Damping Rings and Ring Colliders

General Linear Beam Dynamics

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Outline

- A3.1 - DR Basics: Introduction to Damping Rings
  - Role of the damping rings in the ILC accelerator complex
  - Review parameters and constraints of CLIC and ILC damping rings
  - Identify key challenges
- A3.2 - DR Basics: General Linear Beam Dynamics
  - Review the basic physics of storage rings including the linear beam dynamics
- A3.3 - LER Design: Radiation Damping and Equilibrium Emittance
- A3.4 - LER Design: Damping Ring Lattices
- A3.5 – DR Technical systems
- A3.6 – Beam Dynamics
- A3.7 – R&D Challenges and Test Facilities
- A3.8 – Circular Colliders

These slides have been presented at the 2010 LC school by Mark Palmer

Storage Ring Basics

Now we will begin our review of storage ring basics. In particular, we will cover:

– Ring Equations of Motion
– Betatron Motion
– Emittance
– Transverse Coupling
– Dispersion and Chromaticity
– Momentum Compaction Factor
– Radiation Damping and Equilibrium Beam Properties

Equations of Motion

Particle motion in electromagnetic fields is governed by the Lorentz force:
$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

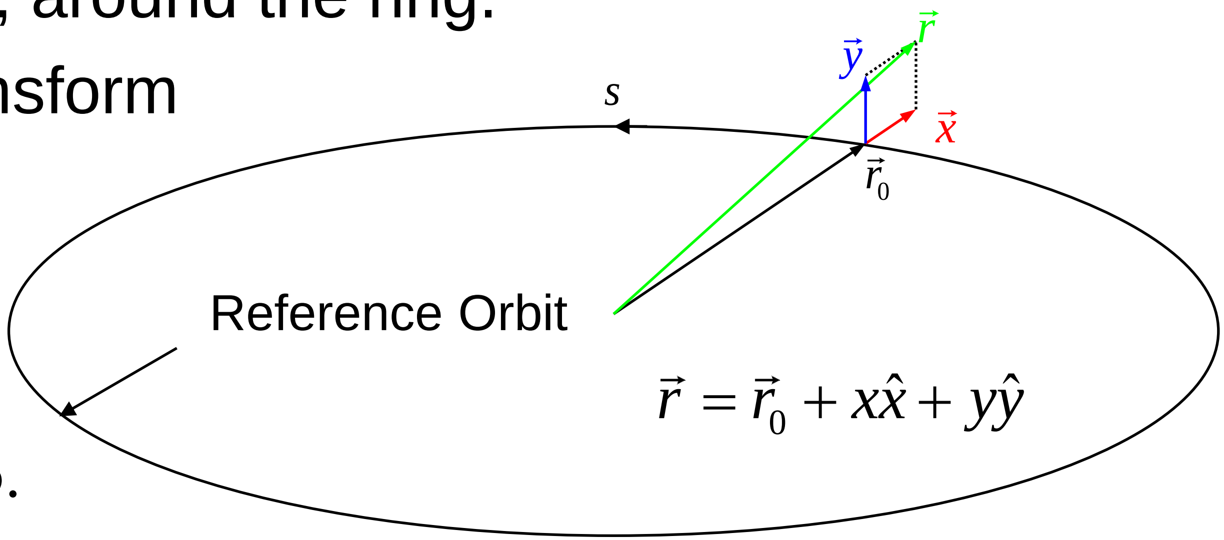
with the corresponding Hamiltonian:
$$\mathcal{H} = c\left[m^2c^2 + \left(\vec{P} - e\vec{A}\right)^2\right]^{1/2} + e\Phi$$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial P_x},\ \dot{P}_x = -\frac{\partial \mathcal{H}}{\partial x},\ldots$$

For circular machines, it is convenient to convert to a curvilinear coordinate system and change the independent variable from time to the location, s-position, around the ring.

In order to do this we transform to the *Frenet-Serret* coordinate system.

The local radius of curvature is denoted by ρ.



