Using the Hadronic Recoil Cross Section Measurement in Higgs Coupling Fits

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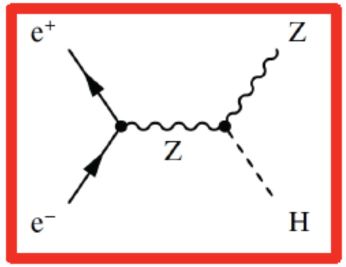
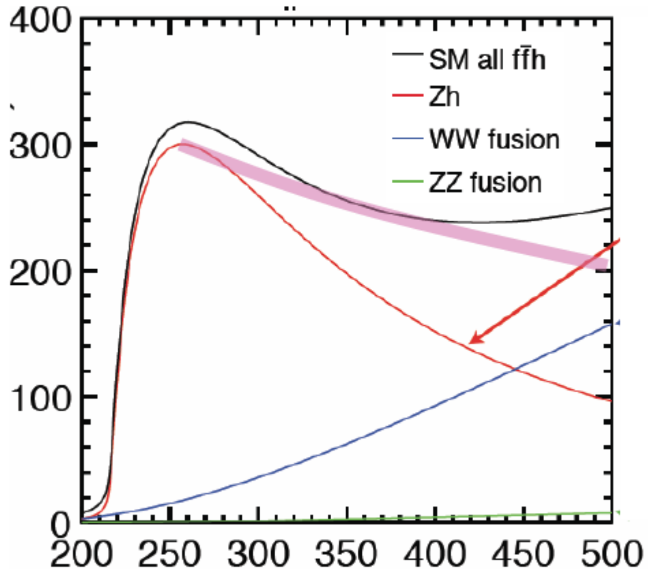
Jan 12, 2015

★ **HZ is essential for unique Model Independent Higgs programme at the ILC**



→ g_{HZZ}



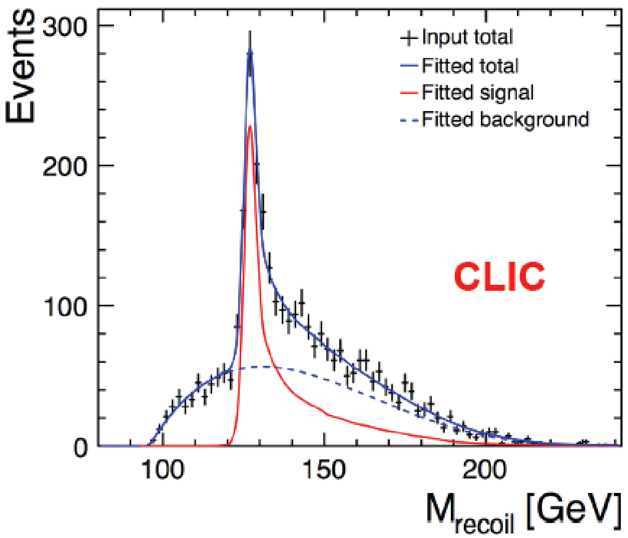
★ **No need to run at peak of cross section**

- **Event rate** $\propto \sigma \times \mathcal{L}$
- $\mathcal{L} \propto \gamma_e \propto \sqrt{s}$

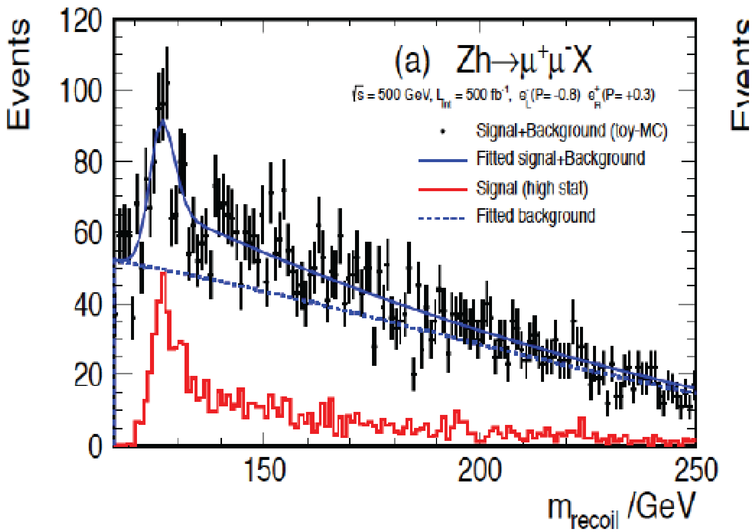
★ **Can we make a M.I. measurement of s(HZ) at** $\sqrt{s} > 250\,\mathrm{GeV}$ **?**

