Course B: rf technology
Normal conducting rf
Part 4a: Introduction to Wakefields

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Wakefields

We will now discuss how charges, which travel through rf structures (and beam pipes with features) , leave behind fields and how these fields act on following charges.

In the previous section we restricted ourselves only to the interaction between a current and a single (the main one) mode of an rf structure. We focused on *acceleration*.

Now we will consider the interaction with many modes, in particular on transverse ones. We address for the purposes of studying *beam stability*. A beam picks up energy spread and transverse kicks from wakefields.

Basic assumptions

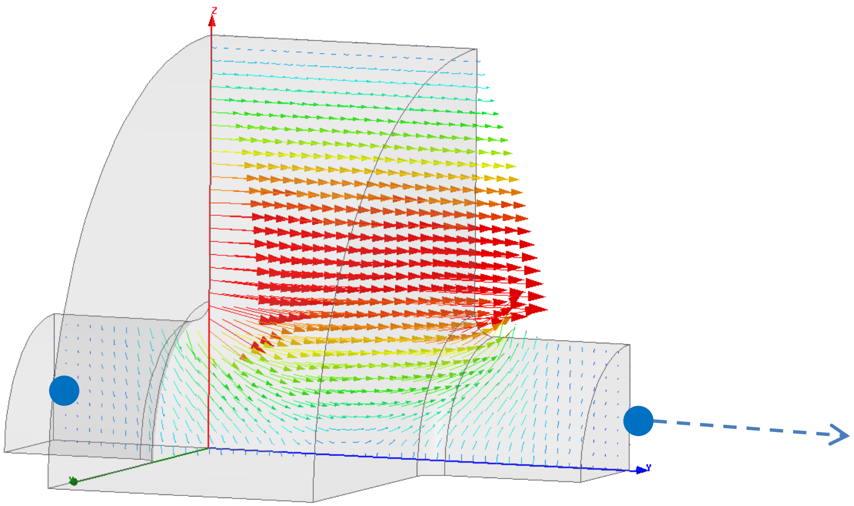
In this section we are going to consider only highly relativistic driving and following bunches:

- using the (really good) approximation that $v=c$
- that the beam trajectories through the structures we study are not affected by the fields, i.e. they follow straight unbendable lines
- we will calculate the momentum “kicks” the particles are subject to and pass them on to the beam dynamics gang. They will deal with the trajectories on bigger scales through particle tracking.

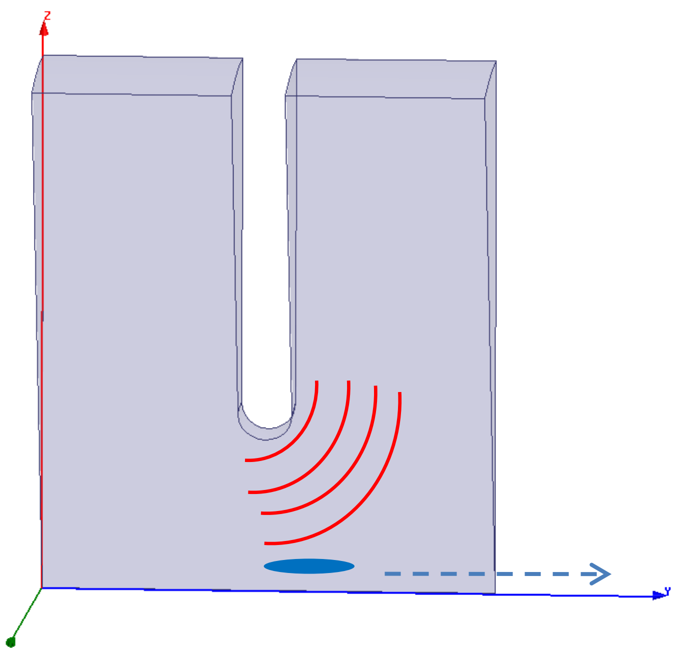
This is a very special case of moving charges, radiated fields along with conductors – luckily for us the charge movement is fixed before hand and we only need to understand the fields.

This is for example not the case in the injector where the beam is not relativistic yet. There you need to solve field and particle trajectories self consistently using PIC (Particle In Cell) codes.

Two kinds of wakefields

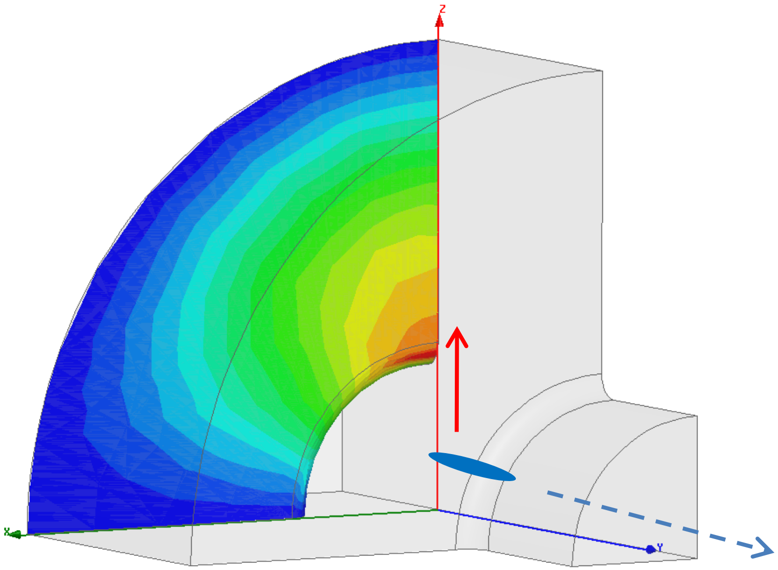


We consider a train of bunches. Here a bunch acts on following bunches. We call this effect the **long-range wakefield**.
We often analyze this case by considering the series of modes the driving beam excites in the cavity or DLWG.



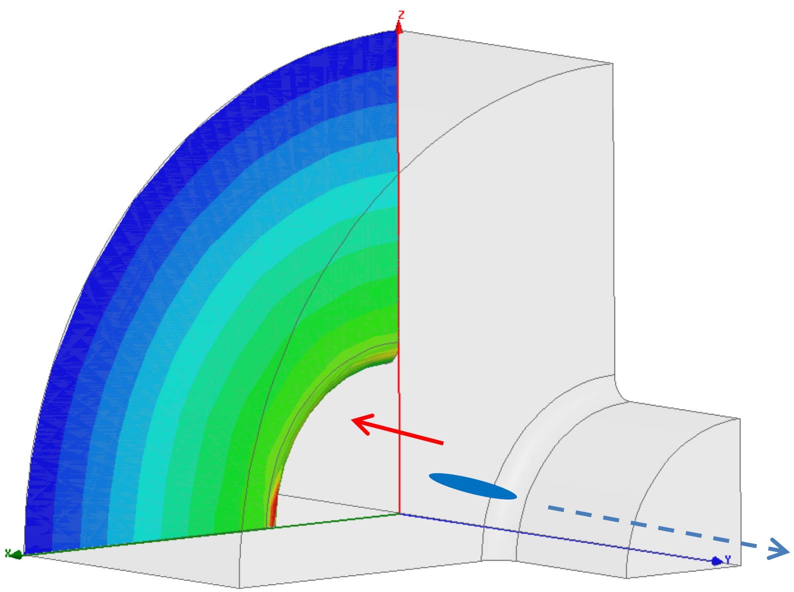
We consider a single bunch, and all real bunches have finite length. Here the head of the bunch acts on the tail of the bunch. We call this effect the **short-range wakefield**.
This one we analyze by considering the diffraction of the fields of the bunch from the discontinuities in the walls of our cavity or DLWG.

Another distinction among wakefields



Transverse wakefields – a radially offset bunch (head) excites azimuthally varying modes which gives a transverse momentum to following bunches (tail).

The induced kick is represented by the red arrow.



Longitudinal wakefields – a bunch (head) excites m=0 modes which gives an acceleration/deceleration momentum to following bunches (tail). We've already started working on this type of wakefield in section 2.

Actually solving something

These days there are a number of programs - GdfidL, Microwave Studio, ACE3P, HFSS - to get spectacularly accurate solutions for the wake potentials.

They operate mainly in time-domain, calculating fields step by step as the particle flies through the structure and doing for you the necessary integrals.

You can also use frequency domain codes to get detailed understanding of what's happening to individual modes and do design work. There is also a rich history of semi-analytic methods like circuit models and mode matching techniques.

Still it is important to be familiar with the underlying theory in order to understand the results and what the origin of different features are.

I won't be fully rigorous, there isn't nearly enough time, but I'll try to pick out the highlights of derivations and the characteristics for solutions for special cases.

In the next section we will study how to design a cavity to reduce wakefield effects.

We will now look at the 'frequency spectrum' of a finite length bunch.

Bunches in linear colliders are short – with a $\sigma \cong 50\ \mu m$ for CLIC

Radiation from bunches starts to be suppressed at frequencies where, roughly, the half wavelength is less than the bunch length.

This point is important since it puts an upper limit on the frequencies that we need to consider.

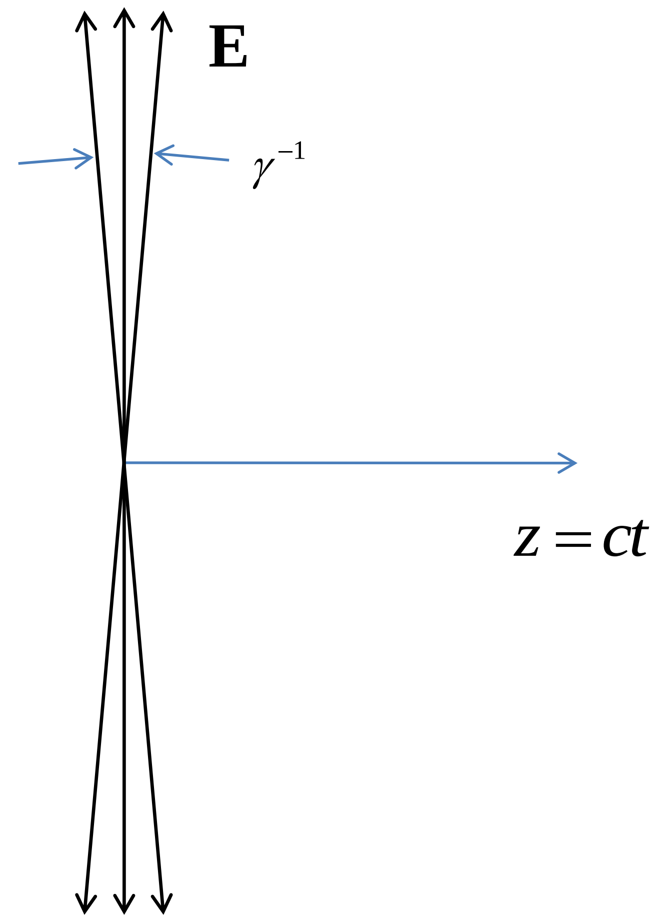
Let's look at this question in a bit more detail.

The "pancake" field pattern of a relativistic point charge

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$E_r(z,r) = \frac{q\gamma r}{4\pi\varepsilon_0 \left(z^2\gamma^2 + r^2\right)^{\frac{3}{2}}}$$



Example: 1 TeV charge at 3 mm radius has field region only 6 nm to be compared to CLIC bunch length of roughly 100 µm.

$$d = \frac{1TeV}{0.511MeV} 3mm = 6nm$$

Points to emphasize

No fields before or after charge - we do not need to consider direct charge-to-charge forces.

We need our conducting boundaries to communicate to following charges.

And of course it's impossible to communicate forward, you'd have to go faster than the speed of light.

Because the 'opening angle' of the field of a relativistic beam is so narrow, our field pattern in free space exactly matches our bunch charge pattern for real linear collider type bunches.

Before we look at the excitations of modes, which have specific frequencies, let's get a feeling for the frequencies a typical linear collider bunch contains.

