New methods for computing helicity amplitudes

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Part I: **Techniques for many external legs**

Part II: **Twistors and MHV vertices**

Part III: **Extension to massive quarks**

in collaboration with Christian Schwinn

Colour decomposition

Amplitudes in QCD may be decomposed into group-theoretical factors carrying the colour structures multiplied by kinematic functions called partial amplitudes.

The partial amplitudes do not contain any colour information and are gauge-invariant. Each partial amplitude has a fixed cyclic order of the external legs.

Examples: The n-gluon amplitude:

$$\mathcal{A}_n(1,2,...,n) \quad = \quad g^{n-2} \sum_{\sigma \in S_n/Z_n} \underbrace{2 \, \text{Tr}\left(T^{a_{\sigma(1)}} ... T^{a_{\sigma(n)}}\right)}_{\text{Chan Patton factors}} \underbrace{A_n\left(\sigma(1),...,\sigma(n)\right)}_{\text{partial amplitudes}}.$$

P. Cvitanovic, P. G. Lauwers, and P. N. Scharbach,

F. A. Berends and W. Giele,

M. L. Mangano, S. J. Parke, and Z. Xu,

D. Kosower, B.-H. Lee, and V. P. Nair,

Z. Bern and D. A. Kosower.

The spinor helicity method

- Basic objects: Massless two-component Weyl spinors

$$|p\pm\rangle, \qquad \langle p\pm|$$

- Gluon polarization vectors (Z. Xu, D.-H. Zhang, and L. Chang) :

$$\varepsilon_\mu^+(k,q) = \frac{\langle k+|\gamma_\mu|q+\rangle}{\sqrt{2}\langle q-|k+\rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle k-|\gamma_\mu|q-\rangle}{\sqrt{2}\langle k+|q-\rangle}$$

  q is an arbitrary null reference momentum. Dependency on q drops out in gauge invariant quantities.

- A clever choice of the reference momentum can reduce significantly the number of diagrams which need to be calculated.

Bra-ket notation versus dotted-undotted indices

Two different notations for the same thing:

$$|p+\rangle = p_B \qquad \langle p+| = p_{\dot{A}}$$

$$|p-\rangle = p^{\dot{B}} \qquad \langle p-| = p^A$$

Supersymmetric relations

In an unbroken supersymmetric theory, the supercharge annihilates the vacuum.

$$\langle 0 | [Q, \Phi_1 \Phi_2 ... \Phi_n] | 0 \rangle = 0$$

The supercharge transforms bosons into fermions and vice versa. It relates therefore amplitudes with a pair of fermions to the pure gluon amplitude:

$$A_n^{tree}(q_1^+, g_2^+, ..., g_j^-, ..., g_{n-1}^+, \bar{q}_n^-) = \frac{\langle p_1 - | p_j + \rangle}{\langle p_j - | p_n + \rangle} A_n^{tree}(g_1^+, g_2^+, ..., g_j^-, ..., g_{n-1}^+, g_n^-).$$

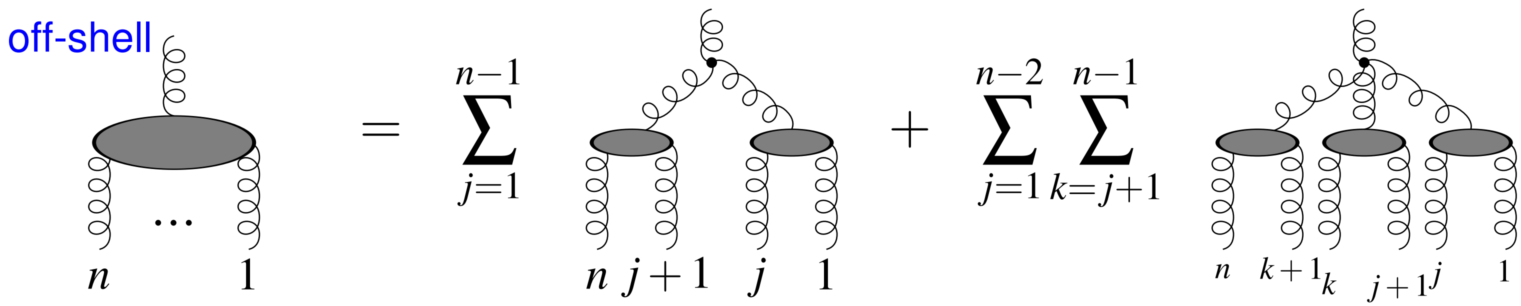
After the colour structure has been stripped off, nothing distinguishes a massless quark from a gluino.