Leptonic Recoil Mass

$500\ \mathrm{fb}^{-1}\ @\ \sqrt{s} = 350\ \mathrm{GeV}$



$500\ \mathrm{fb}^{-1}\ @\ \sqrt{s} = 500\ \mathrm{GeV}$



$$\frac{\Delta\sigma}{\sigma} = 4.7\ \%$$

μμ only

$$\frac{\Delta\sigma}{\sigma} = 6.5\ \%$$

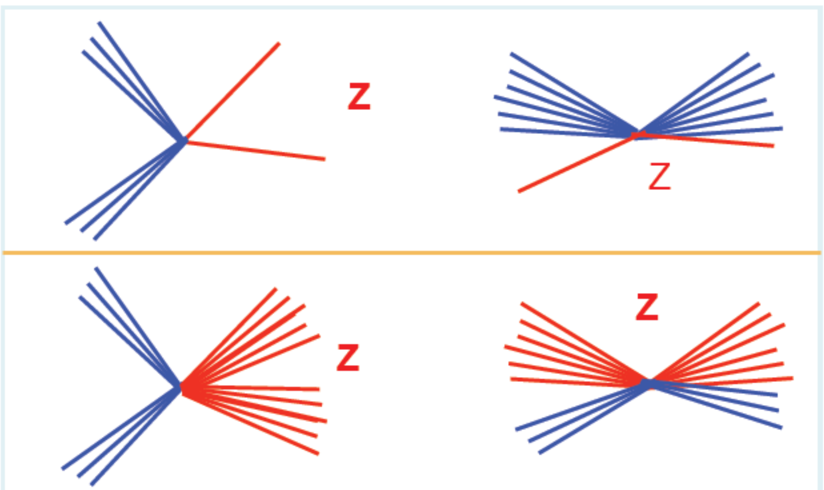
cf $\frac{\Delta\sigma}{\sigma} = 3.1\%$

for $250\ \mathrm{fb}^{-1}\ @\sqrt{s} = 250$ GeV,

$\mu^+\mu^-$ only

HZ Hadronic Recoil

- ★ Argument hinges on ability to exploit HZ production: Z → qq
  - Much larger branching ratio:
    - 60 % Z → qq
    - 3.5 % Z → μμ
- ★ But model independence is the issue…



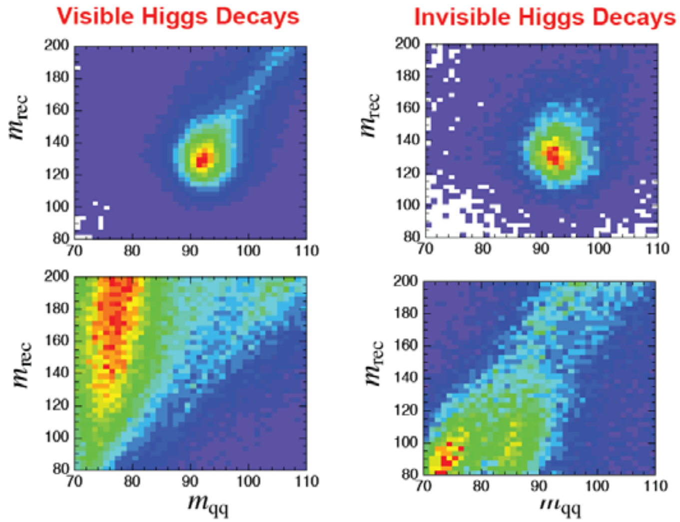
Muons "always" obvious

Here jet finding blurs separation between H and Z

⇒ Different efficiencies for different Higgs decays

SIGNAL

SM Back.



