R}\right)$$

$$H_r = H_z = 0 \qquad H_\varphi = -i\omega\varepsilon E_0 \frac{R}{x_{01}} J_1\left(x_{01}\frac{r}{R}\right)$$

$$\omega = x_{01}\frac{c}{R} \qquad x_{01} = 2.405$$

$$R = \frac{x_{01}}{2\pi}\lambda = 0.383\lambda$$

TM_{010} Mode in a Pill Box Cavity

Energy content

$$U = \varepsilon_0 E_0^2 \frac{\pi}{2} J_1^2(x_{01}) L R^2$$

Power dissipation

$$P = E_0^2 \frac{R_s}{\eta^2} \pi J_1^2(x_{01})(R+L)R$$

$$x_{01} = 2.40483$$

$$J_1(x_{01}) = 0.51915$$

Geometrical factor

$$G = \eta \frac{x_{01}}{2} \frac{L}{(R+L)}$$

TM010 Mode in a Pill Box Cavity

Energy Gain

$$\Delta W = E_0 \frac{\lambda}{\pi} \sin \frac{\pi L}{\lambda}$$

Gradient

$$E_{acc} = \frac{\Delta W}{\lambda / 2} = E_0 \frac{2}{\pi} \sin \frac{\pi L}{\lambda}$$

Shunt impedance

$$R_{sh} = \frac{\eta^2}{R_s} \frac{1}{\pi^3 J_1^2(x_{01})} \frac{\lambda^2}{R(R+L)} \sin^2\left(\frac{\pi L}{\lambda}\right)$$

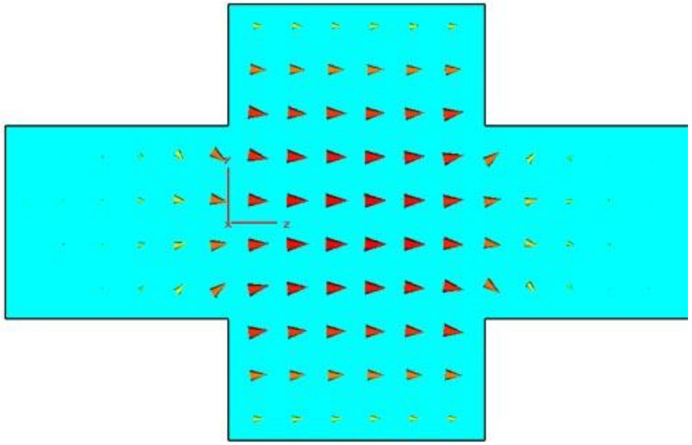
Real Cavities

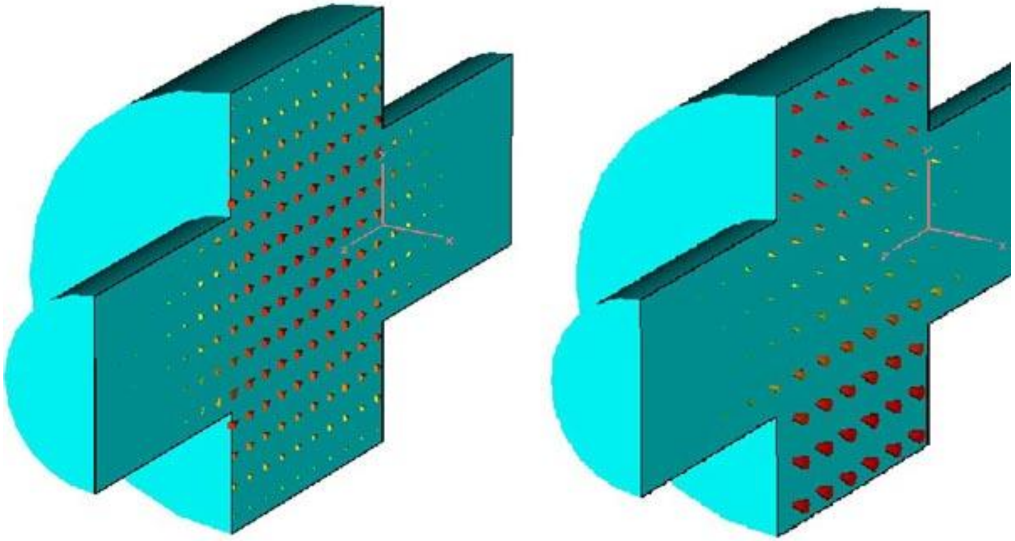
Beam tubes reduce the electric field on axis