S. J. Parke and T. R. Taylor,

M. T. Grisaru and H. N. Pendleton.

Recurrence relations

Off-shell currents provide an efficient way to calculate amplitudes:



No Feynman diagrams are calculated in this approach !

F. A. Berends and W. T. Giele,

D. A. Kosower.

The Parke-Taylor formulae

For specific helicity combinations the amplitudes have a remarkably simple analytic formula or vanish altogether:

$$\begin{aligned} A_n^{tree}(g_1^+, \dots, g_n^+) &= 0, \\ A_n^{tree}(g_1^+, \dots, g_j^-, \dots, g_n^+) &= 0, \\ A_n^{tree}(g_1^+, \dots, g_j^-, \dots, g_k^-, \dots, g_n^+) &= i\left(\sqrt{2}\right)^{n-2} \frac{\langle jk\rangle^4}{\langle 12\rangle \dots \langle n1\rangle}. \end{aligned}$$

The n-gluon amplitude with $n-2$ gluons of positive helicity and 2 gluons of negative helicity is called a maximal-helicity violating amplitude (MHV amplitude).

F. A. Berends and W. T. Giele,

S. J. Parke and T. R. Taylor.

The CSW construction

Cachazo, Svrček and Witten proposed that the gluonic Born amplitude with an arbitrary helicity configuration can be calculated from diagrams with scalar propagators and new vertices, which are MHV-amplitudes continued off-shell.

$$A_n(1^+, ..., j^-, ..., k^-, ..., n^+) \quad = \quad i\left(\sqrt{2}\right)^{n-2} \frac{\langle jk \rangle^4}{\langle 12 \rangle ... \langle n1 \rangle}.$$

Off-shell continuation:

$$P \quad = \quad p^\flat + \frac{P^2}{2Pq} q.$$

Propagators are scalars:

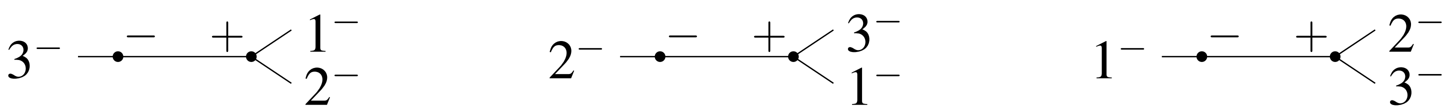
$$\frac{-i}{P^2}$$

Cachazo, Svrček and Witten, JHEP 0409:006, (hep-th/0403047)

Example: Six-gluon amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

The first non-trivial example: The six-gluon amplitude with 3 positive helicity gluons and 3 negative helicity gluons.

One starts with stripped diagrams:



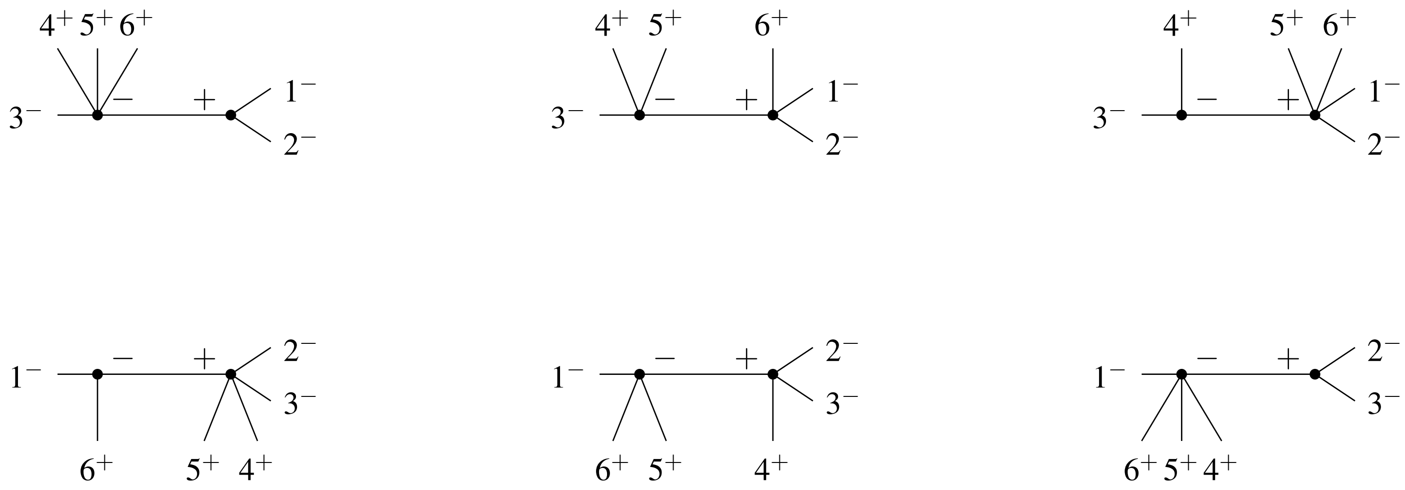
The second diagram will be dressed with all positive helicty gluons inserted between leg 3 and leg 1.