Equations of Motion

With a suitable canonical transformation, we can re-write the Hamiltonian as:

$$\tilde{\mathcal{H}} = -\left(1+\frac{x}{\rho}\right)\left[\frac{\left(\mathcal{H} - e\Phi\right)^2}{c^2} - m^2c^2 - \left(p_x - eA_x\right)^2 - \left(p_y - eA_y\right)^2\right]^{1/2} - eA_s$$

Using the relations $E = \mathcal{H} - e\Phi\,, \quad p = \sqrt{\frac{E^2}{c^2} - m^2c^2}$

and expanding to 2nd order in p_x and p_y yields:

$$\boxed{\tilde{\mathcal{H}} \approx -p\left(1+\frac{x}{\rho}\right) + \frac{1+x/\rho}{2p}\left[\left(p_x - eA_x\right)^2 - \left(p_y - eA_y\right)^2\right] - eA_s}$$

which is now periodic in s.

Equations of Motion

Thus, in the absence of synchrotron motion, we can generate the equations of motion with:

$$x' = \frac{\partial \tilde{\mathcal{H}}}{\partial p_x}, \quad p_x' = -\frac{\partial \tilde{\mathcal{H}}}{\partial x}, \quad y' = \frac{\partial \tilde{\mathcal{H}}}{\partial p_y}, \quad p_y' = -\frac{\partial \tilde{\mathcal{H}}}{\partial y}$$

which yields:

$$\boxed{x'' - \frac{\rho + x}{\rho^2} = \pm \frac{B_y}{B\rho} \frac{p_0}{p} \left(1 + \frac{x}{\rho}\right)^2, \quad \text{top / bottom sign for + / - charges}}$$

and

$$\boxed{y'' = \mp \frac{B_x}{B\rho} \frac{p_0}{p} \left(1 + \frac{x}{\rho}\right)^2}$$

Note: $1/B\rho$ is the *beam rigidity* and is taken to be positive

Specific field configurations are applied in an accelerator to achieve the desired manipulation of the particle beams. Thus, before going further, it is useful to look at the types of fields of interest via the multipole expansion of the transverse field components.

Magnetic Field Multipole Expansion

Magnetic elements with 2-dimensional fields of the form

$$\vec{B} = B_x(x,y)\,\hat{x} + B_y(x,y)\,\hat{y}$$

can be expanded in a complex multipole expansion:

$$B_y(x,y) + iB_x(x,y) = B_0 \sum_{n=0}^{\infty} (b_n + ia_n)(x+iy)^n$$

with $b_n = \frac{1}{n!B_0} \left.\frac{\partial^n B_y}{\partial x^n}\right|_{(x,y)=(0,0)}$ and $a_n = \frac{1}{n!B_0} \left.\frac{\partial^n B_x}{\partial x^n}\right|_{(x,y)=(0,0)}$

In this form, we can normalize to the main guide field strength, -$B\hat{y}$, by setting b_0=1 to yield:

$$\boxed{\frac{1}{B\rho}\left(B_y + iB_x\right) = \frac{e}{p_0}\left(B_y + iB_x\right) = \mp\frac{1}{\rho}\sum_{n=0}^{\infty}(b_n + ia_n)(x+iy)^n \quad \text{for } \pm q}$$

Multipole Moments

Upright Fields

Dipole:

$$\frac{e}{p_0}B_x = 0 \qquad \frac{e}{p_0}B_y = \kappa_x$$

Quadrupole:

$$\frac{e}{p_0}B_x = ky \qquad \frac{e}{p_0}B_y = kx$$

Sextupole:

$$\frac{e}{p_0}B_x = mxy \qquad \frac{e}{p_0}B_y = \frac{1}{2}m\left(x^2 - y^2\right)$$

