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Distribution of Failure Times

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Abstract

This note describes the calculation of probability distribution for failure times for a set of components. The basic single component failure is supposed to follow an exponential failure time distribution. The canonical distributions are derived from products of conditional probabilities.

1 Time of First Failure for N components

Let dP/dt be the failure probability density for a single component. This is the probability that a single component fails in the time interval $[t, t + dt]$:

$$\frac{dP}{dt} = \frac{1}{T}\, e^{-t/T} \equiv p(t). \tag{1}$$

We want to calculate the probability dP_1/dt that the first failure in a group of N components occurs in $[t, t + dt]$. This can be written as follows:

$$\frac{dP_1}{dt} = N\, p(t) \left(\int_t^\infty dt'\, p(t') \right)^{N-1} \tag{2}$$

$$= \frac{N}{T} e^{-tN/T} \tag{3}$$

The factor $p(t)$ in Eq. (2) is the probability that a component fails in $[t, t + dt]$. The term in parentheses is the probability that $N - 1$ components fail in $[t, \infty]$. N is a combinatorial factor giving the number of choices to fix the component which fails in $[t, t + dt]$.

Normalisation and mean time of first failure are calculated from

$$\int_0^\infty dt\, \frac{dP_1}{dt} = 1 \tag{4}$$

and

$$\int_0^\infty dt\, t\, \frac{dP_1}{dt} = T/N. \tag{5}$$

The probability dP_1/dt is plotted in Figure 1 for $N = 10$ and $T = 1$.[1] Note that Eq. (3) assumes the the form of a failure distribution $p(t)$ with $T \rightarrow T/N$ which is the fundamental expectation for components that fail independently.

[1] In all plots, the time is shown in units of T.

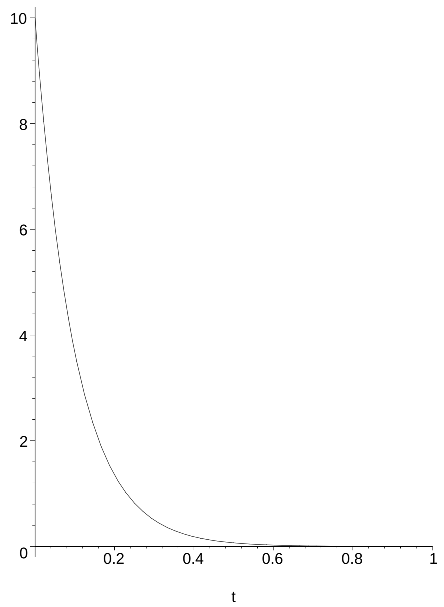

Figure 1: Probability that the first failure within $N = 10$ components occurs in the interval $[t, t + dt]$ when $T = 1$ (all times in arbitrary units).

2 Time of Second failure for N components

We want to calculate the probability dP_2/dt that the second failure in a group of N components happens in $[t, t + dt]$. This can be written as follows:

$$\frac{dP_2}{dt} = N\,(N-1)\;p(t)\int_0^t dx\;p(x)\left(\int_t^\infty dy\;p(y)\right)^{N-2} \tag{6}$$

The first integral is the probability that a component fails in the interval $[0, t]$. The term in parentheses is the probability that $N - 2$ components fail in $[t, \infty]$. There are N possible choices of finding the component which fails in $[t, t + dt]$. For the component which fails in $[0, t]$ there are then only $N - 1$ possible choices.

The probability that the second failure happens in $[0, \infty]$ is unity (normalisation):

$$\int_0^\infty dt\,\frac{dP_2}{dt} = 1. \tag{7}$$

The mean time of the second failure is given by

$$\int_0^\infty dt\,t\,\frac{dP_2}{dt} = \frac{(2N-1)}{(N-1)N}\,T. \tag{8}$$

For example, for $N = 2$ the mean time of the second failure is $\frac{3}{2}\,T$. For large N the repair of a single component is irrelevant and

$$\int_0^\infty dt\,t\,\frac{dP_2}{dt} \approx 2\frac{T}{N}. \tag{9}$$

The probability dP_2/dt is plotted in Figure 2 for $N = 10$ and $T = 1$.

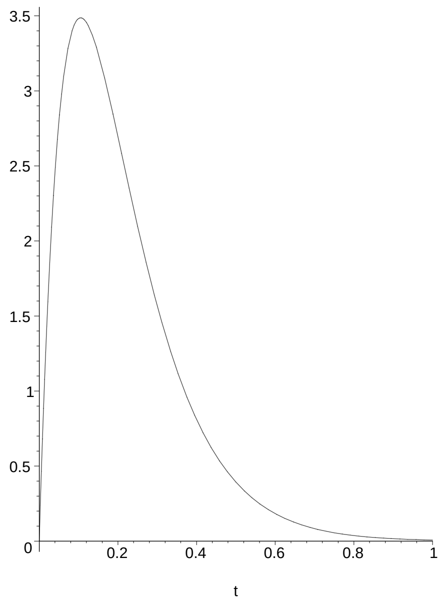

Figure 2: Probability that the second failure within $N = 10$ components occurs in the interval $[t, t + dt]$ when $T = 1$.