★ **Leptonic recoil at 250 GeV:**

$$\frac{\Delta\sigma}{\sigma} = 2.6\,\%$$

ILC: **250 fb^{-1}**

★ **Hadronic recoil at 350 GeV:**

$$\frac{\Delta\sigma}{\sigma} = 1.7\,\%$$

ILC: **350 fb^{-1}**

Mark Thomson's analysis of $\sigma(ZH)$ with $Z \to q\bar{q}$ uses two measurements to obtain the cross section:

$$\sigma(ZH) = \sigma(ZH) \bullet BR(visible) + \sigma(ZH) \bullet BR(invisible)$$

$\sigma(ZH) \bullet BR(visible)$

Model Independent?

- Combining visible + invisible analysis: wanted M.I.
  - i.e. efficiency independent of Higgs decay mode

Decay mode	$\epsilon^{vis}_{\mathcal{L}>0.65}$	$\epsilon^{vis}_{\mathcal{L}>0.60}$	$\epsilon^{vis} + \epsilon^{invis}$
$H \to$ invis.	<0.1 %	22.0 %	22.0 %
$H \to q\bar{q}/gg$	22.2 %	<0.1 %	22.2 %
$H \to WW^*$	21.6 %	0.1 %	21.7 %
$H \to ZZ^*$	20.2 %	1.0 %	21.2 %
$H \to \tau^+\tau^-$	24.7 %	0.3 %	24.9 %
$H \to \gamma\gamma$	25.8 %	<0.1 %	25.8 %
$H \to Z\gamma$	18.5 %	0.3 %	18.8 %

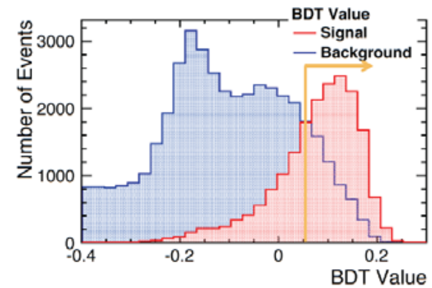
Very similar efficiencies

Mark Thomson CERN, June 2014 20

$\sigma(ZH) \bullet BR(invisible)$

BDT Selection

- Preliminary results (7 variable BDT selection)



Signal

Channel	Efficiency
Z H → qq invis.	20.7 %

Backgrounds

Channel	Efficiency	Events
qqlv	<0.1 %	900
qqll	<0.1 %	4
qqvv	1.5 %	2414

- Assuming no invisible decays (1 sigma stat. error):

$$\Rightarrow \Delta\sigma_{\text{invis}} = \pm 0.57\ \%$$

(CLIC beam spectrum, 500 fb^{-1} @ 350 GeV, no polarisation)

In order to use this cross section measurement in our Higgs analyses we have to quantify the penalty associated with the fact that $\sigma(ZH) \bullet BR(visible)$ is "almost model independent". By how much must we blow up $\Delta\sigma(ZH) \bullet BR(visible)$ to account for the fact that the efficiencies differ by as much as 7%?

Model Independent?

★ **Combining visible + invisible analysis: wanted M.I.**

- **i.e. efficiency independent of Higgs decay mode**

Decay mode	$\varepsilon^{vis}_{\mathscr{L}>0.65}$	$\varepsilon^{vis}_{\mathscr{L}>0.60}$	$\varepsilon^{vis} + \varepsilon^{invis}$
$H \to$ invis.	<0.1 %	22.0 %	22.0 %
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$H \to \gamma\gamma$	25.8 %	<0.1 %	25.8 %
$H \to Z\gamma$	18.5 %	0.3 %	18.8 %
$H \to WW^* \to q\bar{q}q\bar{q}$	21.3 %	<0.1 %	21.3 %
$H \to WW^* \to q\bar{q}l\nu$	21.9 %	<0.1 %	21.9 %
$H \to WW^* \to q\bar{q}\tau\nu$	22.1 %	<0.1 %	22.1 %
$H \to WW^* \to l\nu l\nu$	24.8 %	0.1 %	25.0 %
$H \to WW^* \to l\nu\tau\nu$	20.5 %	0.8 %	22.1 %
$H \to WW^* \to \tau\nu\tau\nu$	16.4 %	2.5 %	18.9 %

Very similar efficiencies

Look at wide range of WW topologies

We have used an approach where we use all of our $\sigma \bullet BR$ measurements for visible Higgs decays to obtain an estimate of the average signal efficiency for $\sigma(ZH) \bullet BR(visible)$. It is then straightforward to propagate the $\sigma \bullet BR$ errors, as well as the systematic errors on the individual decay mode efficiencies for the $\sigma(ZH) \bullet BR(visible)$ selection, to the error on $\sigma(ZH) \bullet BR(visible)$.