Octupole:

$$\frac{e}{p_0}B_x = \frac{1}{6}r\left(3x^2y - y^3\right)$$

$$\frac{e}{p_0}B_y = \frac{1}{6}r\left(x^3 - 3xy^2\right)$$

Skew Fields

Dipole ($\theta = 90°$):

$$\frac{e}{p_0}B_x = -\kappa_y \qquad \frac{e}{p_0}B_y = 0$$

Quadrupole ($\theta = 45°$):

$$\frac{e}{p_0}B_x = -k_{skew}x \qquad \frac{e}{p_0}B_y = k_{skew}y$$

Sextupole ($\theta = 30°$):

$$\frac{e}{p_0}B_x = -\frac{1}{2}m_{skew}\left(x^2 - y^2\right)$$

$$\frac{e}{p_0}B_y = m_{skew}xy$$

Octupole ($\theta = 22.5°$):

$$\frac{e}{p_0}B_x = -\frac{1}{6}r_{skew}\left(x^3 - 3xy^2\right)$$

$$\frac{e}{p_0}B_y = \frac{1}{6}r_{skew}\left(3x^2y - y^3\right)$$

Equations of Motion (Hill's Equation)

We next want to consider the equations of motion for a ring with only guide (dipole) and focusing (quadrupole) elements:

$$B_y = \mp B_0 + \frac{p_0}{e} kx = B_0\left(\rho kx \mp 1\right) \quad \text{and} \quad B_x = \frac{p_0}{e} ky = B_0 \rho kx$$

Taking $p=p_0$ and expanding the equations of motion to first order in x/ρ and y/ρ gives:

$$\boxed{\begin{aligned} &x'' + K_x(s)\,x = 0, \qquad && K_x(s) = \frac{1}{\rho^2(s)} \mp k(s) \\ &y'' + K_y(s)\,y = 0, \qquad && K_y(s) = \pm k(s) \end{aligned}}$$

also commonly denoted as k_1

where the upper/low signs are for a positively/negatively charged particle.

The focusing functions are periodic in s: $\boxed{K_{x,y}(s+L) = K_{x,y}(s)}$

Solutions to Hill's Equation

Some introductory comments about the solutions to Hill's equations:

- The solutions to Hill's equation describe the particle motion around a reference orbit, the *closed orbit*. This motion is known as betatron motion. We are generally interested in small amplitude motions around the closed orbit (as has already been assumed in the derivation of the preceding pages).
- Accelerators are generally designed with discrete components which have locally uniform magnetic fields. In other words, the focusing functions, $K(s)$, can typically be represented in a piecewise constant manner. This allows us to locally solve for the characteristics of the motion and implement the solution in terms of a *transfer matrix*. For each segment for which we have a solution, we can then take a particle's initial conditions at the entrance to the segment and transform it to the final conditions at the exit.

Solutions to Hill's Equation

Let's begin by considering constant $K=k$:

$$x'' + kx = 0$$

where x now represents either x or y. The 3 solutions are:

$$x(s) = a\sin\left(\sqrt{k}s\right) + b\cos\left(\sqrt{k}s\right), \qquad k > 0 \qquad \text{Focusing Quadrupole}$$

$$x(s) = as + b, \qquad k = 0 \qquad \text{Drift Region}$$

$$x(s) = a\sinh\left(\sqrt{|k|}s\right) + b\cosh\left(\sqrt{|k|}s\right), \qquad k < 0 \qquad \text{Defocusing Quadrupole}$$

For each of these cases, we can solve for initial conditions and recast in 2×2 matrix form:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\vec{x} = \mathbf{M}\left(s \middle| s_0\right)\vec{x}_0$$

Transfer Matrices

We can now re-write the solutions of the preceding page in transfer matrix form:

$$\mathbf{M}\left(s|s_0\right) = \begin{cases} \begin{pmatrix} \cos\left(\sqrt{k}\ell\right) & \dfrac{1}{\sqrt{k}}\sin\left(\sqrt{k}\ell\right) \\ -\sqrt{k}\sin\left(\sqrt{k}\ell\right) & \cos\left(\sqrt{k}\ell\right) \end{pmatrix} & \text{Focusing Quadrupole} \\ \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} & \text{Drift Region} \\ \begin{pmatrix} \cosh\left(\sqrt{|k|}\ell\right) & \dfrac{1}{\sqrt{|k|}}\sinh\left(\sqrt{|k|}\ell\right) \\ \sqrt{|k|}\sinh\left(\sqrt{|k|}\ell\right) & \cosh\left(\sqrt{|k|}\ell\right) \end{pmatrix} & \text{Defocusing Quadrupole} \end{cases}$$

where $\ell = s - s_0$.