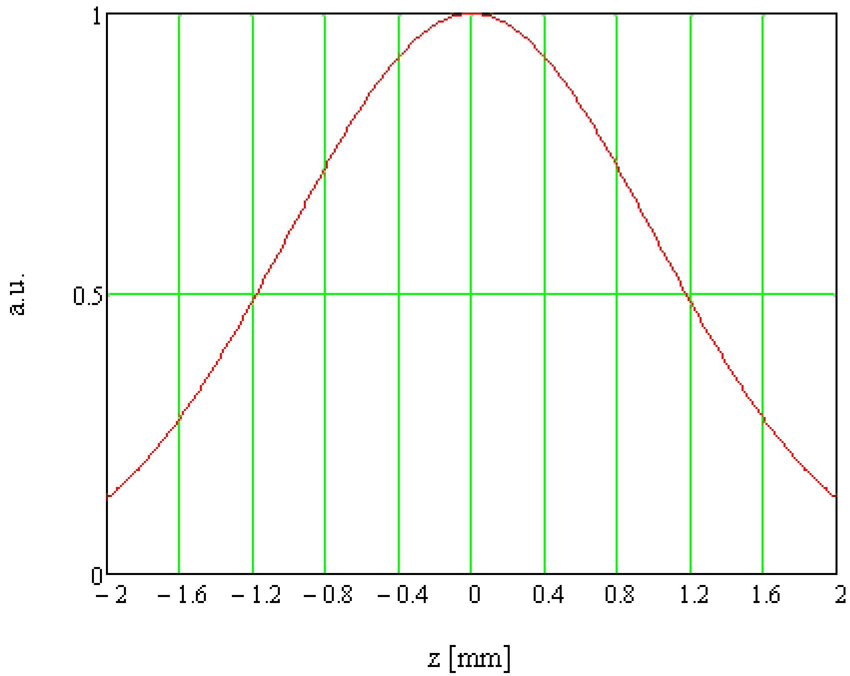
Frequency content of a Gaussian bunch

speed of light　　　　Fourier transform

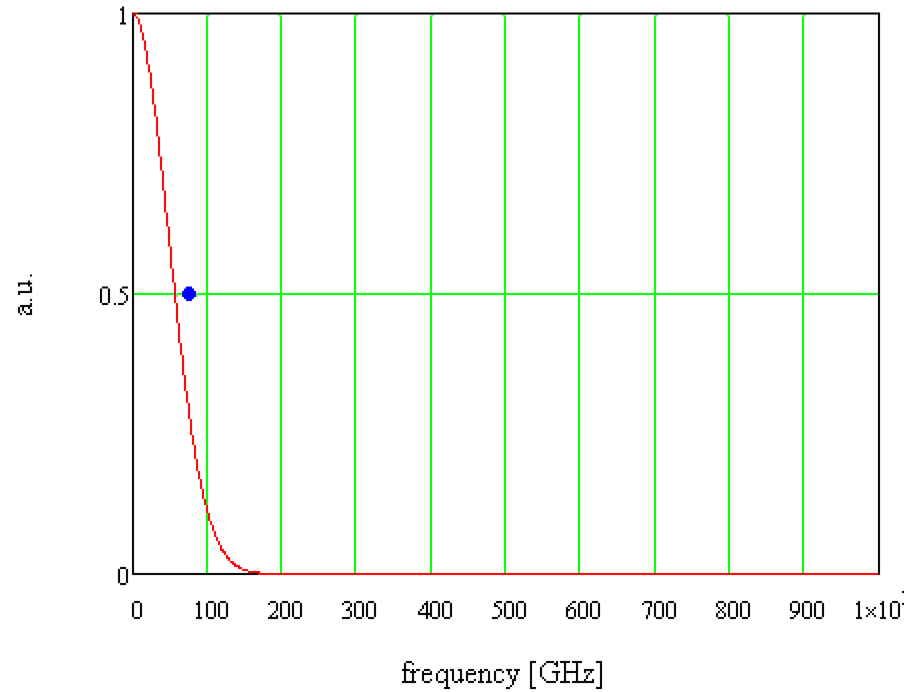
$$\rho(z) = e^{-\frac{z^2}{2\sigma^2}} \Rightarrow \rho(t) = e^{-\frac{(ct)^2}{2\sigma^2}} \Rightarrow A(f) = e^{-\frac{1}{2}\left(\frac{\sigma 2\pi}{c}\right)^2 f^2}$$

σz = 1·mm

Blue dot is $f = \frac{c}{4\sigma}$

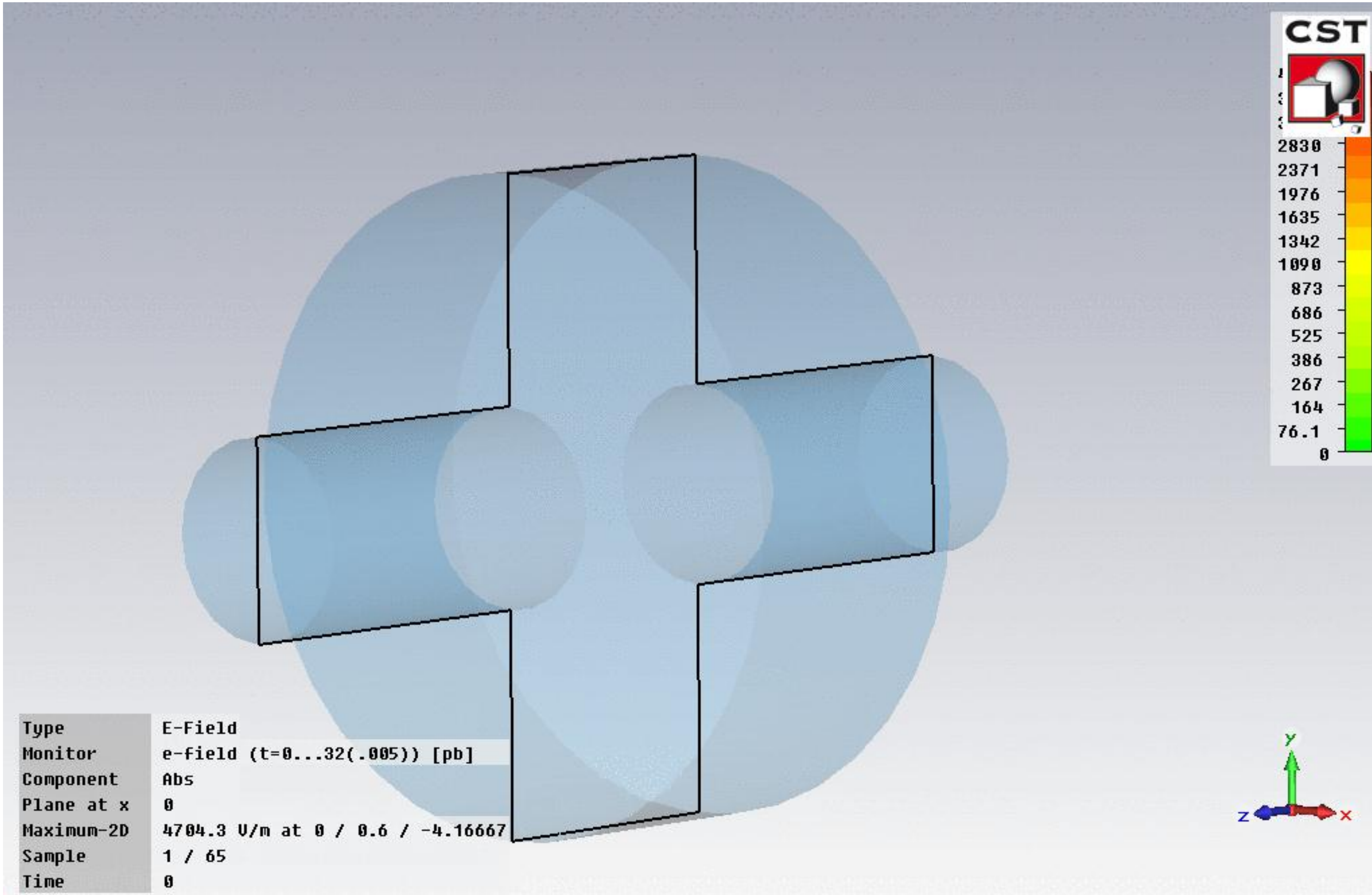


Charge distribution



frequency distribution

The fields left behind a finite length bunch



We are now going to introduce the idea of a **wake potential**.

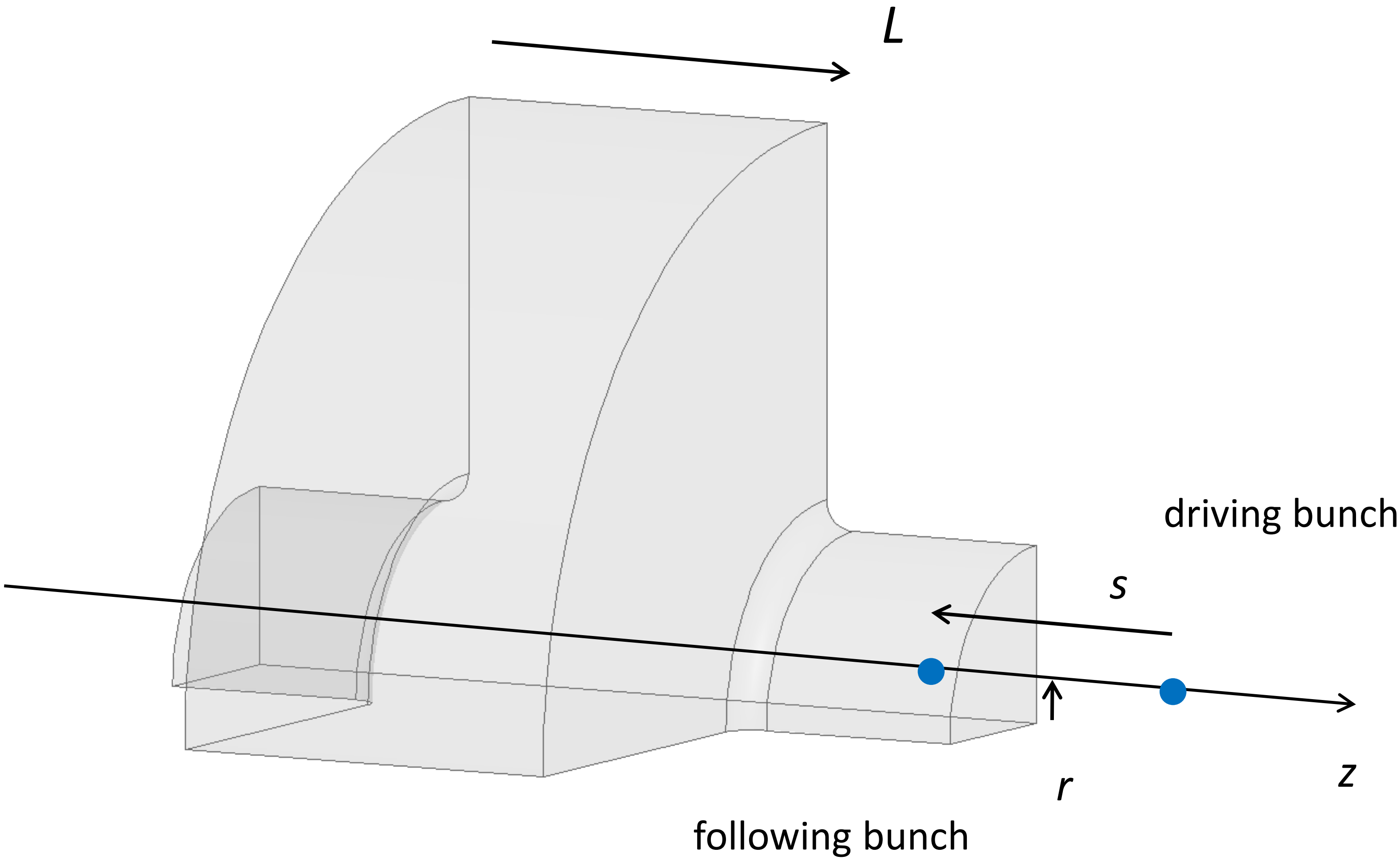
This is the potential, measured in volts, a driving bunch leaves behind it, which is experienced by a witness charge.

In the standard formalism the bunches both travel at the speed of light, along straight lines parallel to the structure axis.

This is the information needed for beam dynamics simulations which calculate instabilities caused by charges acting on following charges.

The wake potential can be given for the whole of a finite length structure or per unit length for a infinitely long periodic structure.

Our coordinate system



The *longitudinal* wake potential

This the total voltage that is lost (or gained) by a test charge following a driving charge q at distance *s.*

It induces an energy spread in the beam which causes emittance growth through effects like chromaticity.

The wake potential is given by the integral of the fields that are left behind by a very short driving bunch,

$$W_z(s) = -\frac{1}{q}\int_0^L E_z\left(z, (z+s)/c\right)dz \qquad \left[\frac{V}{pC}\right]$$

distance   time

As we have seen, high frequency part of the wake will depend on how long the bunch is. It is cut-off at half-wavelengths which correspond to the bunch length.

This comes out rigorously because the delta wake-potential on the previous slide can then be used as a Green's function to get the voltage loss/gain from a bunch of arbitrary shape by convoluting with the shape of the current,

$$W_z(s) = \int_0^{\infty} I(s - s') W_z(s') ds'$$

Now we will look at how you would approach calculating the longitudinal wake for a resonant cavity which has lots of modes, and then look at a numerical example.