Gradient decreases

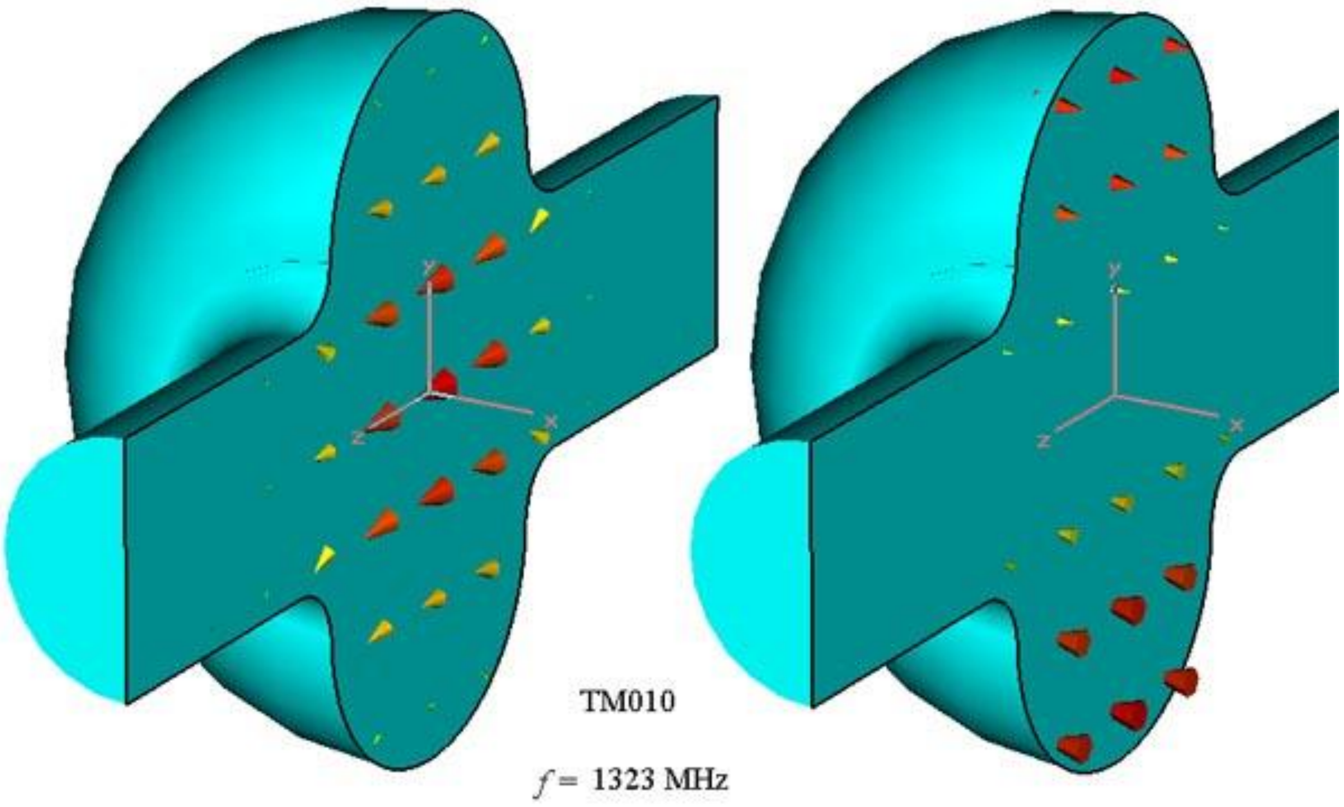
Peak fields increase

R/Q decreases



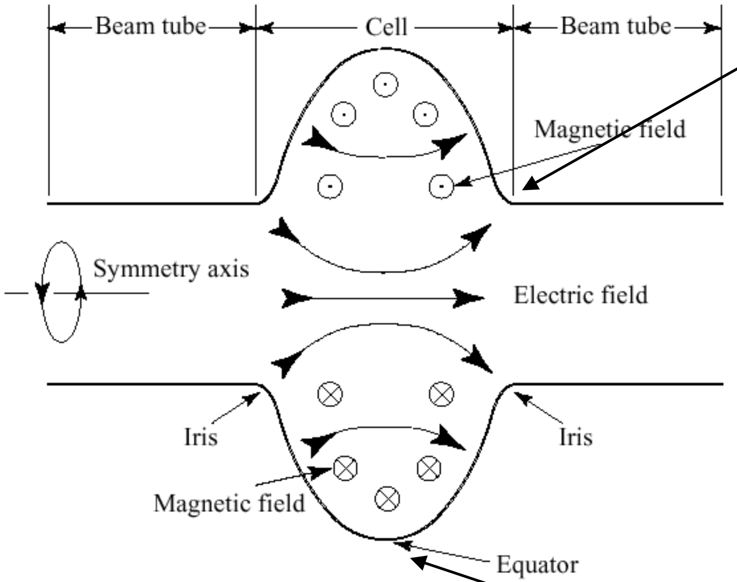


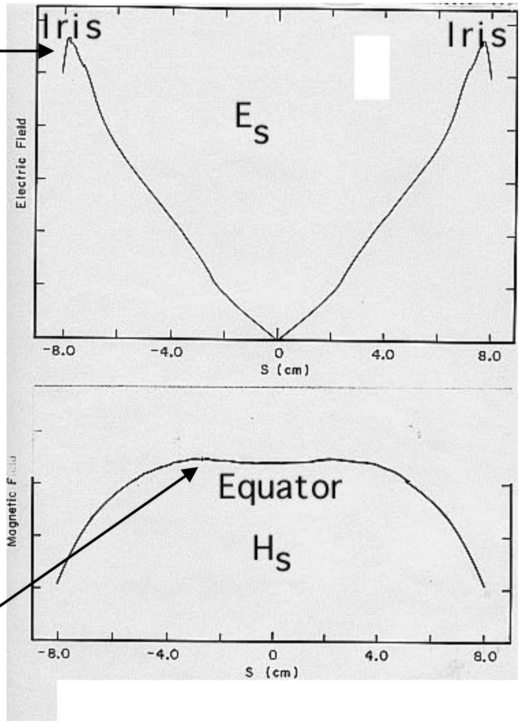
Real Cavities



Single Cell Cavities

Electric field high at iris



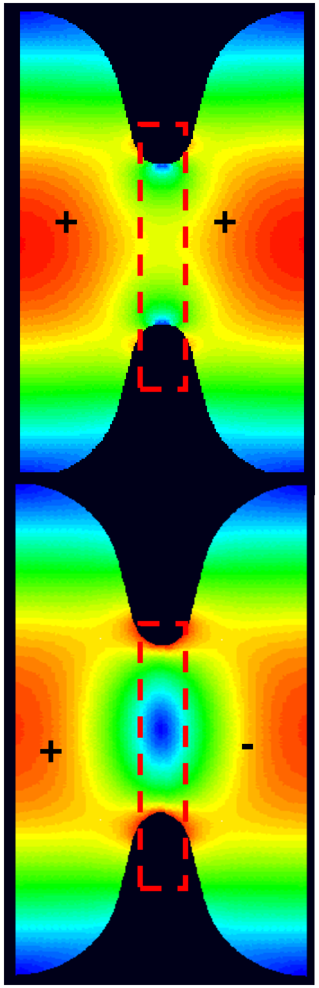


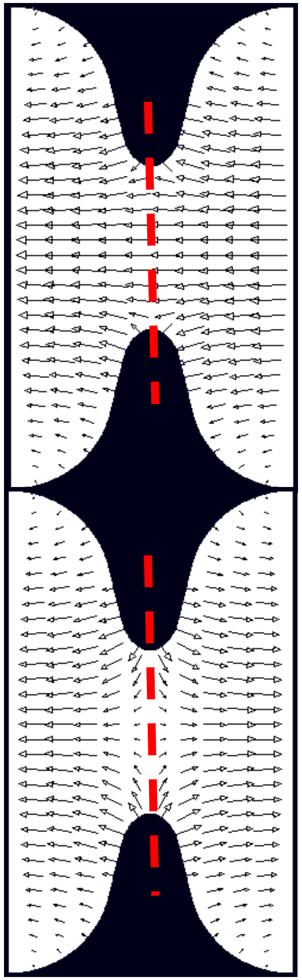
Magnetic field high at equator

Coupling between cells

ω_o

ω_π