Let

$$\Psi \equiv \sigma(ZH)\bullet BR(visible)$$

$\Omega =$ Number of signal + background events in $\sigma(ZH)\bullet BR(visible)$ analysis

$\mathrm{B} =$ Predicted number of background events in $\sigma(ZH)\bullet BR(visible)$ analysis

Ξ = Average efficiency for signal events to pass $\sigma(ZH)\bullet BR(visible)$ analysis

$L =$ luminosity

$$\Psi = \frac{\Omega - \mathrm{B}}{L\,\Xi} = \frac{1}{\Xi}\sum_i \psi_i \xi_i = \sum_i \psi_i \quad \text{where}$$

$\psi_i = \sigma(ZH)\bullet BR_i$

$\xi_i =$ efficiency for events from Higgs decay i to pass $\sigma(ZH)\bullet BR(visible)$ analysis

$$\Xi = \frac{\sum_i \psi_i \xi_i}{\sum_i \psi_i}$$

$$\psi_i = \frac{\omega_i - \beta_i}{L\,\eta_i}$$

ω_i = Number of signal + background events in $\sigma(ZH)\bullet BR_i$ analysis

β_i = Predicted number of background events in $\sigma(ZH)\bullet BR_i$ analysis

η_i = efficiency for Higgs decay i to pass $\sigma\bullet BR_i$ analysis

K_i = number of signal + background events common to had Z recoil and $\sigma\bullet BR_i$ analyses

E = number of signal + background events unique to had Z recoil analysis

ε_i = number of signal + background events events unique to $\sigma\bullet BR_i$ analysis

$$\Omega = \mathrm{E} + \sum_i \mathrm{K}_i \qquad S \equiv \Omega\text{-}\mathrm{B} \qquad \mathrm{T} \equiv \frac{\sqrt{S+\mathrm{B}}}{S}$$

$$\omega_i = \mathrm{K}_i + \varepsilon_i \qquad s_i \equiv \omega_i - \beta_i \qquad \tau_i \equiv \frac{\sqrt{s_i + \beta_i}}{s_i}$$

$$\lambda_i \equiv \frac{\mathrm{K}_i}{\omega_i} \qquad N \equiv L\,\sigma_{ZH} \qquad r_i \equiv BR_i \qquad \delta_i \equiv \xi_i - \Xi$$

$$\left(\frac{\Delta\Psi}{\Psi}\right)^2 = \mathrm{T}^2\left\{1 + \frac{N^2}{\Omega}\sum_i r_i^2\left[\tau_i^2\left(\delta_i^2 - 2\lambda_i\eta_i\delta_i\right) + \Delta\xi_i^2\right]\right\}$$

This is our result for the error on $\sigma(ZH)\bullet BR(visible)$ given the approach outlined on page 8

$$\left(\frac{\Delta\Psi}{\Psi}\right)^2 = \mathrm{T}^2\left\{1+\frac{N^2}{\Omega}\sum_i r_i^2\left[\tau_i^2\delta_i^2+\Delta\xi_i^2\right]\right\}$$

Assume $\sqrt{s} = 350$ GeV and L=500 fb^{-1}

$$\mathrm{N} = L\,\sigma_{ZH} = 45383 \quad r_i = BR_i = (1-BR_{BSM})BR_i(SM) \quad \tau_i(SM) = \frac{\Delta\sigma \bullet \mathrm{BR}_i(SM)}{\sigma \bullet \mathrm{BR}_i(SM)} = \frac{\sqrt{s_i+\beta_i}}{s_i}$$

From Mark Thomson's presentation at the ILD Meeting Oshu City Sep 8, 2014:

$$\mathrm{T} = \frac{\sqrt{S+\mathrm{B}}}{S} = 0.014 \quad \Omega\text{=S+B} = 17738$$

$\xi_i(SM)$ are taken from the table on page 21 of Mark's presentation.