Longitudinal wake by expanding the normal modes in a cavity

Reminder from section 2, the loss factor from a mode λ is,

$$k_\lambda = \frac{|V_\lambda|^2}{4U_\lambda}$$

The total acceleration left behind in the mode is (the bunch sees half its own field),

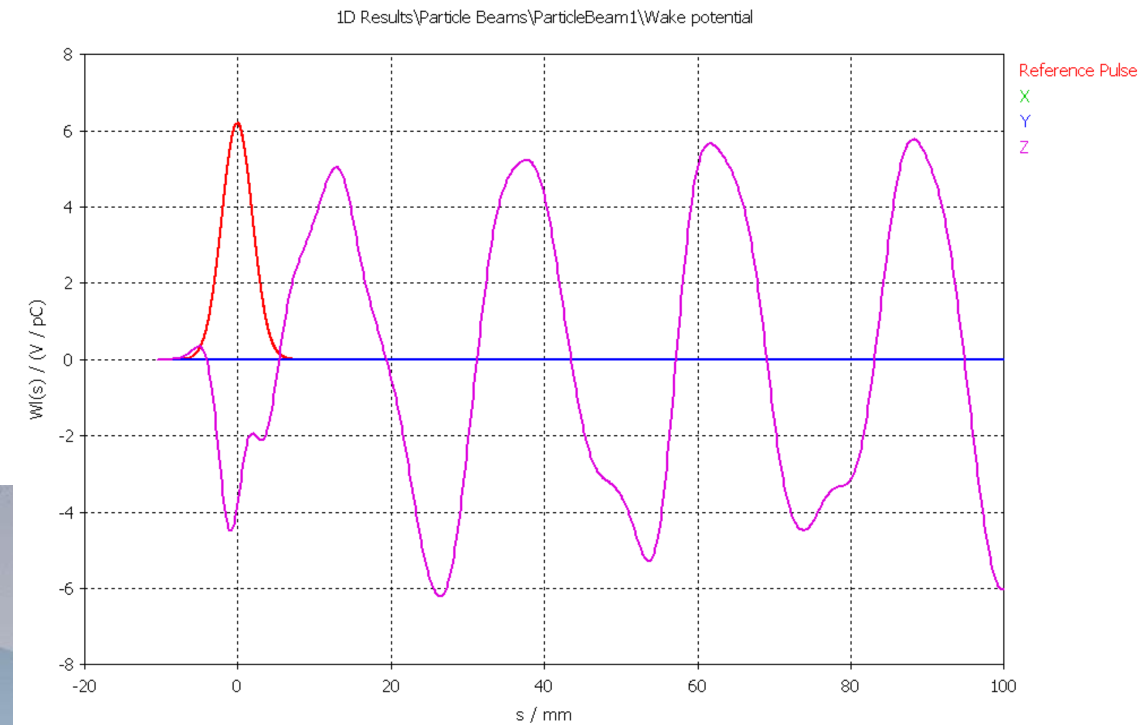
$$2k_\lambda$$

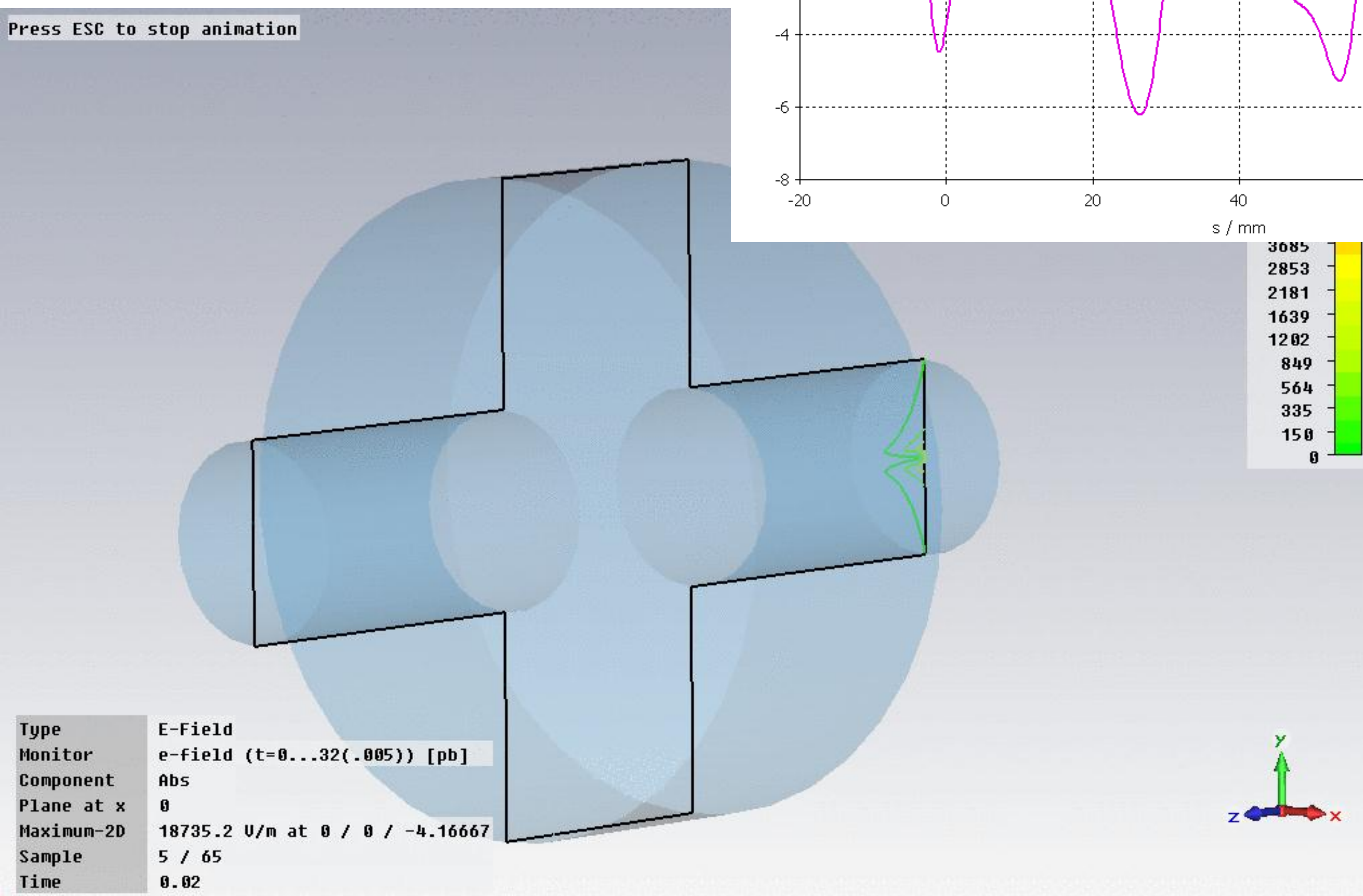
To get the total wake potential we sum over all the modes (that is really about all there is to it),

$$W_z(s) = \sum_\lambda 2k_\lambda \cos\left(\frac{\omega_\lambda s}{c}\right)$$

The terms of the longitudinal wake are cosine-like, which is reasonable, because the driving bunch itself loses energy.

Longitudinal wake potential from in a pillbox cavity





σ=2 mm bunch

Philosophy: Here we see better where the fundamental theorem of beam loading is coming from

Longitudinal wake

Cavity:

r=10 mm radius

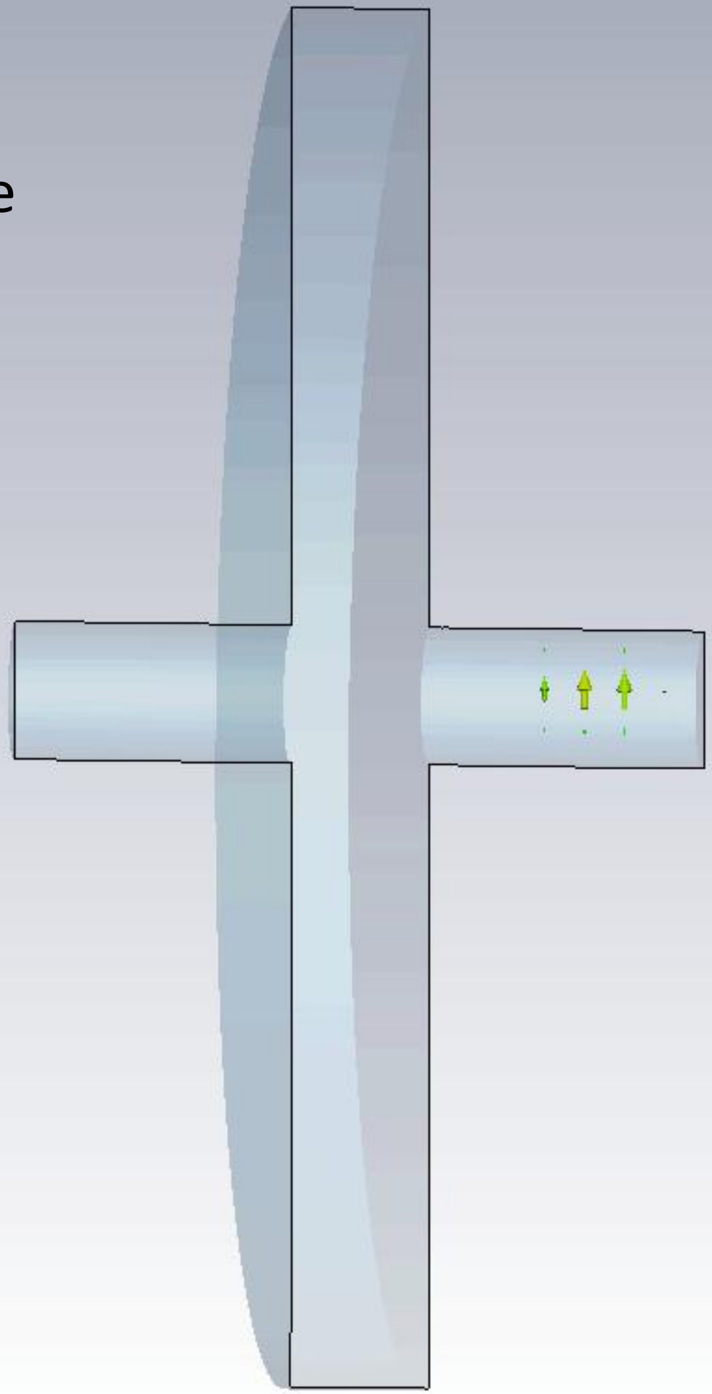
d=2 mm thick

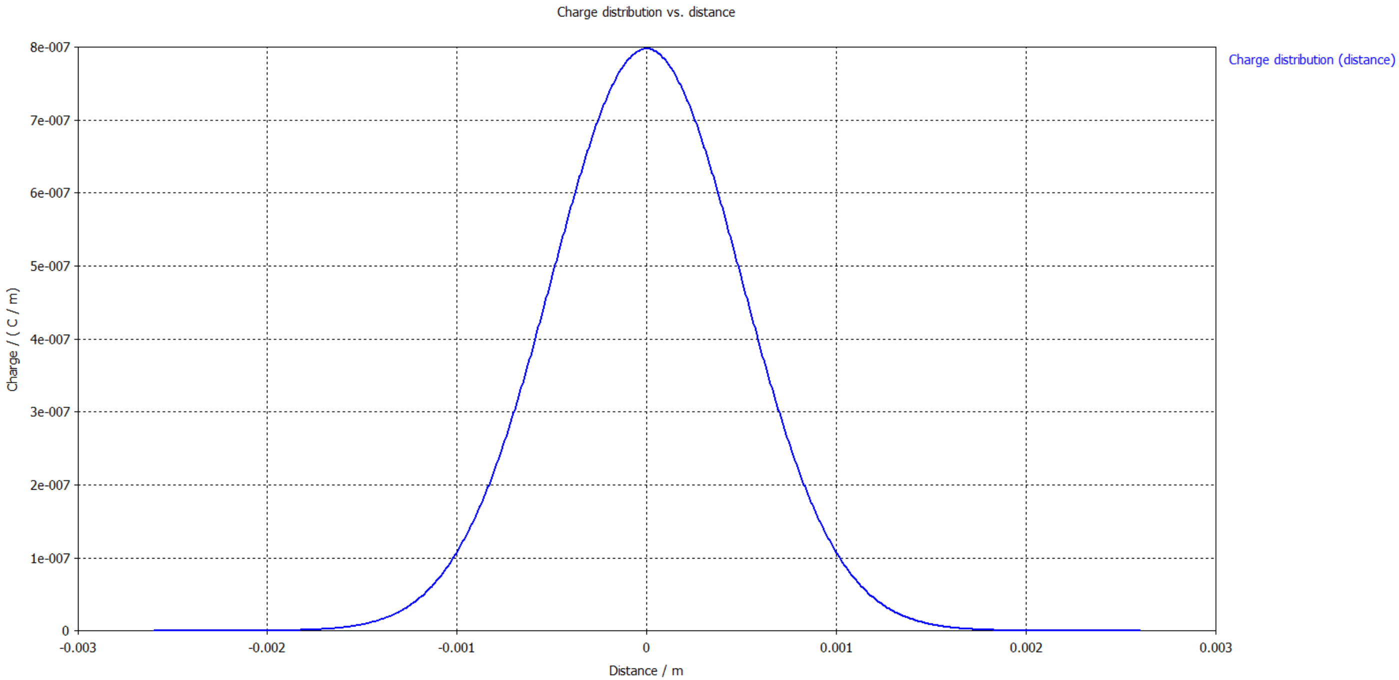
Beam pipe:

1 mm radius

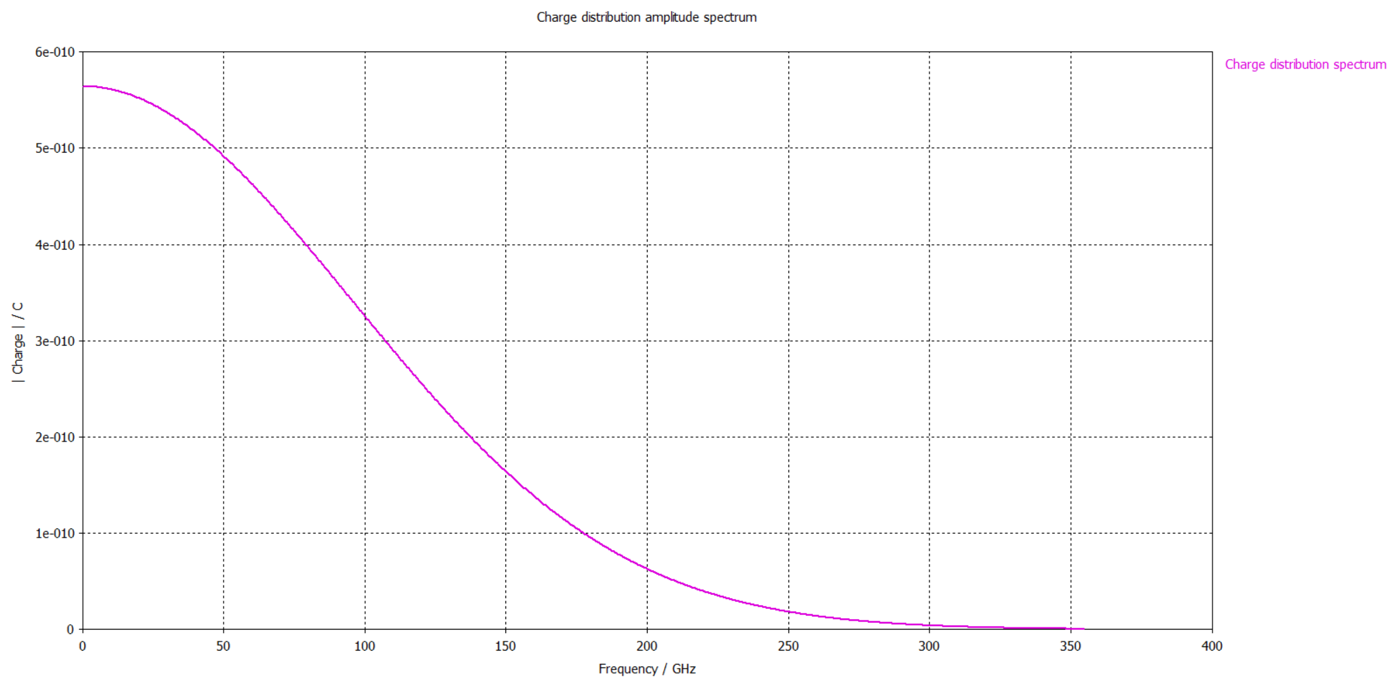
Beam:

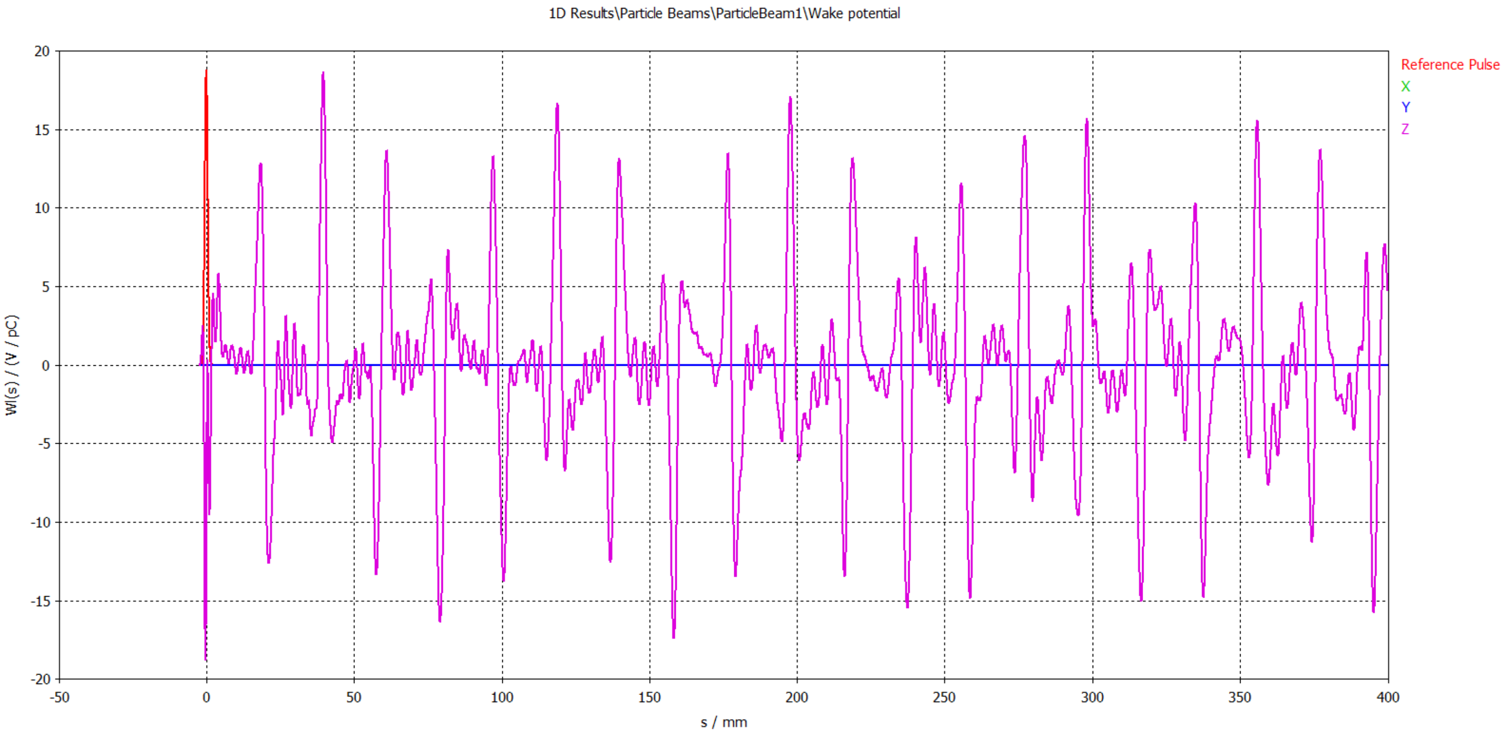
σ = 0.5 mm



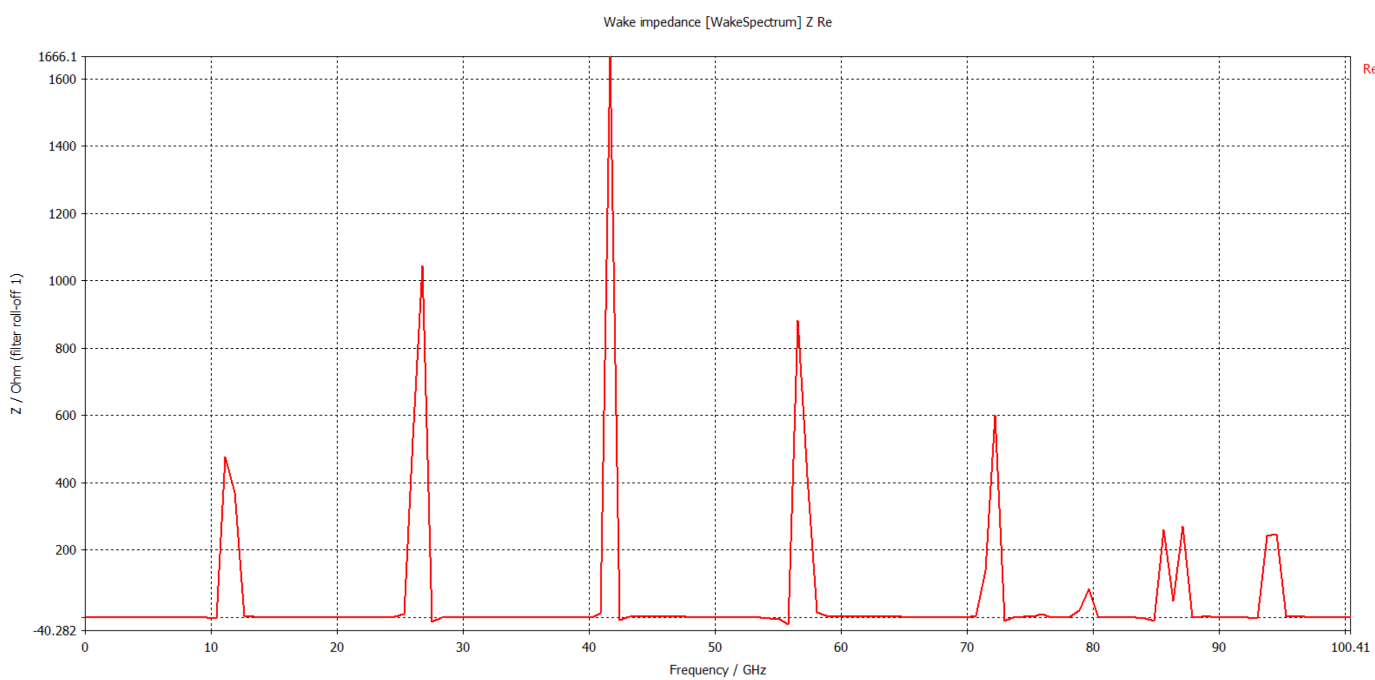

Charge distribution vs. distance
Charge distribution (distance)
8e-007
7e-007
6e-007
5e-007
4e-007
3e-007
2e-007
1e-007
0
Charge / (C / m)
-0.003
-0.002
-0.001
0
0.001
0.002
0.003
Distance / m

$= 0.5\, mm$

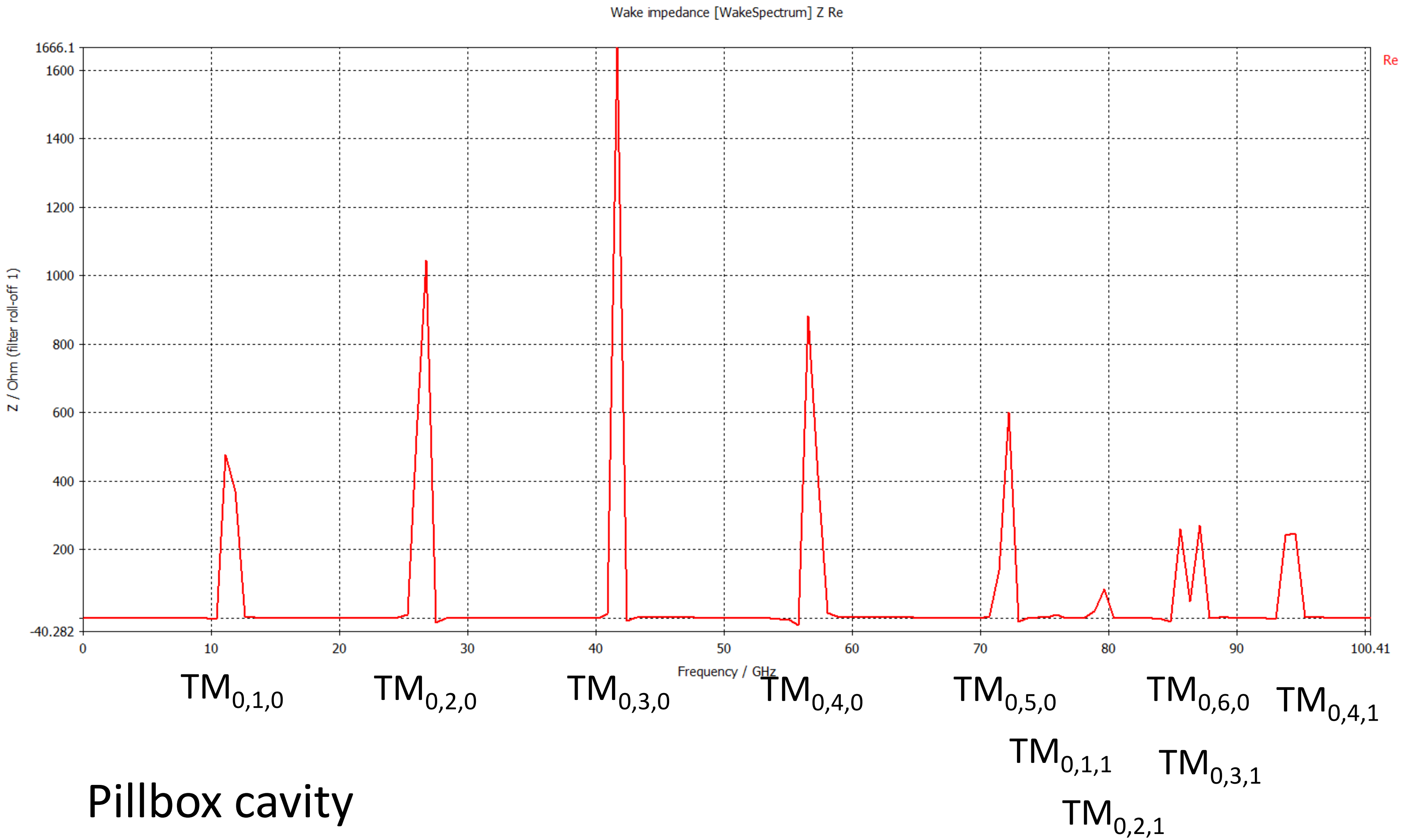

Charge distribution amplitude spectrum
Charge distribution spectrum
6e-010
5e-010
4e-010
3e-010
2e-010
1e-010
0
| Charge | / C
0
50
100
150
200
250
300
350
400
Frequency / GHz



1D Results\Particle Beams\ParticleBeam1\Wake potential
Reference Pulse
X
Y
Z
W(s) / (V / pC)
s / mm



Wake impedance [WakeSpectrum] Z Re
Re
Z / Ohm (filter roll-off 1)
Frequency / GHz



Pillbox cavity

$$f_{n,m,p} = \frac{c}{2\pi}\sqrt{\frac{x_{n,m}^2}{r^2} + \frac{p^2\pi^2}{d^2}}$$

Where for TM modes $x_{n,m}$ is the n^{th} root of $J_m(r)$
r is the cell radius
d is the cell length

Now the *transverse* wake potential

It tells us how a transversely offset bunches gives a transverse kick to following charges.

BUT this one is significantly trickier. We have seen how fields are excited as a function of offset and now we need to understand how they kick following bunches.

We already have all the formalism to deal with the excitation of the bunch, the loss factor *k*:

$$k_\lambda = \frac{|V_\lambda|^2}{4U_\lambda}$$

We just need to do the integral along a path with the correct transverse offset. *k* goes to being $k(r,\theta)$.