Therefore one MHV vertex with two negative helicity gluons and zero positive helicity gluons remains.

Therefore this diagram does not give a contribution.

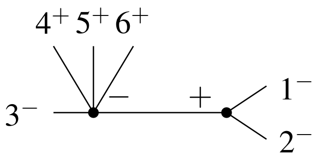
Example: Six-gluon amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

Inserting the gluons with positive helicity:



Example: Six-gluon amplitude $A(1^-,2^-,3^-,4^+,5^+,6^+)$

The first diagram yields:



$=$

$$\left[i\sqrt{2} \frac{\langle 12\rangle^4}{\langle 12\rangle \langle 2\left(-k_{12}^\flat\right)\rangle \langle \left(-k_{12}^\flat\right) 1\rangle} \right] \frac{(-i)}{k_{12}^2} \left[i\left(\sqrt{2}\right)^3 \frac{\langle 3k_{12}^\flat\rangle^4}{\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6k_{12}^\flat\rangle\langle k_{12}^\flat 3\rangle} \right]$$

Similar for the five other diagrams.

Compare this to

- a brute force approach (220 Feynman diagrams)

- colour-ordered amplitudes (36 diagrams)

The BCF recursion relations

R. Britto, F. Cachazo and B. Feng gave a recursion relation for the calculation of the n-gluon amplitude:

$$A_n\left(p_1,p_2,\ldots,p_{n-1}^-,p_n^+\right)=$$
$$\sum_{i=1}^{n-3}\sum_{\lambda=+,-}A_{i+2}\left(\hat{p}_n,p_1,p_2,\ldots,p_i,-\hat{P}_{n,i}^{\lambda}\right)\left(\frac{-i}{P_{n,i}^2}\right)A_{n-i}\left(\hat{P}_{n,i}^{-\lambda},p_{i+1},\ldots,p_{n-2},\hat{p}_{n-1}\right).$$

No off-shell continuation needed. The amplitudes on the r.h.s. are evaluated with shifted momenta.

Britto, Cachazo and Feng, Nucl. Phys. B715, (2005), 499, (hep-th/0412308)

A proof of the BCF recursion relations

Consider the amplitude

$$A(z) \quad = \quad A\left(p_1, ..., p_k(z), ..., p_{n-1}, p_n(z)\right)$$

with shifted momenta

$$\begin{aligned} p_{k,A\dot{B}}(z) &= p_{k,A}\left(p_{k,\dot{B}} - z p_{n,\dot{B}}\right), \\ p_{n,A\dot{B}}(z) &= \left(p_{n,A} + z p_{k,A}\right) p_{n,\dot{B}}. \end{aligned}$$

- $A(z)$ is a rational function of z.
- $A(z)$ has only simple poles as a function of z.

A proof of the BCF recursion relations

- If $A(z)$ vanishes at inifinty, it can be written as

$$A(z) \quad = \quad \sum_{i,j} \frac{c_{ij}}{z - z_{ij}}$$

- The residues c_{ij} are related to the factorization on particle poles:

$$A(z) \quad = \quad \sum_{i,j} \sum_{\lambda} \frac{A_L^{\lambda}(z_{ij}) A_R^{-\lambda}(z_{ij})}{P_{ij}(z)}$$

- The physical amplitude is obtained by setting $z = 0$ in the denominator. Therefore

$$A \quad = \quad \sum_{i,j} \sum_{\lambda} \frac{A_L^{\lambda}(z_{ij}) A_R^{-\lambda}(z_{ij})}{P_{ij}}$$

Britto, Cachazo, Feng and Witten, Phys. Rev. Lett. 94:181602, (2005), (hep-th/0501052)

Axial gauge

Polarisation sum, continued off-shell:

$$\sum_{\lambda=+/-} \varepsilon_\mu^\lambda(k^\flat, q)\varepsilon_\nu^{-\lambda}(k^\flat, q) \quad = \quad -g_{\mu\nu} + 2\frac{k_\mu^\flat q_\nu + q_\mu k_\nu^\flat}{2kq}.$$