We assume that Mark's vis+invis efficiency values on page 21 cover all possible BSM decays since they cover all SM decays from completely invisible to fully hadronic multi-jet decays. Assuming no knowledge of the properties of the BSM decays we can then set

$$\xi_{BSM} = 0.5 * [\xi_{vis+invis}(\text{max}) + \xi_{vis+invis}(\text{min})] = 0.5 * [0.258 + 0.188] = 0.22$$
$$\Delta\xi_{BSM} = 0.5 * [\xi_{vis+invis}(\text{max}) - \xi_{vis+invis}(\text{min})] = .035$$

$$\left(\frac{\Delta\Psi}{\Psi}\right)^2 = T^2\left\{1+\frac{N^2}{\Omega}\sum_i r_i^2\left[\tau_i^2\delta_i^2+\Delta\xi_i^2\right]\right\}$$

We next obtain the error $\tau_{BSM} = \dfrac{\Delta\sigma \bullet BR_{BSM}}{\sigma \bullet BR_{BSM}}$ from Michael Peskin's Higgs coupling fit program. We do not use the $\sum_i BR_i = 1$ constraint, and to begin with we only use the leptonic recoil σ_{ZH} measurement. This provides a model independent measurement of g_{BSM}. For $\sqrt{s} = 350$ GeV, L=500 fb^{-1} Michael's program gives $\dfrac{\Delta g_{BSM}}{g_{BSM}} = 0.032$ which we multiply by two to get $\tau_{BSM} = \dfrac{\Delta\sigma \bullet BR_{BSM}}{\sigma \bullet BR_{BSM}} = 0.064$. We assume that $r_{BSM}(true) = 0$ and therefore set the measured $r_{BSM} = \tau_{BSM} = 0.064$. This gives a model independent $\dfrac{\Delta\Psi}{\Psi} = 0.014 * 1.27 = 0.018$.

We then add this new model indepdendent hadronic recoil σ_{ZH} measurement as input to Michael's program to obtain a new error $\tau_{BSM} = 0.041$. Setting $r_{BSM} = \tau_{BSM} = 0.041$ we then obtain a new model independent hadronic recoil σ_{ZH} error of $\dfrac{\Delta\Psi}{\Psi} = 0.014 * 1.12 = 0.016$.

Iterating again we arrive at $r_{BSM} = \tau_{BSM} = 0.039$ and $\dfrac{\Delta\Psi}{\Psi} = 0.014 * 1.11 = 0.016$. Further interations don't give any improvement. Our best model independent hadronic recoil cross section error is $\Delta\sigma_{ZH} = 0.016$.

$$\left(\frac{\Delta\Psi}{\Psi}\right)^2 = \mathrm{T}^2\left\{1+\frac{N^2}{\Omega}\sum_i r_i^2\left[\tau_i^2\delta_i^2+\Delta\xi_i^2\right]\right\}$$

We have shown that $\frac{1}{2}\frac{N^2}{\Omega}\sum_i r_i^2\left[\tau_i^2\delta_i^2+\Delta\xi_i^2\right]=0.11$ for $\sqrt{s}=350$ GeV, L=500 fb^{-1}.

How does this scale with luminosity?

$\frac{N^2}{\Omega}\propto L$ $\quad\tau_i^2\propto L^{-1}$ $\quad r_i^2$ is independent of lumi except $r_{BSM}^2=\tau_{BSM}^2\propto L^{-1}$.

If we assume $\Delta\xi_i=0$ except $\Delta\xi_{BSM}=0.035$ then

$\frac{1}{2}\frac{N^2}{\Omega}\sum_i r_i^2\left[\tau_i^2\delta_i^2+\Delta\xi_i^2\right]=0.11$ independent of the luminosity at $\sqrt{s}=350$ GeV.

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Caveats for hadronic recoil systematic error calculation :

These results assume that the true $r_{BSM} = BR(H \to \text{BSM}) = 0.$

As the true r_{BSM} grows we need to keep the product $r_{BSM}\Delta\xi_{BSM}$ constant to maintain the same systematic error. For example

true r_{BSM}	required $\Delta\xi_{BSM}$
.05	0.027
.10	0.014
.15	0.0091
.20	0.0068

These $\Delta\xi_{BSM}$ requirements may seem stringent for the larger values of true r_{BSM} . However as r_{BSM} grows we will have more *BSM* decays to analyze and the required improvement in Monte Carlo modelling of the *BSM* decays should follow.