The physics of how rf fields kick a particle is of course all completely contained in the Lorenz force,

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$

But there is a simpler way of looking at the kick from rf cavities.
You can put in some properties which come from E and B being related through Maxwell's equations,
and that we are considering the special case that *v*=c,
and derive something called the **Panofsky-Wenzel theorem**.

The Panofsky-Wenzel Theorem

We don't have time to do the derivation, I will just state the result, and then we will look at the consequences of it in a couple of special cases.

The Panofsky-Wenzel theorem says that you can get the total transverse kick of an rf cavity by integrating the *radial* variation of the *longitudinal* acceleration,

$$p_{\perp} = \frac{-ie}{\omega_0} \int_0^L \nabla_{\perp} E_z dz$$

The Panofsky-Wenzel theorem relates the transverse wake to changes in the longitudinal one.

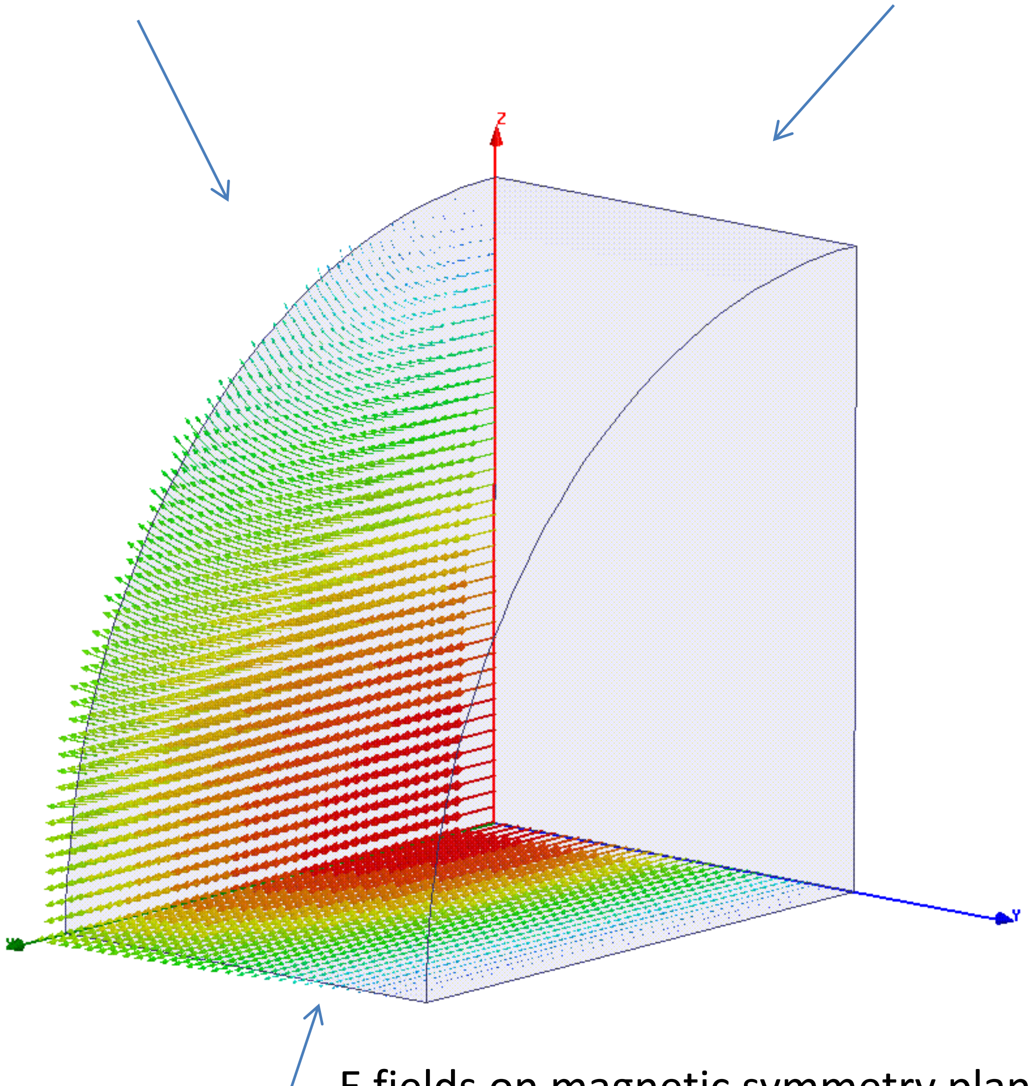
It also tends to be computationally simpler than calculating the forces directly and can be very, very useful in quickly settling arguments with your colleagues. You can easily see what the fields will do.

Let's look at some special cases to get a feeling for this equation.

A surprise - the $TE_{1,1,1}$ mode cannot kick the beam!

E fields on magnetic symmetry plane

H fields on electric symmetry plane



E fields on magnetic symmetry plane

Because it has no longitudinal electric field so Panofsky-Wenzel says it won't!

But certainly the transverse electric field must kick the beam?

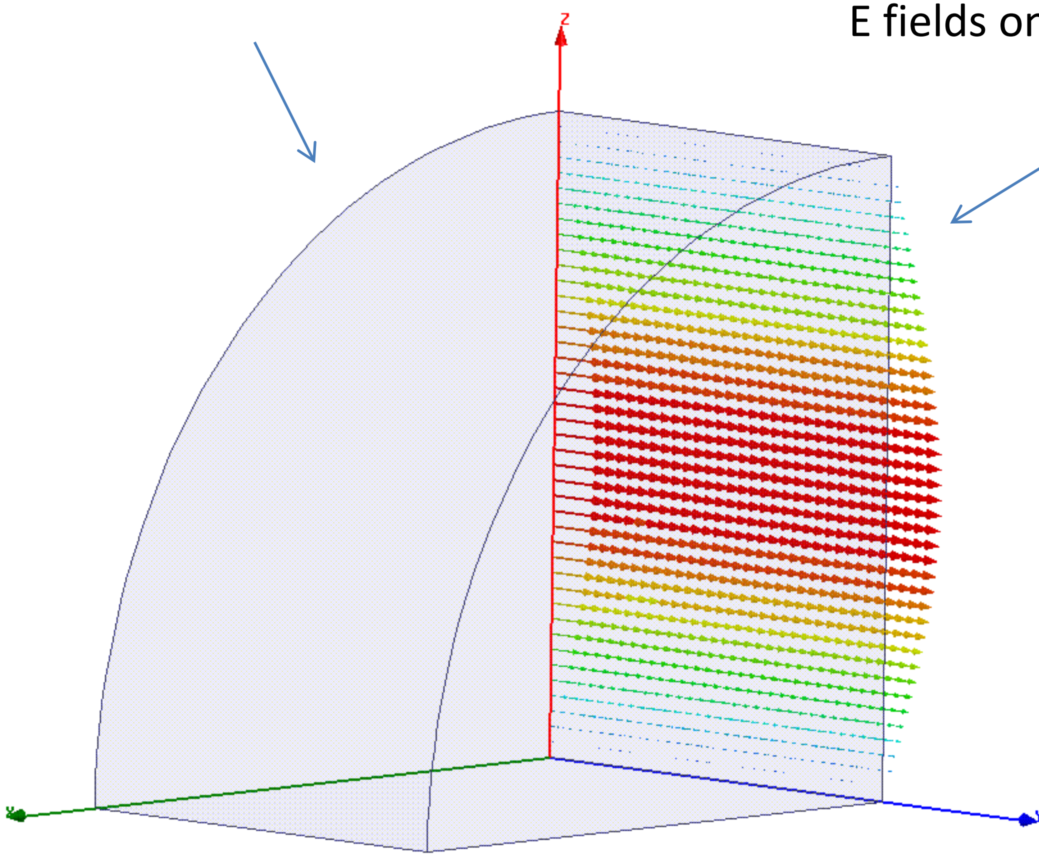
You can look at the problem more closely and in this specific case convince yourself the magnetic field magnetic field cancels the kick from the electric field.

But it's easier just to use the Panofsky-Wenzel theorem, no E_z so no variation in E_z so no kick!

But the $TM_{1,1,0}$ mode does deflect

H fields on conducting end wall

E fields on magnetic symmetry plane



$$\nabla_{\perp} E_z$$

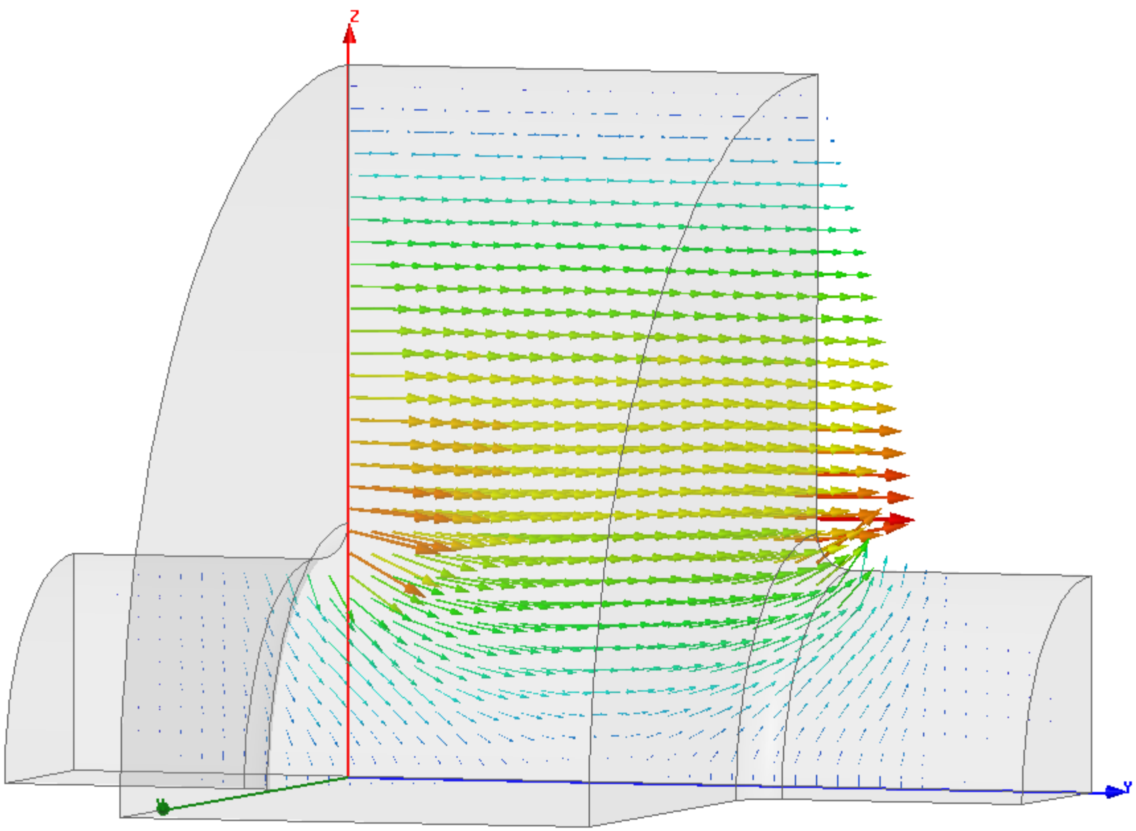
There's both E_z and a transverse gradient of it – hence the mode kicks the beam.

The excitation by the beam is proportional to E so increases 'linearly' with transverse offset.

This is the "cavity BPM" and "rf kicker" mode.

Now there's an amazing thing about beam apertures.

It's one of the rare occasions where a necessary, practical, feature (you've got to get the beam through the cavity) gives you a simplified behavior.



What do I mean by this?

Consider the acceleration by the $TM_{0,1,0}$ mode in a pillbox cavity.

The field pattern is J_0, a Bessel function. Consequently you have a radial dependence of the integral and you expect to have rf (de)focusing.

BUT

over a circular beam aperture there is no variation of the integral of E_z so no focusing!

Why? I would need to study this proof more to be in a position to teach it… But the essential element is that with a beam pipe, the bunch does not cross any conducting charges. This means that there are conservation of flux integrals over volumes inside the beam aperture.

For the $TM_{0,1,0}$ mode the flux lines in the center of the cavity only end on the cavity beam pipe, so you get the same projected and integrated E_z at all radii inside the beam pipe.