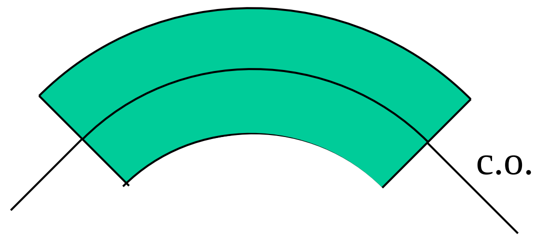
Transfer Matrices

Examples:

- Thin lens approximation: $\ell \to 0, \qquad f = \lim_{\ell \to 0} \frac{1}{|K|\,\ell}$

$$\mathbf{M}_{\text{focusing}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \qquad \mathbf{M}_{\text{defocusing}} = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

- Sector dipole (entrance and exit faces ⊥ to closed orbit):



$$\mathbf{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix} \approx \begin{pmatrix} 1 & \ell \\ -\frac{\ell}{\rho^2} & 1 \end{pmatrix}$$

where $\theta = \frac{\ell}{\rho}$

Transfer Matrices

Transport through an interval $s_0 \rightarrow s_2$ can be written as the product of 2 transport matrices for the intervals $s_0 \rightarrow s_1$ and $s_1 \rightarrow s_2$:

$$\mathbf{M}\left(s_2 \middle| s_0\right) = \mathbf{M}\left(s_2 \middle| s_1\right)\mathbf{M}\left(s_1 \middle| s_0\right)$$

and the determinant of each transfer matrix is: $\left|\mathbf{M}_i\right| = 1$

Many rings are composed of repeated sets of identical magnetic elements. In this case it is particularly straightforward to write the *one-turn matrix* for P superperiods, each of length L, as:

$$\mathbf{M}_{ring} = \left[\mathbf{M}\left(s+L \middle| s\right)\right]^P$$

with the boundary condition that: $\mathbf{M}\left(s+L \middle| s\right) = \mathbf{M}(s)$

The multi-turn matrix for m revolutions is then: $\left[\mathbf{M}(s)\right]^{mP}$

Twiss Parameters

The generalized one turn matrix can be written as:

$$\boxed{\mathbf{M}=\begin{pmatrix}\cos\Phi+\alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi-\alpha\sin\Phi\end{pmatrix}=\mathbf{I}\cos\Phi+\mathbf{J}\sin\Phi}$$

Identity matrix

This is the most general form of the matrix. α, β, and γ are known as either the Courant-Snyder or Twiss parameters (note: they have nothing to do with the familiar relativistic parameters) and Φ is the betatron phase advance. The matrix **J** has the properties:

$$\mathbf{J}=\begin{pmatrix}\alpha & \beta \\ -\gamma & -\alpha\end{pmatrix}, \qquad \mathbf{J}^2=-\mathbf{I}\Leftrightarrow\boxed{\beta\gamma=1+\alpha^2}$$

The n-turn matrix can be expressed as: $\mathbf{M}^n=\mathbf{I}\cos(n\Phi)+\mathbf{J}\sin(n\Phi)$

which leads to the stability requirement for betatron motion:

$$\boxed{\left|\mathrm{Trace}(\mathbf{M})\right|=2\cos\Phi\le 2}$$

The Envelope Equations

We will look for 2 independent solutions to Hill's Equation of the form:

$$x(s) = aw(s)e^{i\psi(s)} \quad \text{and} \quad x^*(s) = aw(s)e^{-i\psi(s)}$$

Then w and ψ satisfy:

$$\boxed{\begin{aligned} & w'' + Kw - \frac{1}{w^3} = 0 \\ & \psi' = \frac{1}{w^2} \end{aligned}}$$