1st Five Years of ILC Running

Model Independent Higgs Couplings $\Delta g_i / g_i$

	Scenario B	Scenario D-500	
$\sqrt{s}$	250 GeV	350 GeV	
L	360 fb^{-1}	470 fb^{-1}	
σ_{ZH} meas.	l^+l^- only	l^+l^- only	$l^+l^- + q\bar{q}$
$\gamma\gamma$	14.9 %	11.0	11.0 %
gg	5.2 %	3.3	3.1 %
WW	4.0 %	1.7	1.0 %
ZZ	1.1 %	1.5	0.72 %
$b\bar{b}$	4.4 %	2.4	2.0 %
$\tau^+\tau^-$	4.7 %	3.0	2.8 %
$c\bar{c}$	5.6 %	4.1	3.9 %
$\Gamma_T(h)$	9.6 %	7.1	4.9 %

Further improvement in the Higgs coupling measurements can be obtained using the constraint $\sum_i BR_i = 1$ as first noted by Michael Peskin.

This constraint is model independent so long as the error in $BR(H \to BSM)$ is included in the fit. What error in $BR(H \to BSM)$ is required to produce an improvement in Higgs coupling measurements ?

1st Five Years of ILC Running

Model Independent Higgs Couplings $\Delta g_i / g_i$

$\sqrt{s}$ L	Scenario D-500 350 GeV 470 fb^{-1}					
σ_{ZH} meas.	$l^+l^- + q\bar{q}$	$l^+l^- + q\bar{q}$.	$l^+l^- + q\bar{q}$	$l^+l^- + q\bar{q}$	$l^+l^- + q\bar{q}$	$l^+l^- + q\bar{q}$
$BR(H \to BSM)$ (95% CL)	no meas.	<7.2%	<3.6%	<1.8%	<0.9%	<0.09%
$\gamma\gamma$	11.0 %	10.9 %	10.9 %	10.9 %	10.9 %	10.9 %
gg	3.1 %	3.0 %	2.9 %	2.9 %	2.9 %	2.9 %
WW	1.0 %	0.94 %	0.82 %	0.71 %	0.67 %	0.65 %
ZZ	0.72 %	0.67 %	0.60 %	0.53 %	0.51 %	0.50 %
$b\bar{b}$	2.0 %	1.8 %	1.6 %	1.5 %	1.4 %	1.4 %
$\tau^+\tau^-$	2.8 %	2.7 %	2.6 %	2.5 %	2.5 %	2.4 %
$c\bar{c}$	3.9 %	3.8 %	3.7 %	3.7 %	3.7 %	3.7 %
$\Gamma_T(h)$	4.9 %	4.4 %	3.6 %	2.8 %	2.5 %	2.3 %

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215 page "Exotic Decays of the 125 GeV Higgs Boson" arXiv:1312.4992 : Is this a starting point for a complete $\sigma \bullet \mathrm{BR}(H \to BSM)$ analysis?

Contents

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more Table of Contents from 215 page "Exotic Decays of the 125 GeV Higgs Boson" arXiv:1312.4992:

Summary

- The systematic error for the model dependence of Mark Thomson's hadronic recoil Higgstrahlung cross section measurement has been shown to be 11% of the statistical error assuming no knowledge of the properties of any BSM Higgs decays. This result is tailored for the context where BR(H->BSM) is small.

- If BR(H->BSM) is not small then analysis of BSM decays will improve the error on the efficiency for such events to pass the hadronic recoil analysis. It may be possile to maintain the 11 % systematic error using the improved efficiency error. Of course we have a different Higgs physics program if BR(H->BSM) is not small.

- A good understanding of $\sigma \bullet BR(H \rightarrow BSM)$ is required to squeeze the last little bit of model independent Higgs coupling precision out of the data.