The consequence of this is that, for a circular geometry, with a circular beam pipe, you can expand the E_z integral as,

$$\left(\frac{r}{a}\right)^m e^{im\vartheta}$$

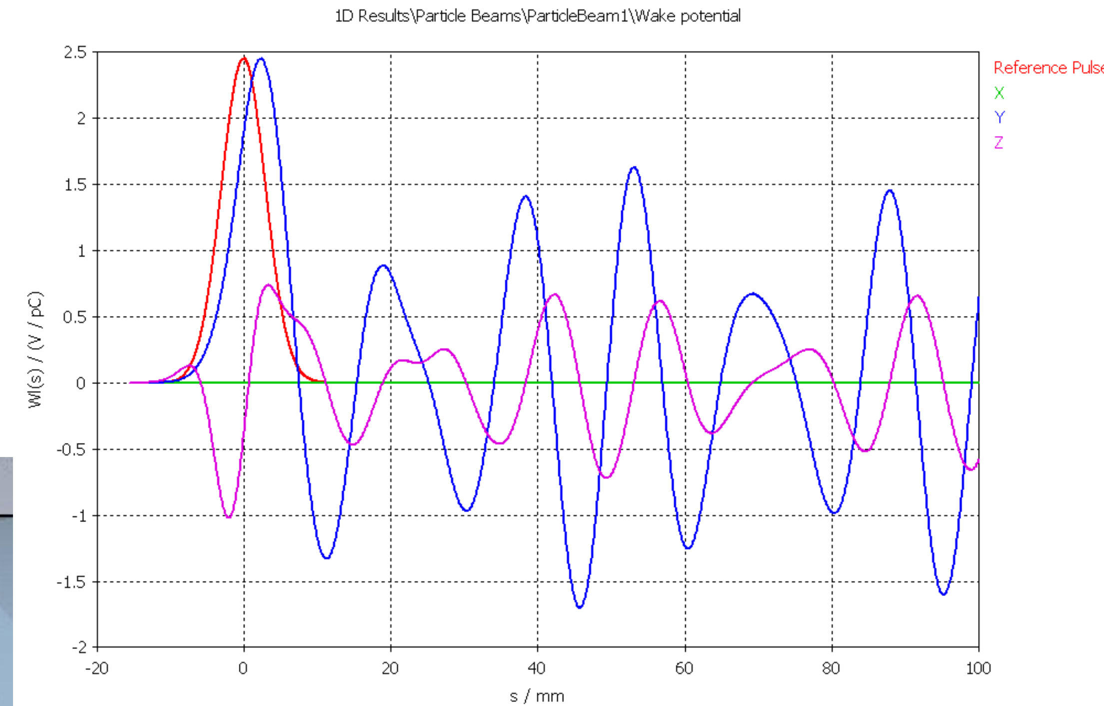
And the consequence of being able to expand the fields in such a way is that for modes with

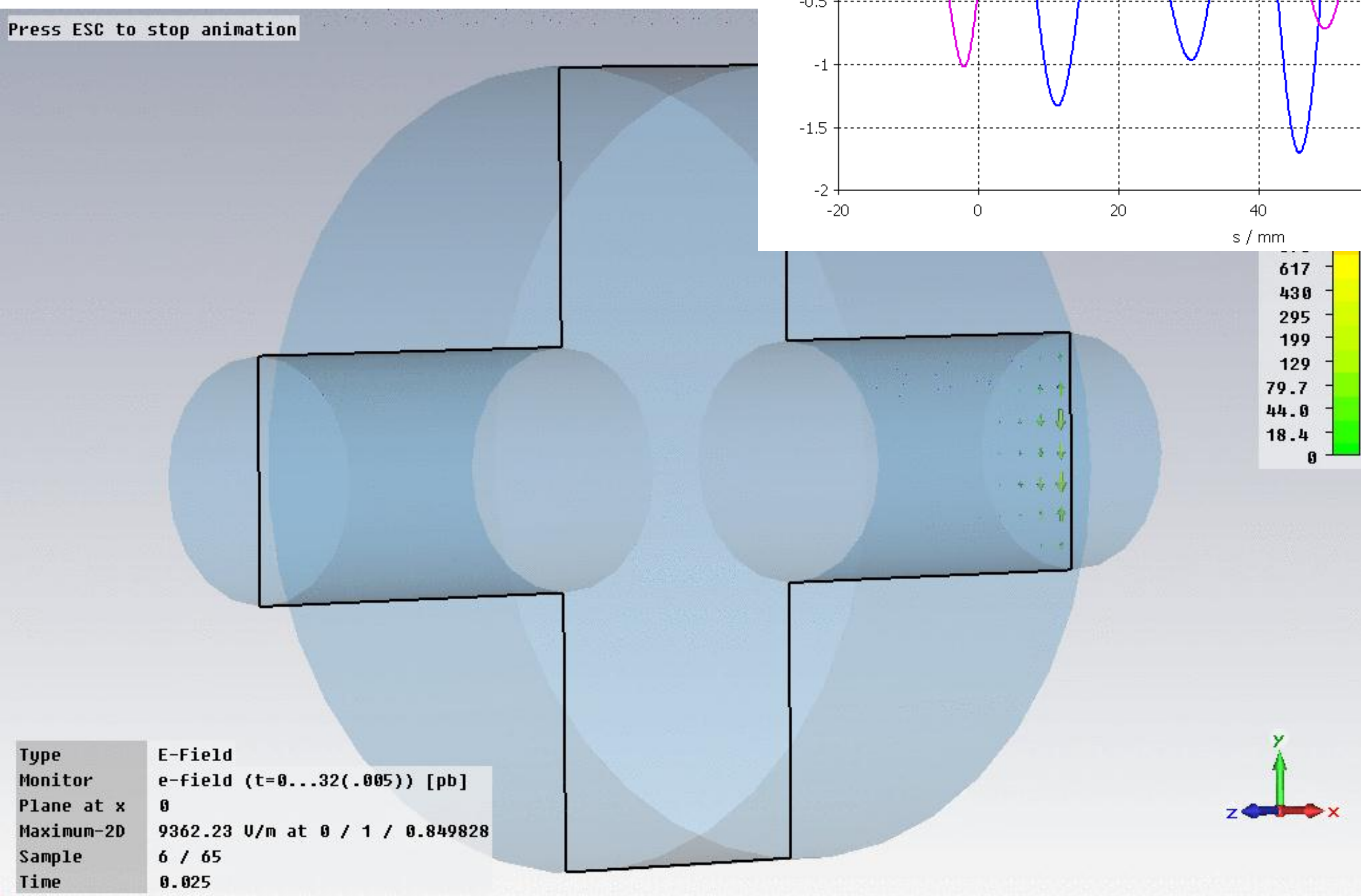
m=0: The E_z integral is constant so no kick.
m=1: The E_z integral varies strictly linearly across the beam aperture
m=2: etc.

On the other hand if the beam aperture, or the cavity, is not circular then you cannot expand the fields like shown above and you get variation in *r* and θ. You can make a rf quadrupole from oval cells or apertures.

Transverse wake potential in pillbox cavity.

3 mm σ beam, 1 mm off-set





Transverse wake

Cavity:

10 mm radius

2 mm thick

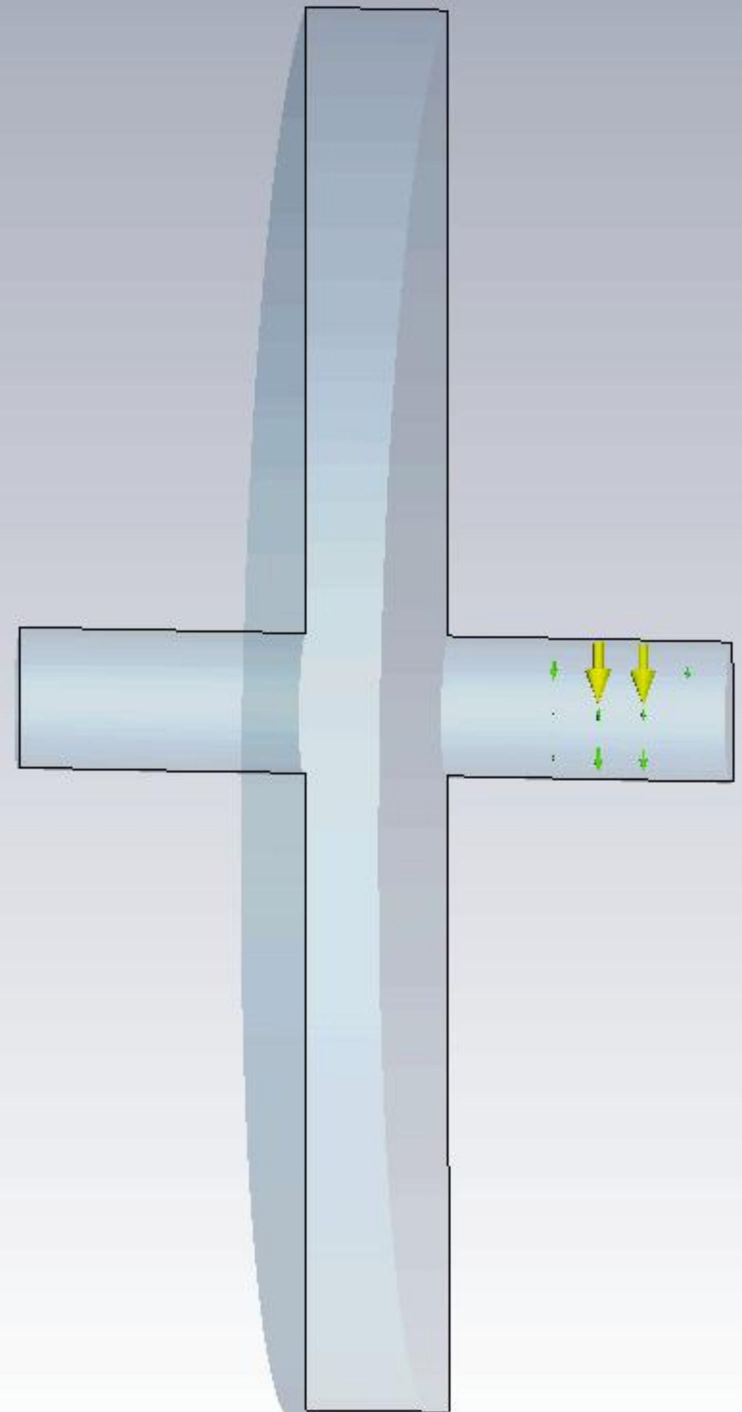
Beam pipe:

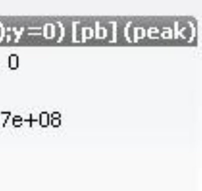
1 mm radius

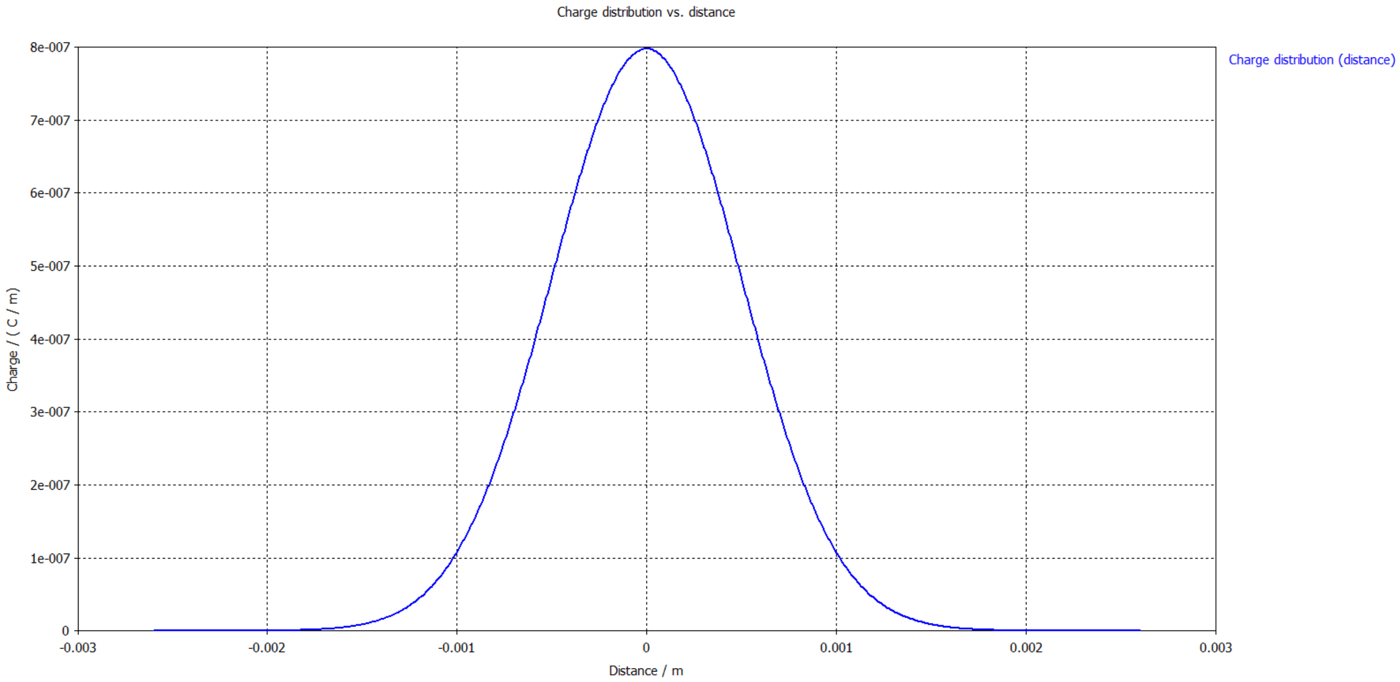
Beam:

σ = 0.5 mm

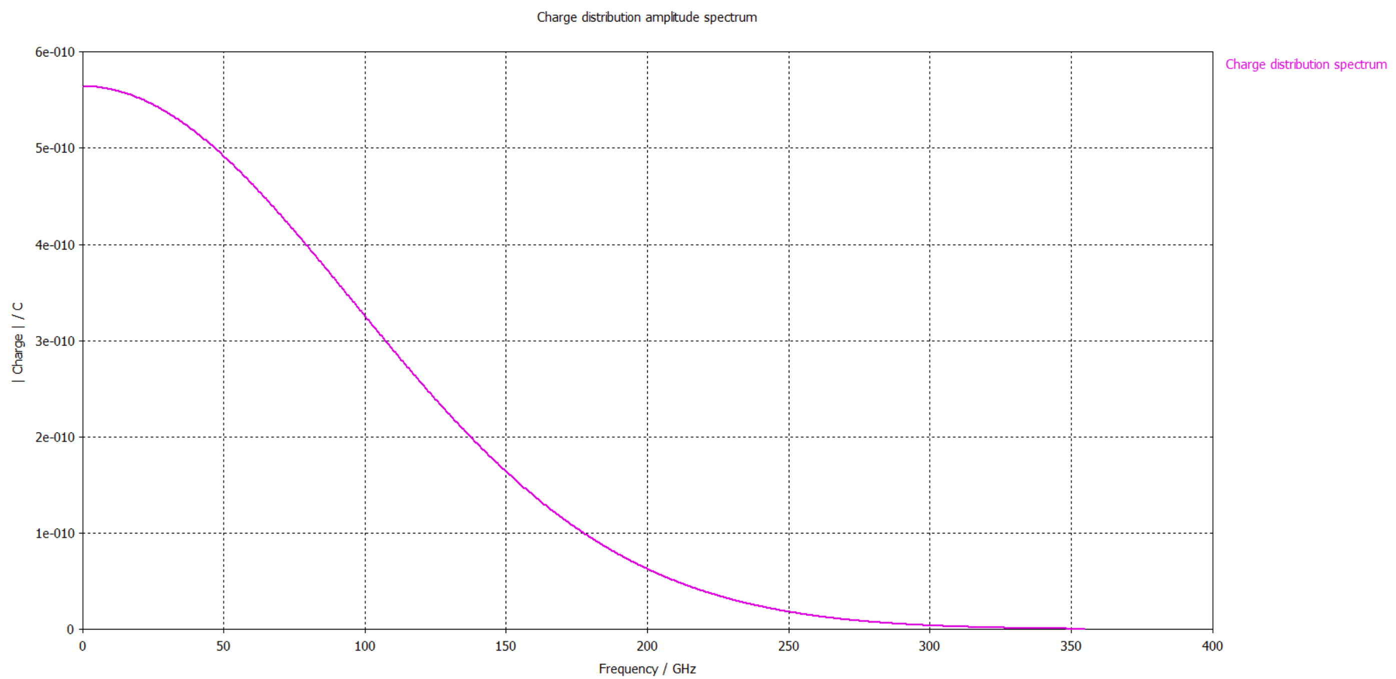
Offset 0.5 mm

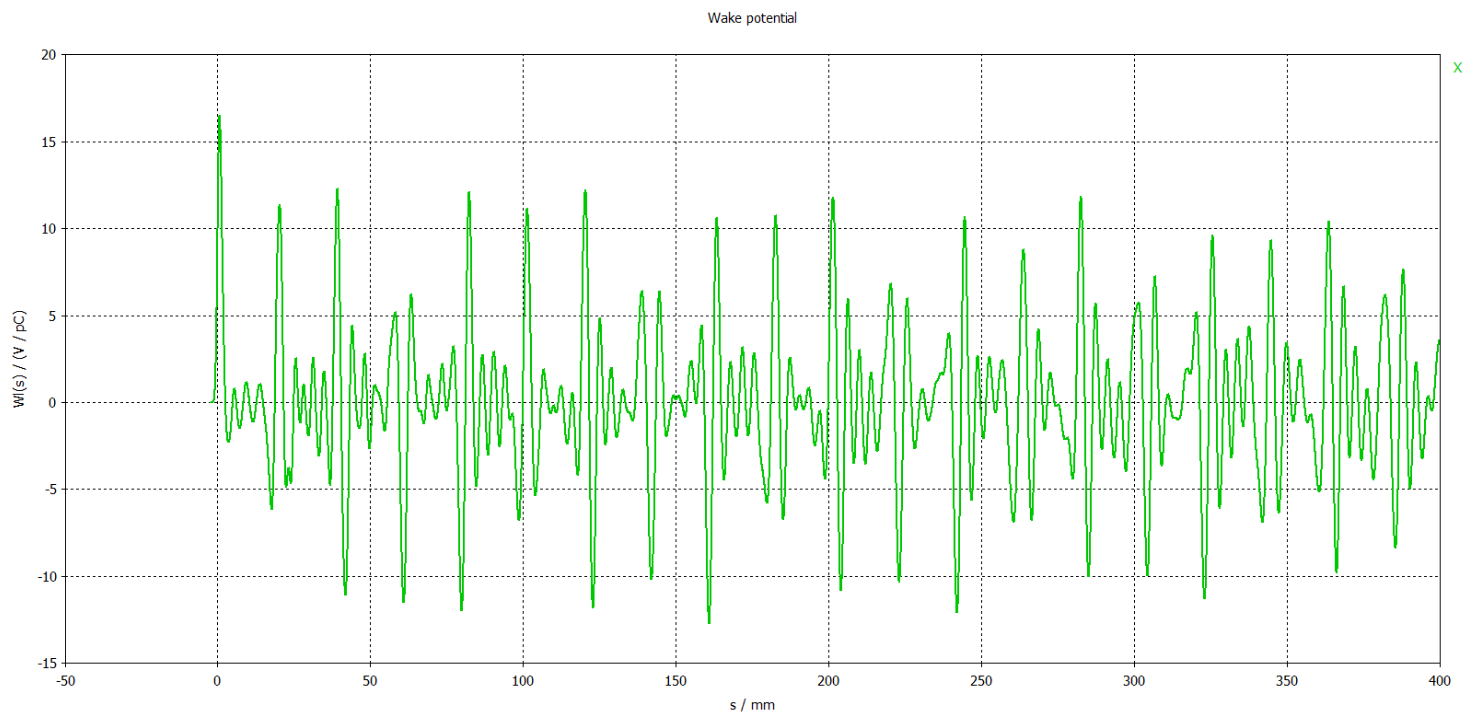


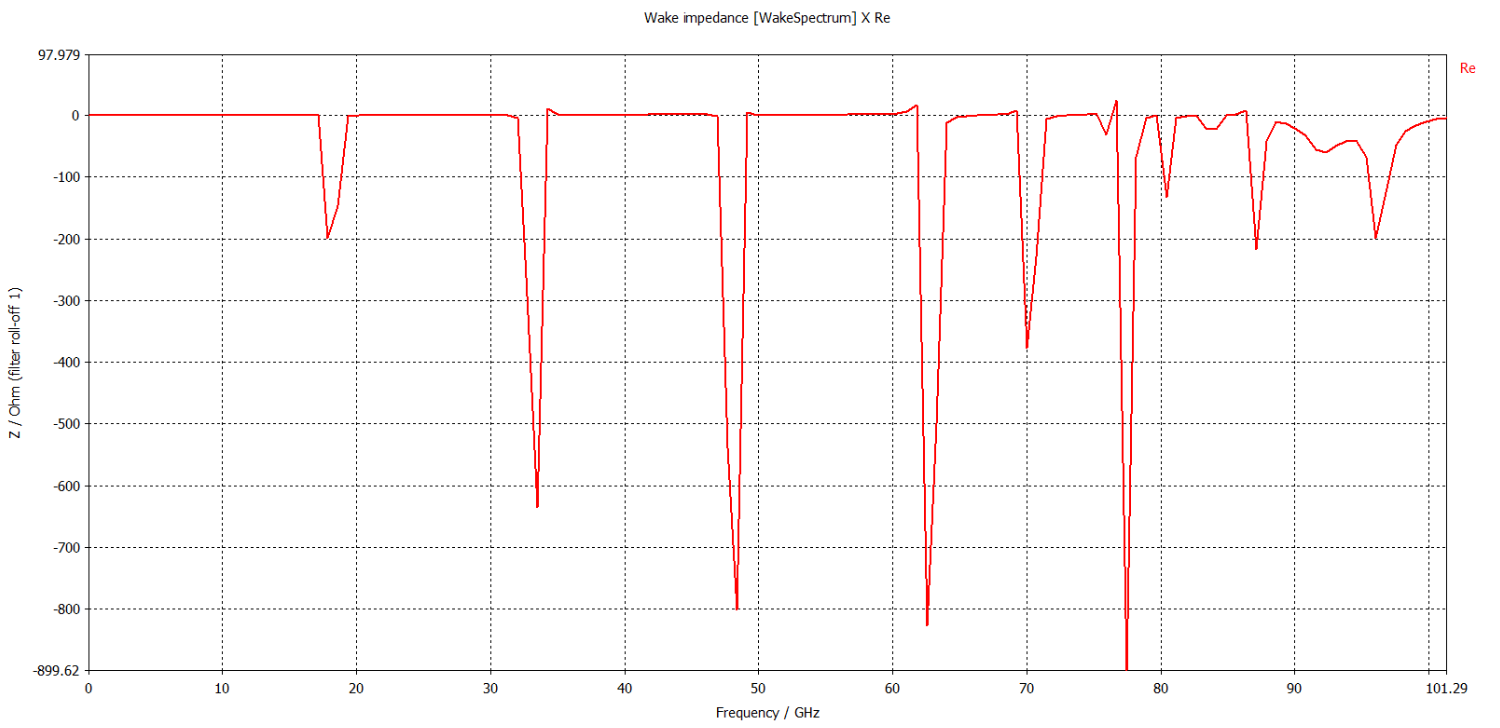


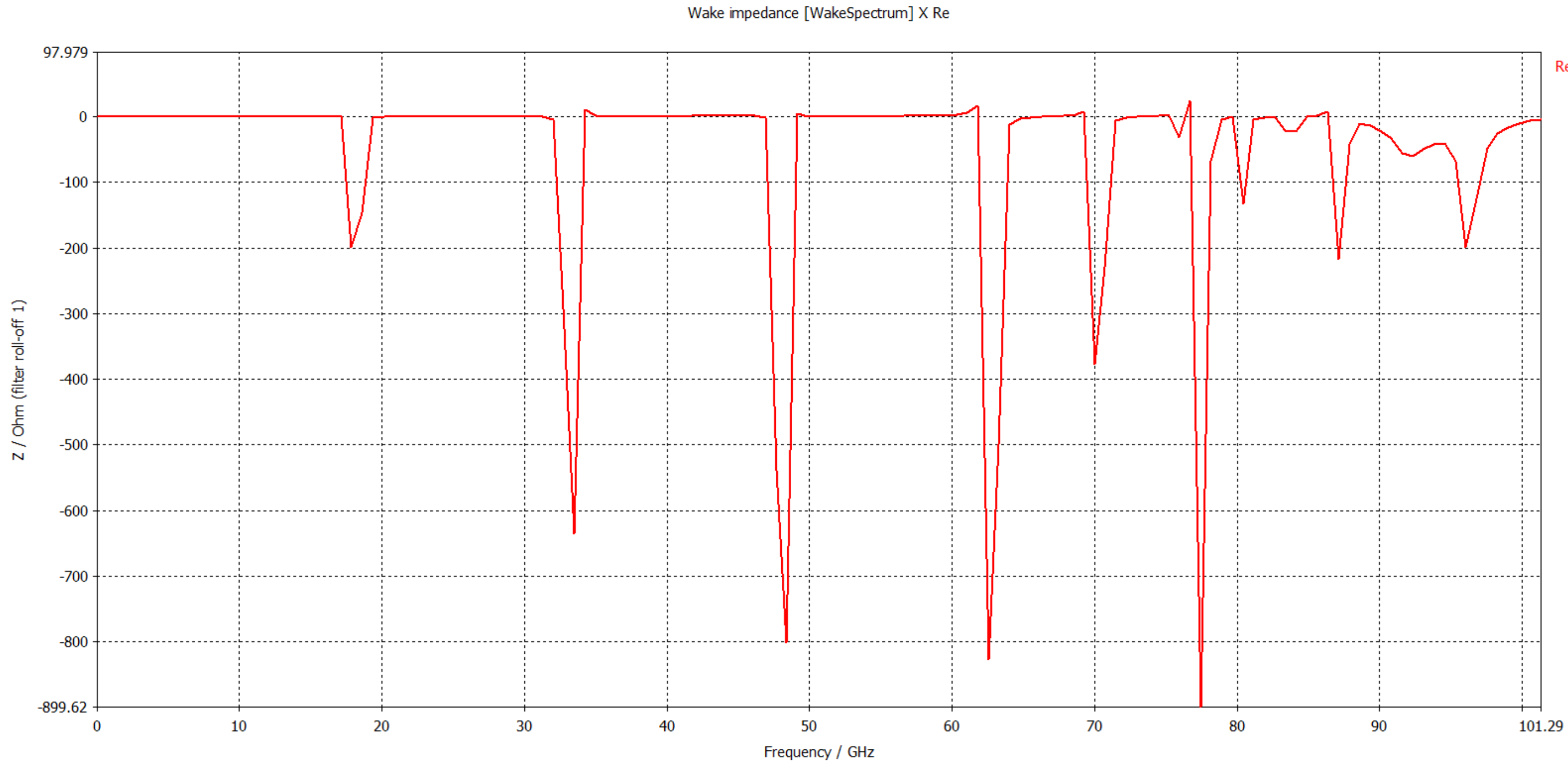

Charge distribution vs. distance
Charge distribution (distance)
Charge / (C / m)
8e-007
7e-007
6e-007
5e-007
4e-007
3e-007
2e-007
1e-007
0
-0.003
-0.002
-0.001
0
0.001
0.002
0.003
Distance / m

$$\sigma = 0.5\,mm$$


Charge distribution amplitude spectrum
Charge distribution spectrum
| Charge | / C
6e-010
5e-010
4e-010
3e-010
2e-010
1e-010
0
0
50
100
150
200
250
300
350
400
Frequency / GHz


Wake potential
X
W(s) / (V / pC)
20
15
10
5
0
-5
-10
-15
-50
0
50
100
150
200
250
300
350
400
s / mm


Wake impedance [WakeSpectrum] X Re
Re
Z / Ohm (filter roll-off 1)
97.979
0
-100
-200
-300
-400
-500
-600
-700
-800
-899.62
0
10
20
30
40
50
60
70
80
90
101.29
Frequency / GHz



$TM_{1,1,0}$ $TM_{1,2,0}$ $TM_{1,3,0}$ $TM_{1,4,0}$ etc. – pillbox does not include beam pipe.