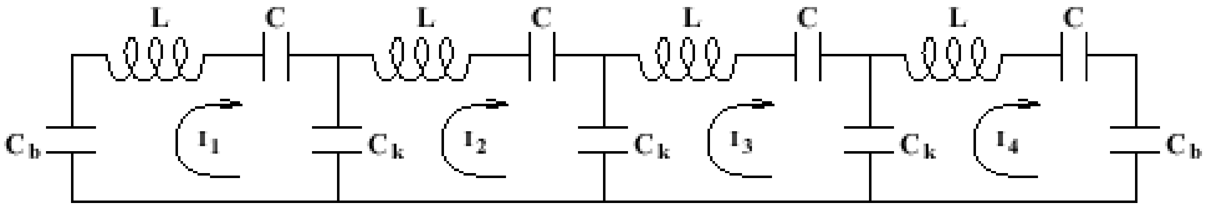
Symmetry plane for the H field

Symmetry plane for the E field which is an additional solution

The normalized difference between these frequencies is a measure of the energy flow via the coupling region

$$k_{cc} = \frac{\omega_\pi - \omega_0}{\frac{\omega_\pi + \omega_0}{2}}$$

Multi-Cell Cavities



$$k = \frac{C}{C_k} \qquad C_b = C_k / 2$$

Mode frequencies:

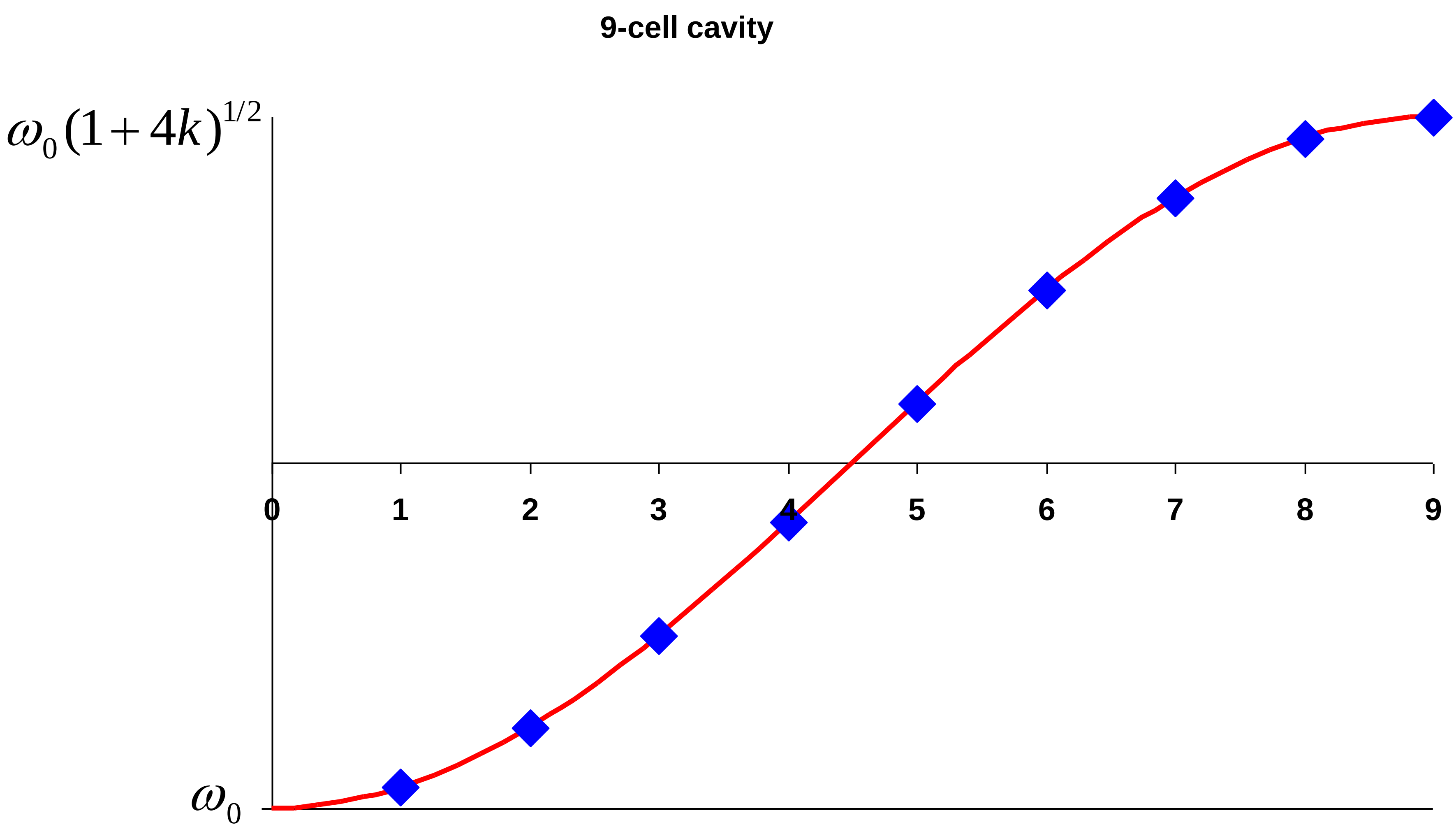
$$\frac{\omega_m^2}{\omega_0^2} = 1 + 2k\left(1 - \cos\frac{\pi m}{n}\right)$$

$$\frac{\omega_n - \omega_{n-1}}{\omega_0} \simeq k\left(1 - \cos\frac{\pi}{n}\right) \simeq \frac{k}{2}\left(\frac{\pi}{n}\right)^2$$