Backup Slides

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Let

$\Psi \equiv \sigma(ZH) \bullet BR(visible)$

$\Omega =$ Number of signal + background events in $\sigma(ZH) \bullet BR(visible)$ analysis

$\mathrm{B} =$ Predicted number of background events in $\sigma(ZH) \bullet BR(visible)$ analysis

Ξ = Average efficiency for signal events to pass $\sigma(ZH) \bullet BR(visible)$ analysis

$L =$ luminosity

$$\Psi = \frac{\Omega - \mathrm{B}}{L\,\Xi} = \frac{1}{\Xi}\sum_i \psi_i \xi_i = \sum_i \psi_i \quad \text{where}$$

$\psi_i = \sigma(ZH) \bullet BR_i$

$\xi_i =$ efficiency for events from Higgs decay i to pass $\sigma(ZH) \bullet BR(visible)$ analysis

$$\Xi = \frac{\sum_i \psi_i \xi_i}{\sum_i \psi_i}$$

$$\psi_i = \frac{\omega_i - \beta_i}{L\,\eta_i}$$

$\omega_i =$ Number of signal + background events in $\sigma(ZH)\bullet BR_i$ analysis

$\beta_i =$ Predicted number of background events in $\sigma(ZH)\bullet BR_i$ analysis

η_i = efficiency for Higgs decay i to pass $\sigma\bullet BR_i$ analysis

K_i = number of signal + background events common to had Z recoil and $\sigma\bullet BR_i$ analyses

E = number of signal + background events unique to had Z recoil analysis

ε_i = number of signal + background events events unique to $\sigma\bullet BR_i$ analysis

$$\Omega = \mathrm{E} + \sum_i \mathrm{K}_i \qquad S \equiv \Omega\text{-}\mathrm{B} \qquad \mathrm{T} \equiv \frac{\sqrt{S + \mathrm{B}}}{S}$$

$$\omega_i = \mathrm{K}_i + \varepsilon_i \qquad s_i \equiv \omega_i - \beta_i \qquad \tau_i \equiv \frac{\sqrt{s_i + \beta_i}}{s_i}$$

$$\lambda_i \equiv \frac{\mathrm{K}_i}{\omega_i} \qquad N \equiv L\,\sigma_{ZH} \qquad r_i \equiv BR_i \qquad \delta_i \equiv \xi_i - \Xi$$

$$(\Delta\Psi)^2 = \left(\frac{\partial\Psi}{\partial\Omega}\right)^2 V_{\Omega\Omega} + \left(\frac{\partial\Psi}{\partial\Xi}\right)^2 V_{\Xi\Xi} + 2\frac{\partial\Psi}{\partial\Omega}\frac{\partial\Psi}{\partial\Xi} V_{\Omega\Xi}$$

$$\frac{\partial\Psi}{\partial\Omega} = \frac{1}{L\,\Xi} = \frac{\Psi}{\Omega}\left(1 - \frac{\mathrm{B}}{\Omega}\right)^{-1} \qquad \frac{\partial\Psi}{\partial\Xi} = -\frac{\Omega - \mathrm{B}}{L\,\Xi^2} = -\frac{\Psi}{\Xi}$$

$$V_{\Omega\Omega} = \mathrm{E} + \sum_i \mathrm{K}_i = \Omega$$

$$V_{\Xi\Xi} = \frac{1}{L^2\Psi^2}\sum_i \frac{(\xi_i - \Xi)^2}{(\eta_i)^2}(\varepsilon_i + \mathrm{K}_i)$$

$$V_{\Omega\Xi} = \frac{1}{L\,\Psi}\sum_i \frac{\xi_i - \Xi}{\eta_i}\mathrm{K}_i$$

$$\left(\frac{\Delta\Psi}{\Psi}\right)^2 = \frac{1}{\Omega^2}\left(1-\frac{\mathrm{B}}{\Omega}\right)^{-2} V_{\Omega\Omega} + \frac{1}{\Xi^2} V_{\Xi\Xi} - \frac{2}{\Omega\Xi}\left(1-\frac{\mathrm{B}}{\Omega}\right)^{-1} V_{\Omega\Xi}$$

$$= \frac{1}{\Omega}\left(1-\frac{\mathrm{B}}{\Omega}\right)^{-2} + \frac{1}{L^2\Xi^2\Psi^2}\sum_i \frac{(\xi_i-\Xi)^2}{(\eta_i)^2}(\varepsilon_i + \mathrm{K}_i) - \frac{2}{L\,\Omega\Xi\Psi}\left(1-\frac{\mathrm{B}}{\Omega}\right)^{-1}\sum_i \frac{\xi_i-\Xi}{\eta_i}\mathrm{K}_i$$