Pillbox cavity

$$f_{n,m,p} = \frac{c}{2\pi}\sqrt{\frac{x_{n,m}^2}{r^2} + \frac{p^2\pi^2}{d^2}}$$

Where for TM modes $x_{n,m}$ is the n^{th} root of $J_m(r)$
r is the cell radius
d is the cell length

The most important special case

- Circularly symmetric structure.
- Small offsets so the transverse wake potential is dominated by dipole, *m*=1, modes.

We can write the transverse wake potential as

$$W_{\perp}(r,s) \approx \left(\frac{r}{a}\right)\sum_{n}\frac{2k_{1n}(a)}{\omega_{1n}a/c}\sin\left(\frac{\omega_{1n}s}{c}\right)$$

a is beam pipe diameter
r is particle path

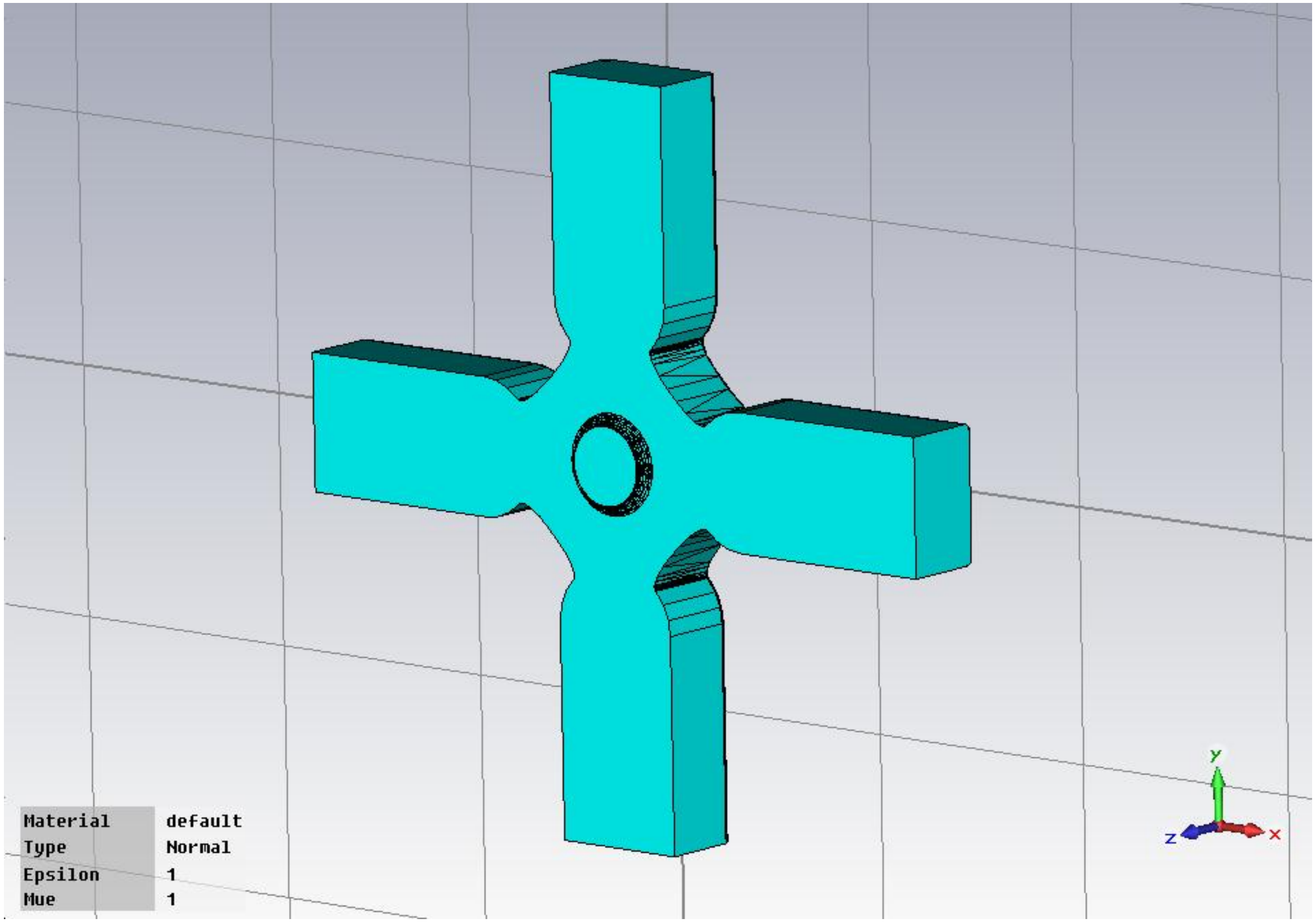
One radial variation due to longitudinal wake.

$$\left[\frac{V}{pCmm}\right]$$

And of course wake for a finite length bunch is the convolution with the delta function wake:

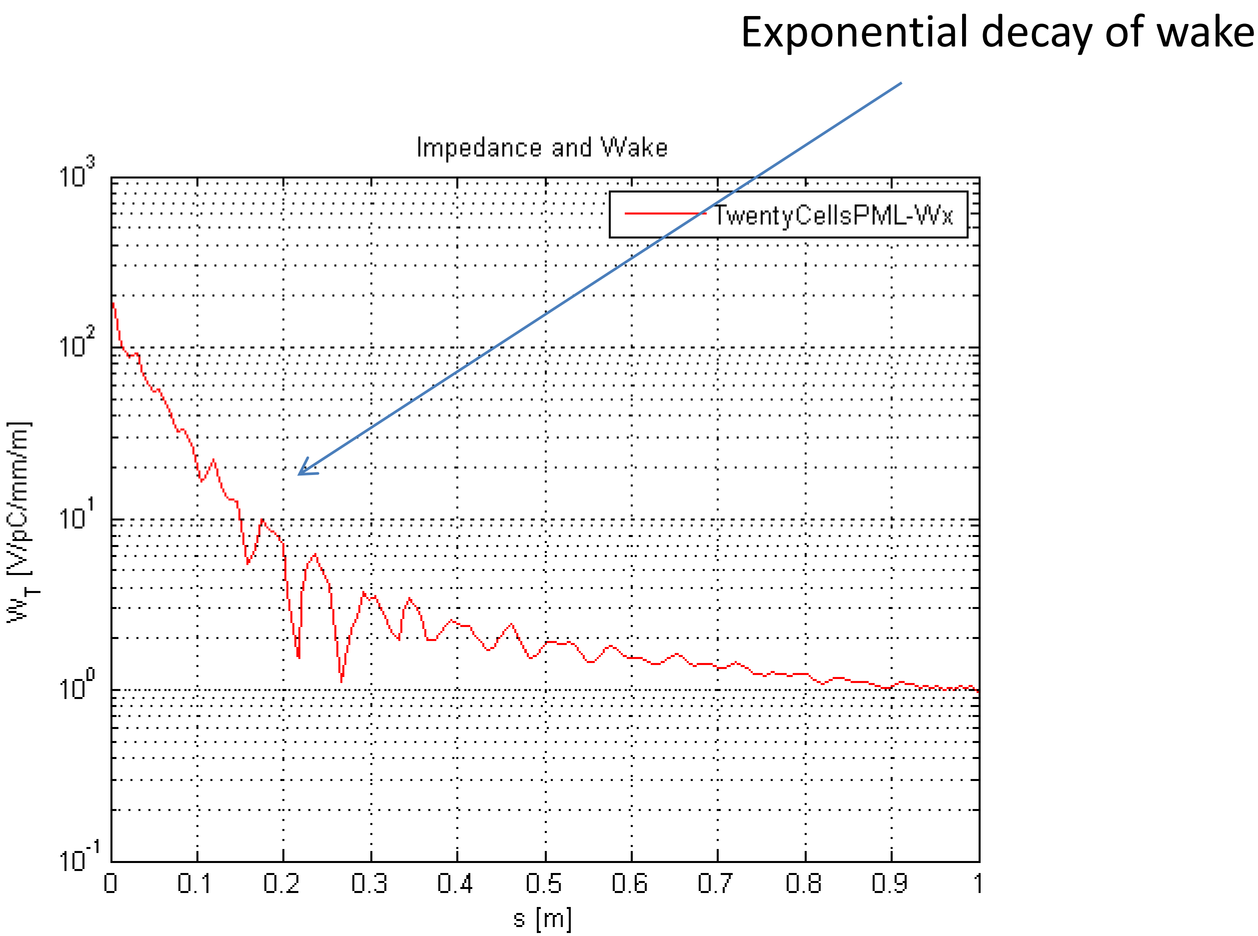
$$W_z(s) = \int_0^{\infty} I(s - s') W_z(s') ds'$$

In the next section we will discuss higher-order-mode suppression and one of the techniques is to lower the Q of dipole modes. The geometry looks typically like this:

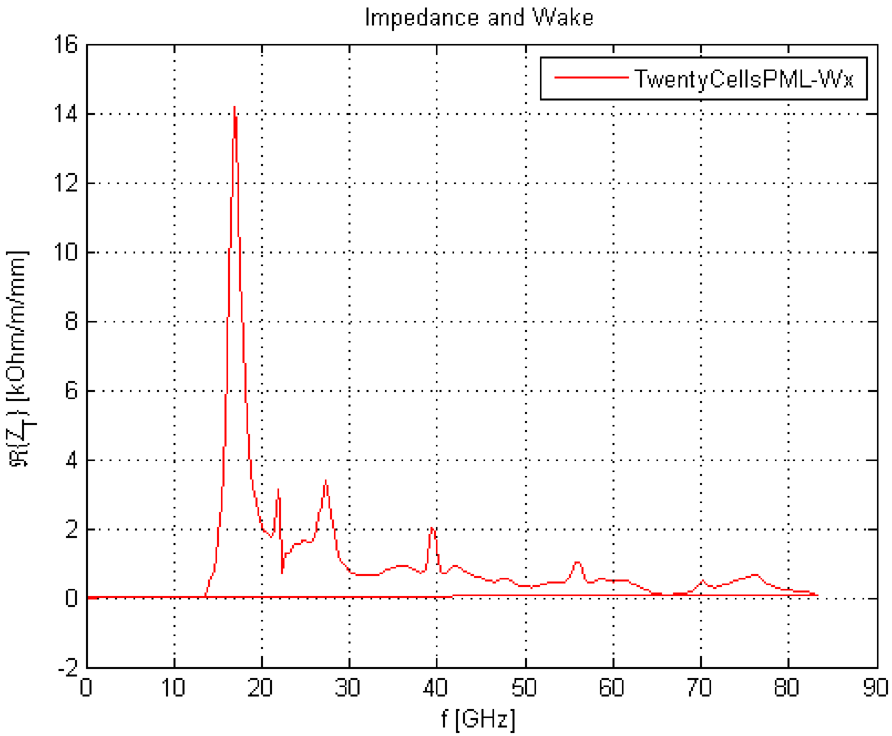


The figures in the next three slides are from Giovanni De Michele.

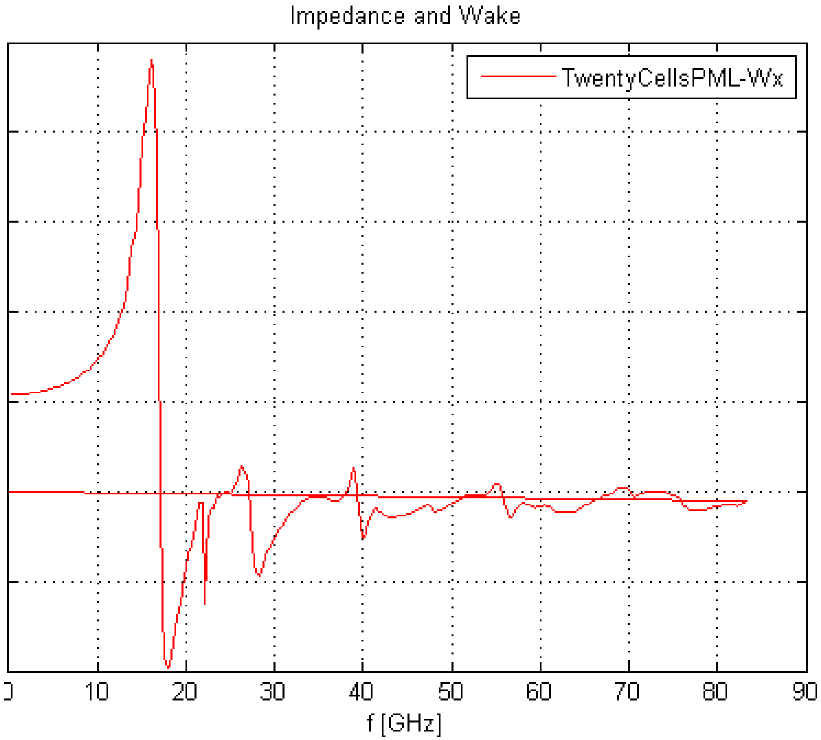
The wakefield from such a structure looks like calculated in the time domain:



In such a case it becomes important to describe the wake as an impedance spectrum, not just as a sum of modes.



real



imaginary