Betatron envelope

and

phase equations

Since any solution can be written as a superposition of the above solutions, we can write [with $w_i = w(s_i)$]:

$$\mathbf{M}(s_2|s_1) = \begin{pmatrix} \dfrac{w_2}{w_1}\cos\psi - w_2 w_1' \sin\psi & w_1 w_2 \sin\psi \\ -\dfrac{(1 + w_1 w_1' w_2 w_2')}{w_1 w_2}\sin\psi - \left(\dfrac{w_1'}{w_2} - \dfrac{w_2'}{w_1}\right)\cos\psi & \dfrac{w_1}{w_2}\cos\psi + w_1 w_2' \sin\psi \end{pmatrix}$$

The Envelope Equations

Application of the previous transfer matrix to a full turn and direct comparison with the Courant-Snyder form yields:

$$w^2 = \beta$$

$$\alpha = -ww' = -\frac{\beta'}{2}$$

the betatron envelope equation becomes

$$\boxed{\frac{1}{2}\beta'' + K\beta - \frac{1}{\beta}\left[1 + \frac{\beta'^2}{4}\right] = 0}$$

and the transfer matrix in terms of the Twiss parameters can immediately be written as:

$$\boxed{\mathbf{M}(s_2|s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\Delta\psi + \alpha_1 \sin\Delta\psi\right) & \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\Delta\psi + \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1\beta_2}}\cos\Delta\psi & \sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\Delta\psi - \alpha_2 \sin\Delta\psi\right) \end{pmatrix}}$$

General Solution to Hill's Equation

The general solution to Hill's equation can now be written as:

$$\boxed{x(s) = A\sqrt{\beta_x(s)}\cos\left[\psi_x(s) + \phi_0\right] \quad \text{where} \quad \psi_x(s) = \int_0^s \frac{ds}{\beta_x(s)}}$$

We can now define the *betatron tune* for a ring as:

$$\boxed{Q_x = \nu_x = \frac{\Phi_{turn}}{2\pi} = \frac{1}{2\pi}\int_s^{s+C} \frac{ds}{\beta_x(s)}} \quad \text{where} \quad C = \text{ring circumference}$$

If we make the coordinate transformation:

$$z = \frac{x}{\sqrt{\beta_x}} \quad \text{and} \quad \xi(s) = \frac{1}{\nu_x}\int_0^s \frac{ds}{\beta_x(s)}$$

we see that particles in the beam satisfy the equation for simple harmonic motion:

$$\frac{d^2 z}{d\xi^2} + \nu_x^2 z = 0$$

The Courant-Snyder Invariant

With K real, Hill's equation is conservative. We can now take

$$x(s) = A\sqrt{\beta_x(s)}\cos\left[\psi_x(s)+\phi_0\right]$$ and

$$x'(s) = -\frac{A}{\sqrt{\beta_x(s)}}\left\{\alpha(s)\cos\left[\psi_x(s)+\phi_0\right]+\sin\left[\psi_x(s)+\phi_0\right]\right\}$$

After some manipulation, we can combine these two equations to give:

Conserved quantity

$$A^2 = \varepsilon = \frac{x^2}{\beta_x(s)} + \left[\frac{\alpha_x(s)}{\sqrt{\beta_x(s)}}x + \sqrt{\beta_x(s)}x'\right]^2$$

Recalling that $\beta\gamma = 1+\alpha^2$ yields:

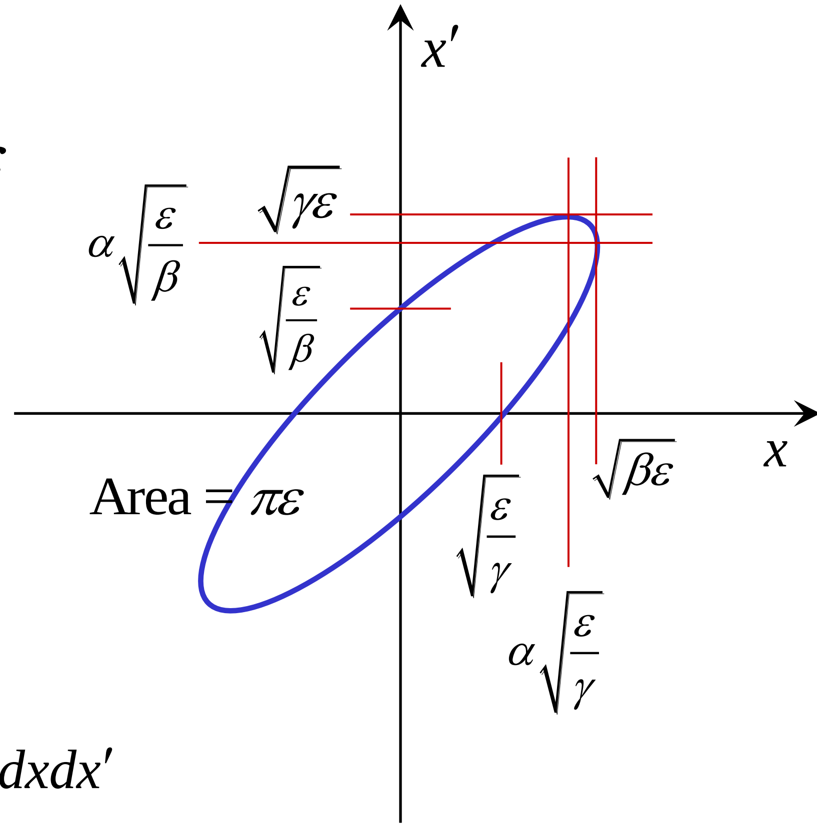
$$\boxed{A^2 = \varepsilon = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s)}$$

Emittance

The equation

$$\gamma(s)x^2(s)+2\alpha(s)x(s)x'(s)+\beta(s)x'^2(s)=\varepsilon$$

describes an ellipse with area $\pi\varepsilon$.



For an ensemble of particles, each following its own ellipse, we can define the moments of the beam as:

$$\langle x\rangle=\int x\rho(x,x')dxdx' \qquad \langle x'\rangle=\int x'\rho(x,x')dxdx'$$

$$\sigma_x^2=\int\left(x-\langle x\rangle\right)^2\rho(x,x')\,dxdx' \qquad \sigma_{x'}^2=\int\left(x'-\langle x'\rangle\right)^2\rho(x,x')\,dxdx'$$

$$\sigma_{xx'}^2=\int\left(x-\langle x\rangle\right)\left(x'-\langle x'\rangle\right)\rho(x,x')\,dxdx'=r\sigma_x\sigma_{x'}$$

The rms emittance of the beam is then which is the area enclosed by the ellipse of an *rms particle*.

$$\varepsilon_{rms}=\sqrt{\sigma_x^2\sigma_{x'}^2-\sigma_{xx'}^2}=\frac{\langle A^2\rangle}{2}$$

Coupling

Up to this point, the equations of motion that we have considered have been independent in x and y. An important issue for all accelerators, and particularly for damping rings which attempt to achieve a very small vertical emittance, is coupling between the two planes. For the damping ring, we are primarily interested in the coupling that arises due to small rotations of the quadrupoles. This introduces a *skew quadrupole* component to the equations of motion.

$$\boxed{\begin{aligned} x'' + K_x(s)x = 0 \quad &\Rightarrow \quad x'' + K_x(s)x + k_{skew}y = 0 \\ y'' + K_y(s)y = 0 \quad &\Rightarrow \quad y'' + K_y(s)y + k_{skew}x = 0 \end{aligned}}$$

Another skew quadrupole term arises from "feed-down" when the closed orbit is displaced vertically in a sextupole magnet. In this case the effective skew quadrupole moment is given by the product of the sextupole strength and the closed orbit offset

$$k_{skew} = my_{co}$$

Coupling

For uncoupled motion, we can convert the 2D (x,x') and (y,y') transfer matrices to 4D form for the vector (x,x',y,y'):

$$\mathbf{M}_{4\mathrm{D}}\left(s\middle|s_0\right)=\begin{pmatrix}\mathbf{M}_{\mathrm{focusing}} & 0\\ 0 & \mathbf{M}_{\mathrm{defocusing}}\end{pmatrix}=\begin{pmatrix}\mathbf{M}_{\mathrm{F}} & 0\\ 0 & \mathbf{M}_{\mathrm{D}}\end{pmatrix}$$