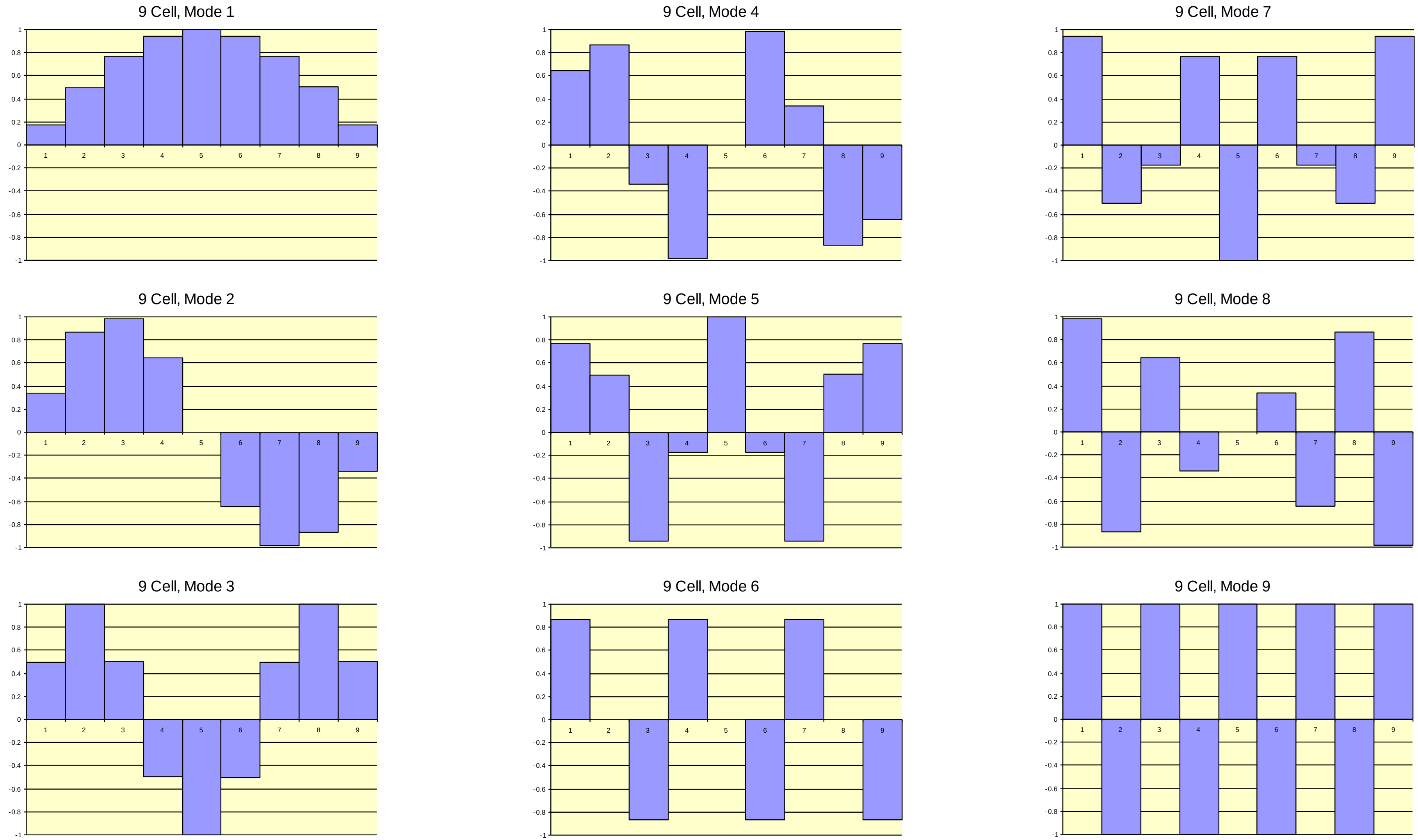
Voltages in cells:

$$V_j^m = \sin\left(\pi m \frac{2j-1}{2n}\right)$$

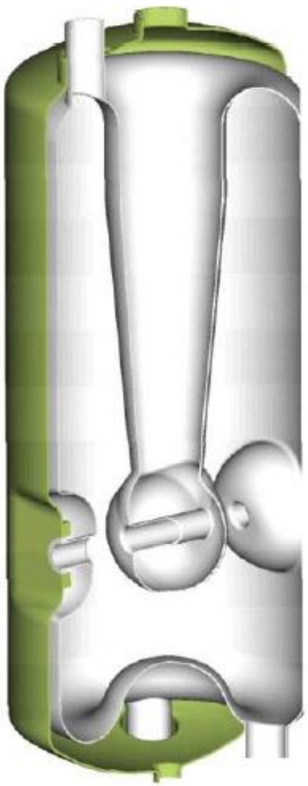
Pass-Band Modes Frequencies

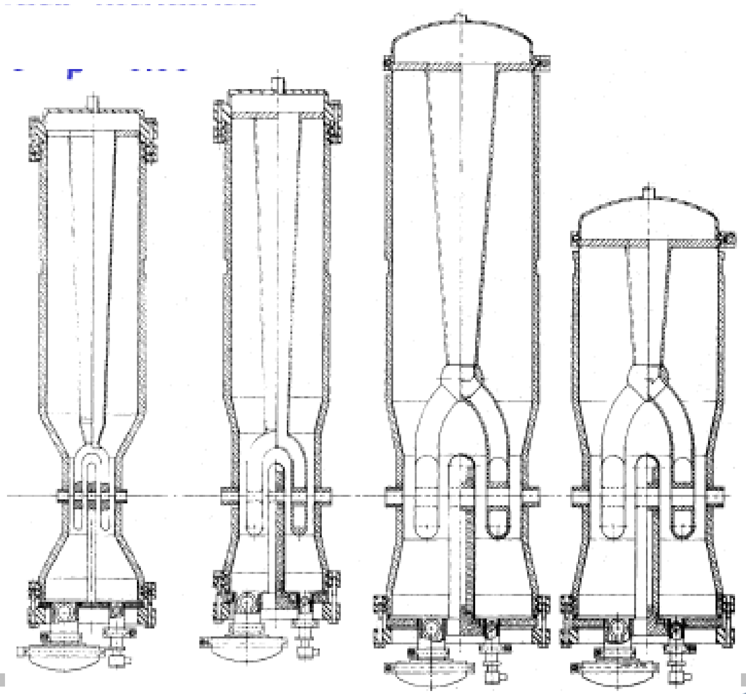


Cell Excitations in Pass-Band Modes

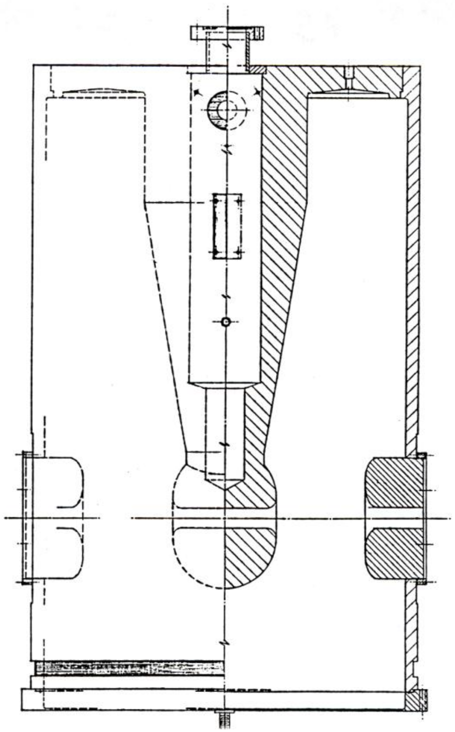


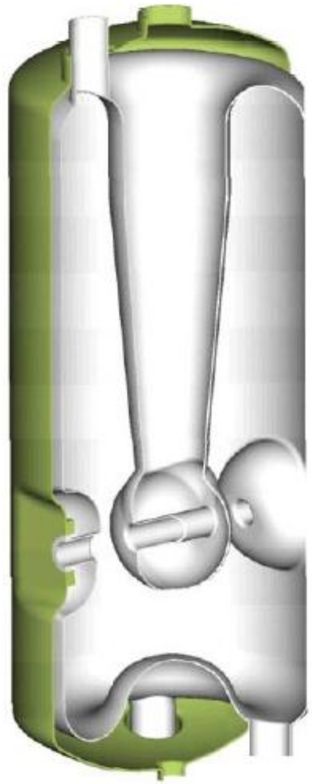
Some Real Geometries (λ/4)