3 Time of second failure for two components with repair

We want to calculate the probability dP_{2r}/dt that the second failure in a group of two components happens in $[t, t + dt]$ and the first failure is repaired. Here we assume the repair time to be zero for simplicity.

There are two possibilities:

a) The same component breaks twice and the other component doesn't fail during this interval.

b) Each of the two components breaks once.

For a), the probability is calculated as (the components are labelled 'A' and 'B'):

$$\frac{dP_{2ra}}{dt} = 2 \int_0^t dx\, p(x)\, p(t-x) \int_t^\infty dy\, p(y) \tag{10}$$

The second integral is the probability that component B doesn't fail in $[0, t]$ (=probability for failure in $[t, \infty]$). The first integral contains the failure probabilities of component A which breaks twice. $p(x)$ is the probability for failure in $[x, x + dx]$. The term $p(t - x)$ is the probability of the second failure and it counts from x. There is a combinatorial factor 2 because we can exchange A and B.

When the integrals are carried out, Eq. (10) reduces to

$$\frac{dP_{2ra}}{dt} = 2\, p(t)^2\, t. \tag{11}$$

For possibility b), the calculation is as follows: Component A breaks in $[x, x+dx]$ with $x < t$ and is instantly repaired. Component B breaks in $[t, t + dt]$. Component A breaks in $[t, \infty]$ but the probability counts from x. A combinatorial factor 2 is needed because we can exchange A and B.

$$\frac{dP_{2rb}}{dt} = 2\, p(t) \int_0^t dx\; p(x) \; \int_t^\infty dy\; p(y - x) \tag{12}$$

When the integrals are carried out, also Eq. (12) reduces to

$$\frac{dP_{2rb}}{dt} = 2\, p(t)^2\, t. \tag{13}$$

The probabilities for a) and b) have to be added, such that the total probability is

$$\frac{dP_{2r}}{dt} = \frac{dP_{2ra}}{dt} + \frac{dP_{2rb}}{dt} = 4\, p(t)^2\, t. \tag{14}$$

This distribution is properly normalised:

$$\int_0^\infty dt\, \frac{dP_{2r}}{dt} = 1. \tag{15}$$

The mean time of the second failure with repair is given by

$$\int_0^\infty dt\, t\, \frac{dP_{2r}}{dt} = T. \tag{16}$$

The probability dP_{2r}/dt is depicted in Figure 3 for $T = 1$.

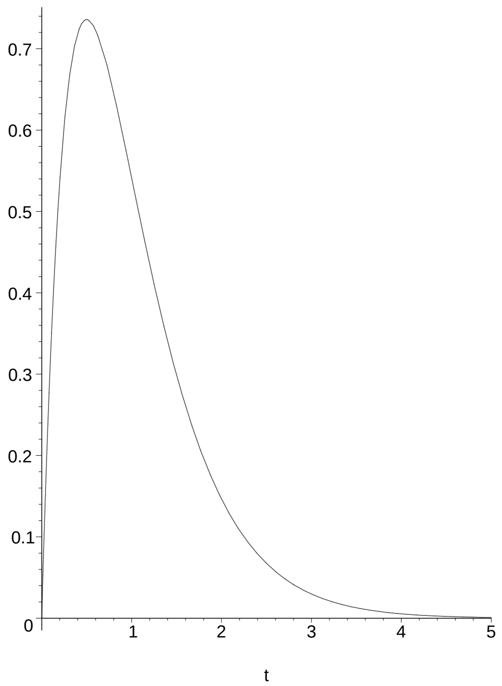

Figure 3: Probability that the second failure within $N = 2$ components occurs in the interval $[t, t + dt]$ when $T = 1$ and when the failed component is instantly repaired.

4 Time of second failure for N components with repair

Eq. (14) can be generalised to N components. The probability dP_{2r}/dt that the second failure in a group of N components occurs at time $[t, t + dt]$ is given by:

$$\frac{dP_{2r}}{dt} = \left(\frac{N}{T}\right)^2 t e^{-tN/T} \tag{17}$$

with mean

$$\int_0^\infty dt\, t \frac{dP_{2r}}{dt} = 2\frac{T}{N} \tag{18}$$

which is the large N limit of Eq. (9) as expected.