where we have arbitrarily chosen this case to be focusing in x. The matrix is block diagonal and there is no coupling between the two planes. If the quadrupole is rotated by angle θ, the transfer matrix becomes:

$$\mathbf{M}_{\mathrm{skew}}=\begin{pmatrix}\mathbf{M}_{\mathrm{F}}\cos^2\theta+\mathbf{M}_{\mathrm{D}}\sin^2\theta & \sin\theta\cos\theta\left(\mathbf{M}_{\mathrm{D}}-\mathbf{M}_{\mathrm{F}}\right)\\ \sin\theta\cos\theta\left(\mathbf{M}_{\mathrm{D}}-\mathbf{M}_{\mathrm{F}}\right) & \mathbf{M}_{\mathrm{D}}\cos^2\theta+\mathbf{M}_{\mathrm{F}}\sin^2\theta\end{pmatrix}$$

and motion in the two planes is coupled.

Coupling and Emittance

Later in this lecture series we will look in greater detail at the sources of vertical emittance for the damping rings.

In the absence of coupling and ring errors, the vertical emittance of a ring is determined by the the radiation of photons and the fact that emitted photons are randomly radiated into a characteristic cone with half-angle $\theta_{1/2}\sim 1/\gamma$. This quantum limit to the vertical emittance is generally quite small and can be ignored for presently operating storage rings.

Thus the presence of betatron coupling becomes one of the primary sources of vertical emittance in a storage ring.

Dispersion

In our initial derivation of Hill's equation, we assumed that the particles being guided had the design momentum, p_0, thus ignoring longitudinal contributions to the motion. We now want to address off-energy particles. Thus we take the equation of motion:

$$x'' - \frac{\rho + x}{\rho^2} = \pm \frac{B_y}{B\rho} \frac{p_0}{p} \left(1 + \frac{x}{\rho}\right)^2$$

and expand to lowest order in $\delta = \frac{\Delta p}{p_0}$ and $\frac{x}{\rho}$ which yields:

$$\boxed{x'' + K(s)\,x = \frac{\delta}{\rho}}$$

We have already obtained a homogenous solution, $x_\beta(s)$. If we denote the particular solution as $D(s)\delta$, the general solution is:

$$x = x_\beta(s) + D(s)\,\delta$$

Dispersion Function and Momentum Compaction

The dispersion function satisfies: $\boxed{D'' + K(s)D = 1/\rho}$

with the boundary conditions: $D(s+L) = D(s);\ D'(s+L) = D'(s)$

The solution can be written as the sum of the solution to the homogenous equation and a particular solution:

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = \mathbf{M}(s_2|s_1)\begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix}$$

which can be expressed in a 3×3 matrix form as:

$$\boxed{\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{M}(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}, \quad \textbf{where} \quad \bar{d} = \begin{pmatrix} d \\ d' \end{pmatrix}}$$

Momentum Compaction

We can now consider the difference in path length experienced by such an off-momentum particle as it traverses the ring. The path length of an on-momentum particle is given by:

$$C = \oint \frac{x_{c.o.}}{\rho} ds$$

For the off-momentum case, we then have:

$$\Delta C = \delta \times \oint \frac{D(s)}{\rho} ds = I_1 \delta$$

I_1 is the first *radiation integral*.

The momentum compaction factor, α_c, is defined as:

$$\boxed{\alpha_c = \frac{\Delta C / C}{\delta} = \frac{I_1}{C}}$$

The Synchrotron Radiation Integrals

I_1 is the first of 5 "radiation integrals" that we will study in this lecture. These 5 integrals describe the key properties of a storage ring lattice including:

- Momentum compaction
- Average power radiated by a particle on each revolution
- The radiation excitation and average energy spread of the beam
- The *damping partition numbers* describing how radiation damping is distributed among longitudinal and transverse modes of oscillation
- The natural emittance of the lattice

In later sections of this lecture we will work through the key aspects of radiation damping in a storage ring

Chromaticity

An off-momentum particle passing through a quadrupole will be under/over-focused for positive/negative momentum deviation. This is chromatic aberration. Hill's equation becomes:

$$x'' + \left[K_0(s)(1-\delta) \right] x = 0$$

We will evaluate the chromaticity by first looking at the impact of local gradient errors on the particle beam dynamics.