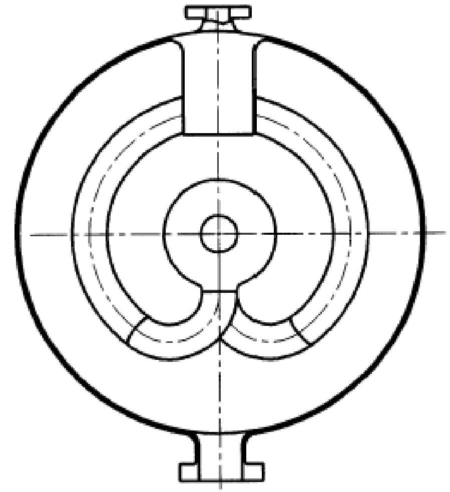


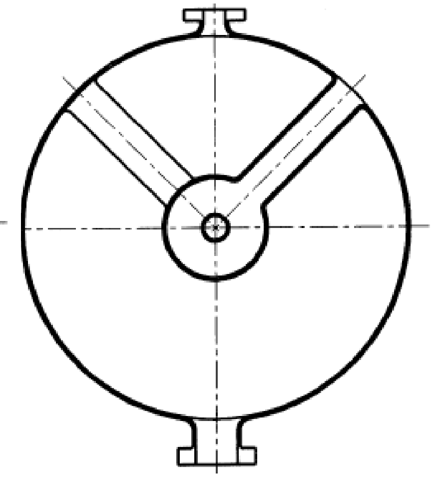


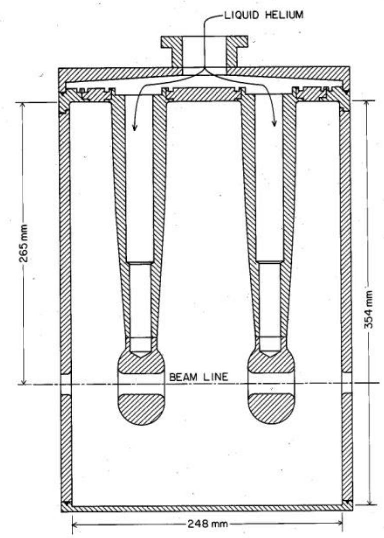
λ/4 Resonant Lines

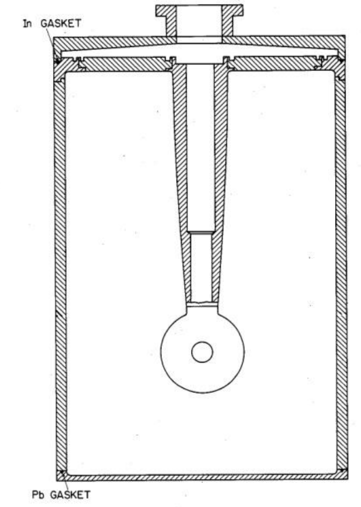




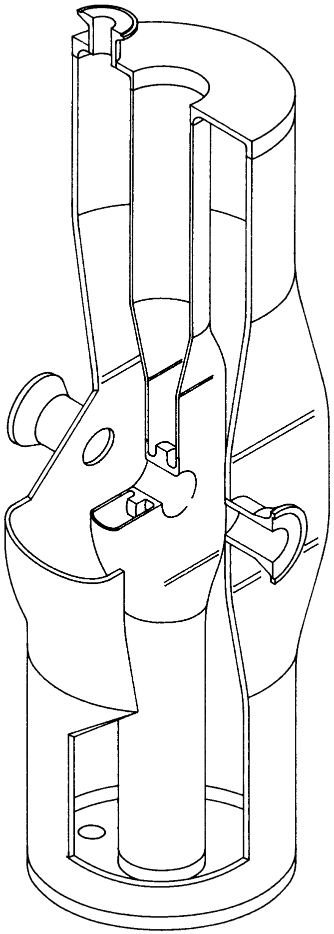


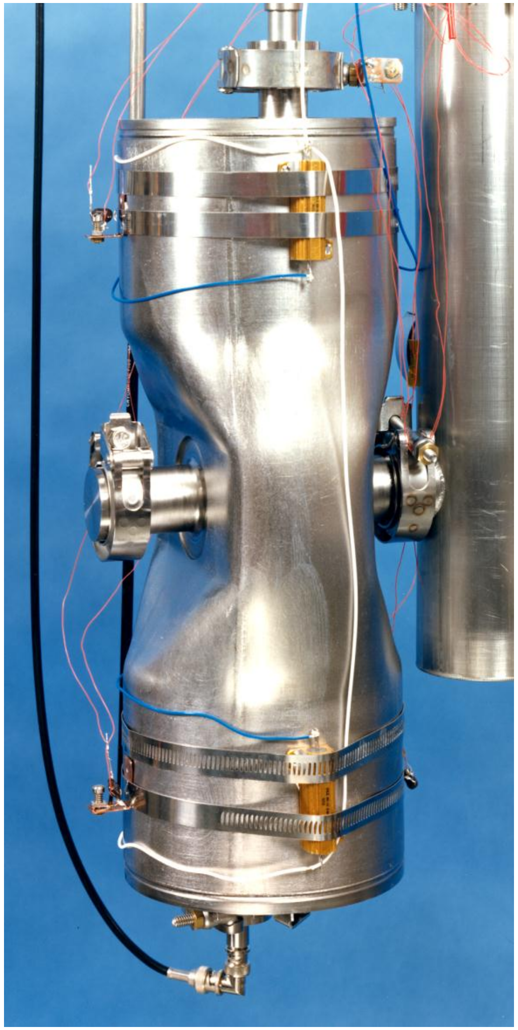


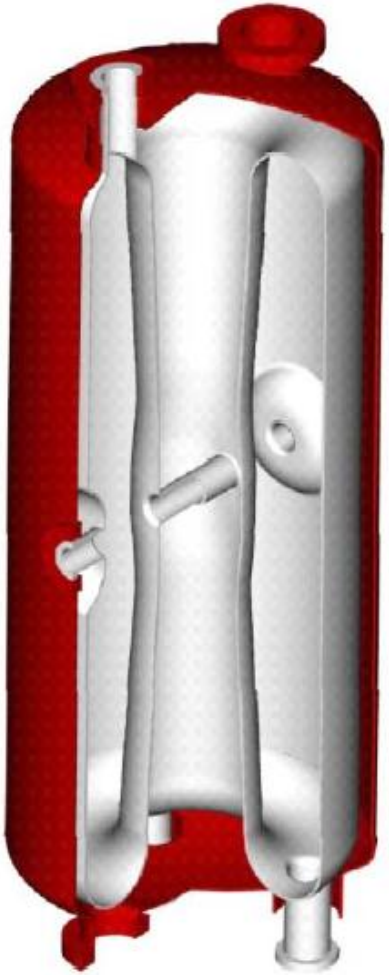




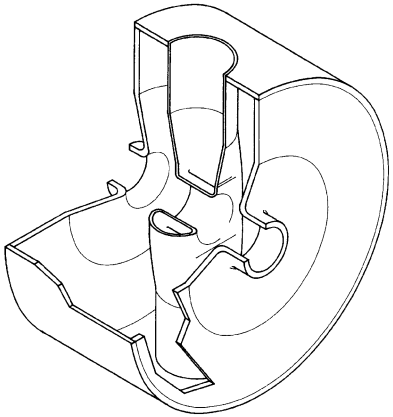