$$= \frac{1}{\Omega}\left(1-\frac{\mathrm{B}}{\Omega}\right)^{-2} + \frac{1}{L^2\Xi^2\Psi^2}\sum_i \frac{(\xi_i-\Xi)^2}{(\eta_i)^2}(L\eta_i\psi_i + \beta_i) - \frac{2}{L\,\Omega\Xi\Psi}\left(1-\frac{\mathrm{B}}{\Omega}\right)^{-1}\sum_i \frac{\xi_i-\Xi}{\eta_i}\lambda_i(L\eta_i\psi_i + \beta_i)$$

$$= \frac{1}{\Omega}\left(1-\frac{\mathrm{B}}{\Omega}\right)^{-2}\left[1+\frac{L}{\Omega}\sum_i \frac{(\xi_i-\Xi)^2}{\eta_i}\psi_i\left(1+\frac{\beta_i}{s_i}\right) - \frac{2L}{\Omega}\sum_i(\xi_i-\Xi)\psi_i\lambda_i\left(1+\frac{\beta_i}{s_i}\right)\right]$$

$$= \frac{S+\mathrm{B}}{S^2}\left\{1+\frac{L}{\Omega}\sum_i(\xi_i-\Xi)\psi_i\left(\frac{s_i+\beta_i}{s_i^2}\right)\left[(\xi_i-\Xi)L\psi_i - 2\lambda_i s_i\right]\right\}$$

$$= \mathrm{T}^2\left\{1+\frac{N^2}{\Omega}\sum_i r_i^2\tau_i^2\left[\delta_i^2 - 2\lambda_i\eta_i\delta_i\right]\right\}$$

What if we don't do a hadronic Z recoil measurement and instead only use $\sigma(ZH)\bullet BR_i$ to calculate $\sigma(ZH)\bullet BR(visible) = \sum_i \sigma(ZH)\bullet BR_i$?

$$\Psi' = \sum_i \psi_i \qquad \psi_i = \frac{\omega_i - \beta_i}{L\,\xi_i}$$

$$(\Delta\Psi')^2 = \sum_i \left(\frac{\partial \Psi'}{\partial \omega_i}\right)^2 \omega_i \ , \qquad \frac{\partial \Psi'}{\partial \omega_i} = \frac{1}{L\eta'_i}$$

$$(\Delta\Psi')^2 = \frac{1}{L^2}\sum_i = \frac{1}{L^2}\sum_i \frac{s_i + \beta_i}{\xi_i^{\,2}}$$

$$\left(\frac{\Delta\Psi'}{\Psi'}\right)^2 = \left(\sum_i \frac{\omega_i - \beta_i}{L\,\xi_i}\right)^{-2} \frac{1}{L^2}\sum_i \frac{s_i + \beta_i}{\xi_i^{\,2}}$$

$$= \frac{S + \mathrm{B}}{S^2}\frac{L}{\Omega}\Xi^2 \sum_i \frac{\psi_i}{\xi_i}\left(1 + \frac{\beta_i}{s_i}\right)$$

Compare this now with our formula for $\left(\frac{\Delta\Psi}{\Psi}\right)^2$ for $\lambda_i = 1$:

$$\left(\frac{\Delta\Psi}{\Psi}\right)^2 = \frac{S + \mathrm{B}}{S^2}\left\{1 + \frac{1}{\Omega}\sum_i \omega_i\left[\left(1 - \frac{\Xi}{\xi_i}\right)^2 - 2\left(1 - \frac{\Xi}{\xi_i}\right)\right]\right\}$$

$$= \frac{S + \mathrm{B}}{S^2}\left\{1 + \frac{1}{\Omega}\sum_i \omega_i\left[1 - \frac{2\Xi}{\xi_i} + \left(\frac{\Xi}{\xi_i}\right)^2 - 2 + 2\frac{\Xi}{\xi_i}\right]\right\} = \left(\frac{\Delta\Psi'}{\Psi'}\right)^2$$