Effect of a Gradient Error

We consider a local perturbation of the focusing strength $K = K_0 + \Delta K$. The effect of ΔK can be represented by including a thin lens transfer matrix in the one-turn matrix. Thus we have

$$\mathbf{M}_{\Delta K} = \begin{pmatrix} 1 & 0 \\ -\Delta K \ell & 1 \end{pmatrix}$$

and

$$\mathbf{M}_{1-turn} = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\Phi_0 + \alpha\sin\Phi_0 & \beta\sin\Phi_0 \\ -\gamma\sin\Phi_0 & \cos\Phi_0 - \alpha\sin\Phi_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Delta K \ell & 1 \end{pmatrix}$$

With $\Phi = \Phi_0 + \Delta\Phi$, we can take the trace of the one-turn matrix to give:

$$\cos(\Phi_0 + \Delta\Phi) = \cos\Phi_0 - \frac{1}{2}\beta\Delta K\ell\sin\Phi_0$$

Effect of a Gradient Error

Using the relation: $\cos\left(\Phi_0 + \Delta\Phi\right) = \cos\Delta\Phi\cos\Phi_0 - \sin\Delta\Phi\sin\Phi_0$

we can identify: $\Delta\Phi \approx \frac{1}{2}\beta\Delta K\ell$

Thus we can write: $\Delta Q = \frac{1}{4\pi}\beta\Delta K\ell$

and we see that the result of gradient errors is a shift in the betatron tune. For a distributed set of errors, we then have:

$$\boxed{\Delta Q = \frac{1}{4\pi}\oint\beta\Delta K ds}$$

which is the result we need for evaluating chromatic aberrations. *Note that the tune shift will be positive/negative for a focusing/defocusing quadrupole.*

Chromaticity

We can now write the betatron tune shift due to chromatic aberration as:

$$\boxed{\Delta Q = \frac{1}{4\pi}\oint \beta \Delta K ds \approx -\frac{\delta}{4\pi}\oint \beta K ds}$$

The chromaticity is defined as the change in tune with respect to the momentum deviation:

$$C = \frac{\partial Q}{\partial \delta}$$

Because the focusing is weaker for a higher momentum particle, the natural chromaticity due to quadrupoles is always negative. This can be a source of instabilities in an accelerator. However, the fact that a momentum deviation results in a change in trajectory (the dispersion) as well as the change in focusing strength, provides a route to mitigate this difficulty.

Sextupoles

Recall that the magnetic field in a sextupole can be written as:

$$\frac{e}{p_0}B_x = mxy \qquad \frac{e}{p_0}B_y = \frac{1}{2}m\left(x^2 - y^2\right)$$

Using the orbit of an off-momentum particle $x = x_\beta(s) + D(s)\delta$

we obtain

$$\frac{e}{p_0}B_x = mD(s)\delta y_\beta(s) + mx_\beta(s)y$$

and

$$\frac{e}{p_0}B_y = mD(s)\delta x_\beta(s) + \frac{1}{2}mD^2(s)\delta^2 + \frac{1}{2}m\left[x_\beta^2(s) - y_\beta^2(s)\right]$$

where the first terms in each expression are a quadrupole feed-down term for the off-momentum particle. Thus the sextupoles can be used to compensate the chromatic error. The change in tune due to the sextupole is

$$\boxed{\Delta Q = \frac{\delta}{4\pi}\oint mD(s)\beta(s)\,ds}$$

Summary

During the last portion of today's lecture, we have begun our walk through the basics of storage/damping ring physics.

We will pick up this discussion tomorrow with the effect known as radiation damping which is central to the operation of all lepton collider, storage and damping rings.

Once we have completed that discussion we will look in greater detail at the lattice choices that have been made for the damping rings and how these lattices are presently being forced to evolve.

In the first part of today's lecture we had an overview of the key design issues impacting the damping ring lattice. The homework problems will provide an opportunity to become more familiar with some of these issues.

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