The gluon propagator in the axial gauge is given by

$$\frac{i}{k^2} d_{\mu\nu} \quad = \quad \frac{i}{k^2}\left(-g_{\mu\nu} + 2\frac{k_\mu q_\nu + q_\mu k_\nu}{2kq}\right) = \frac{i}{k^2}\left(\varepsilon_\mu^+ \varepsilon_\nu^- + \varepsilon_\mu^- \varepsilon_\nu^+ + \varepsilon_\mu^0 \varepsilon_\nu^0\right),$$

where we introduced an unphysical polarisation

$$\varepsilon_\mu^0(k, q) \quad = \quad 2\frac{\sqrt{k^2}}{2kq} q_\mu.$$

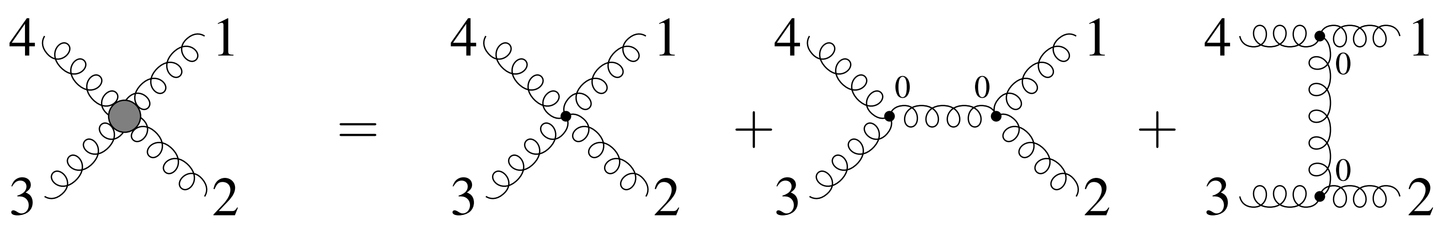
Ch. Schwinn and S.W., JHEP 0505:006, (hep-th/0503015)

Modified vertices

The only non-zero contribution containing ε^0 is obtained from a contraction of a single ε^0 into a three-gluon vertex.

In this case the other two helicities are necessarily ε^+ and ε^-.

The additional polarisation ε^0 can be absorbed into a redefinition of the four-gluon vertex.

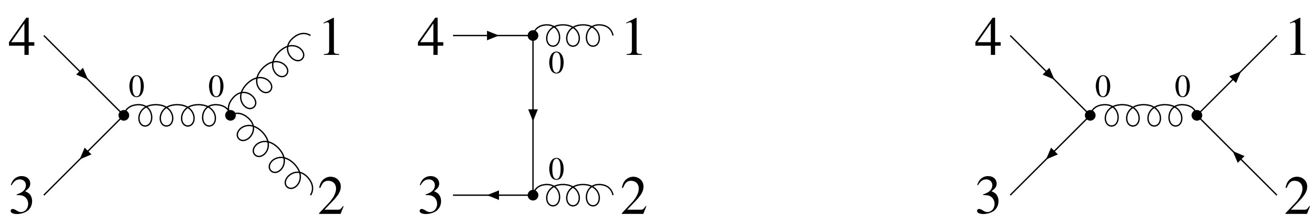


Quarks

Rewrite the quark propagator as

$$i\frac{\not{p}+m}{p^2-m^2} \quad = \quad \frac{i}{p^2-m^2}\left(\sum_{\lambda=+/-} u(-\lambda)\bar{u}(\lambda)+\frac{p^2-m^2}{2pq}\not{q}\right).$$

New vertices:



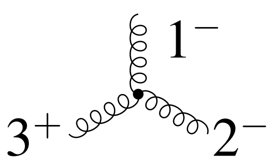
Scalar diagrammatic rules

Extension to massive and massless quarks: Born amplitudes in QCD can be computed from scalar propagators and a set of three- and four-valent vertices. Only vertices of degree zero and one occur.

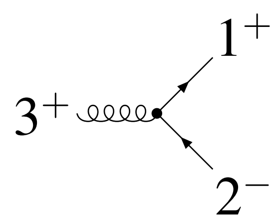
Propagators:

$$\frac{i}{p^2 - m^2}$$

Vertices:



$$= i\sqrt{2}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle},$$



$$= i\sqrt{2}\frac{[13]^2}{[12]}.$$

Ch. Schwinn and S.W., JHEP 0505:006, (hep-th/0503015)

Summary

Tree-level techniques:

- Colour decomposition, spinor methods, supersymmetric relations and recurrence relations
- Twistor space, MHV vertices and BCF recursion relations
- Scalar diagrammatic rules