λ/2 Resonant Lines

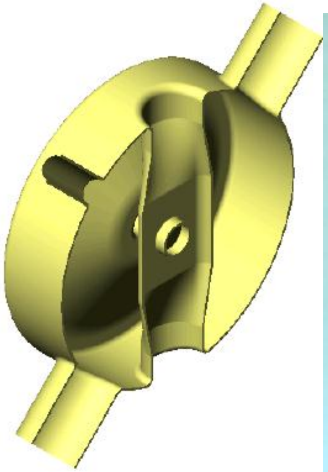


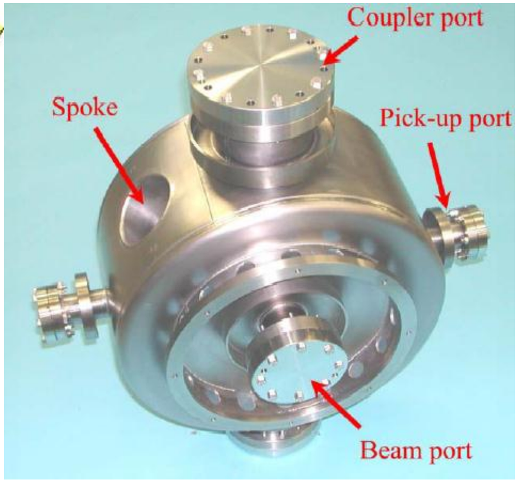




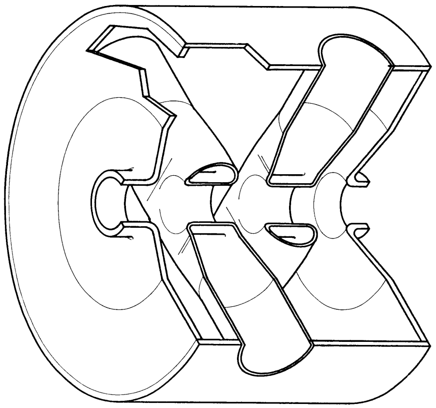
λ/2 Resonant Lines – Single-Spoke

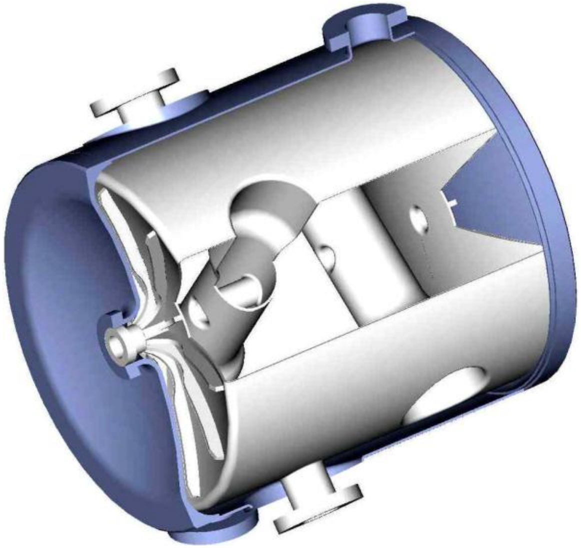


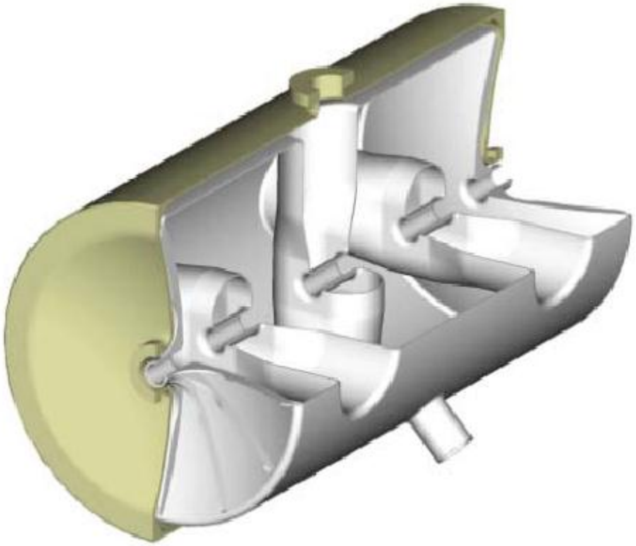




λ/2 Resonant Lines – Double and Triple-Spoke







λ/2 Resonant Lines – Multi-Spoke