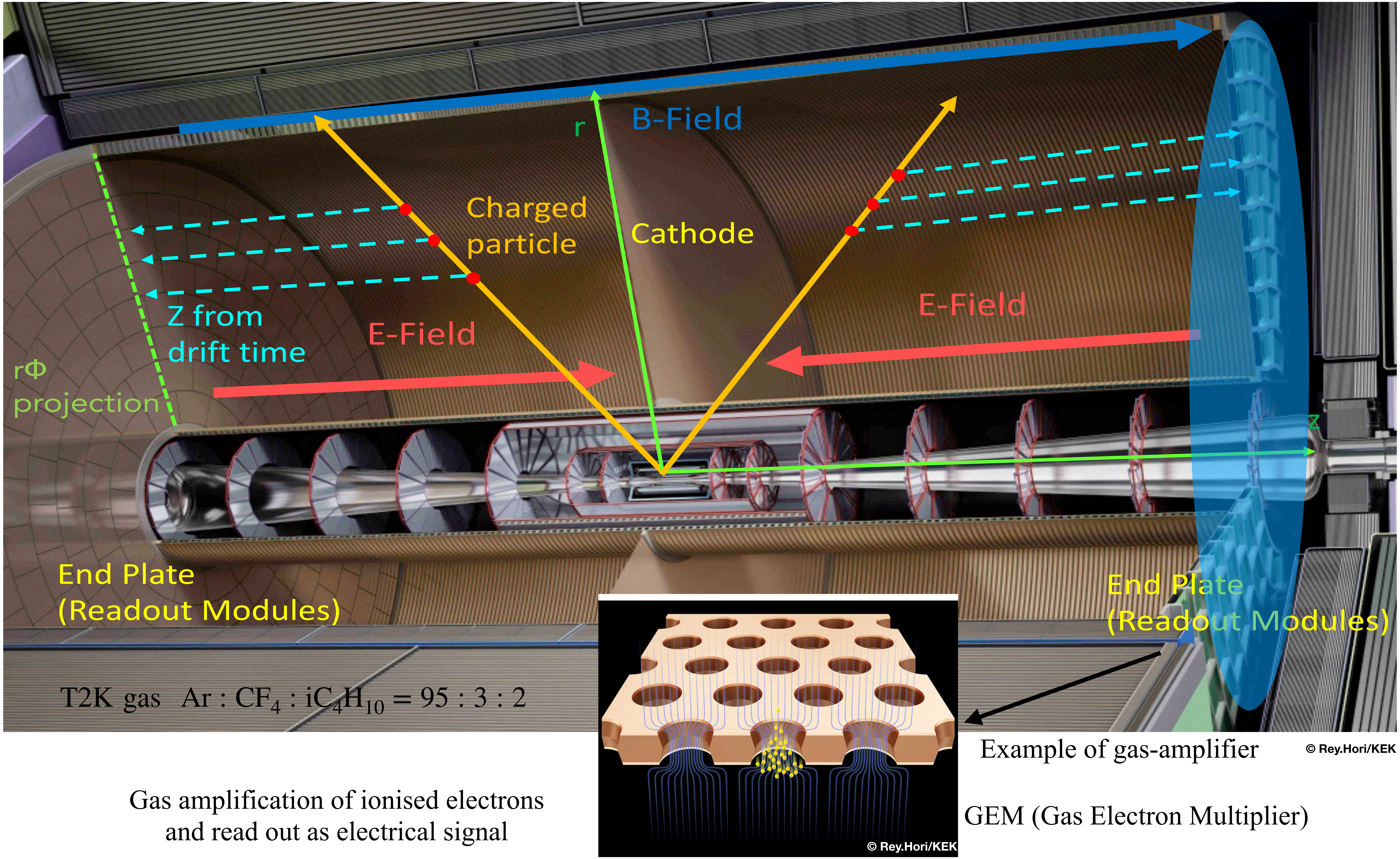
Simulation study for design optimisation of GEM module

Keita Yumino
on behalf of the LCTPC Asia

Mar 09, 2022

Introduction

Inside of ILD-TPC



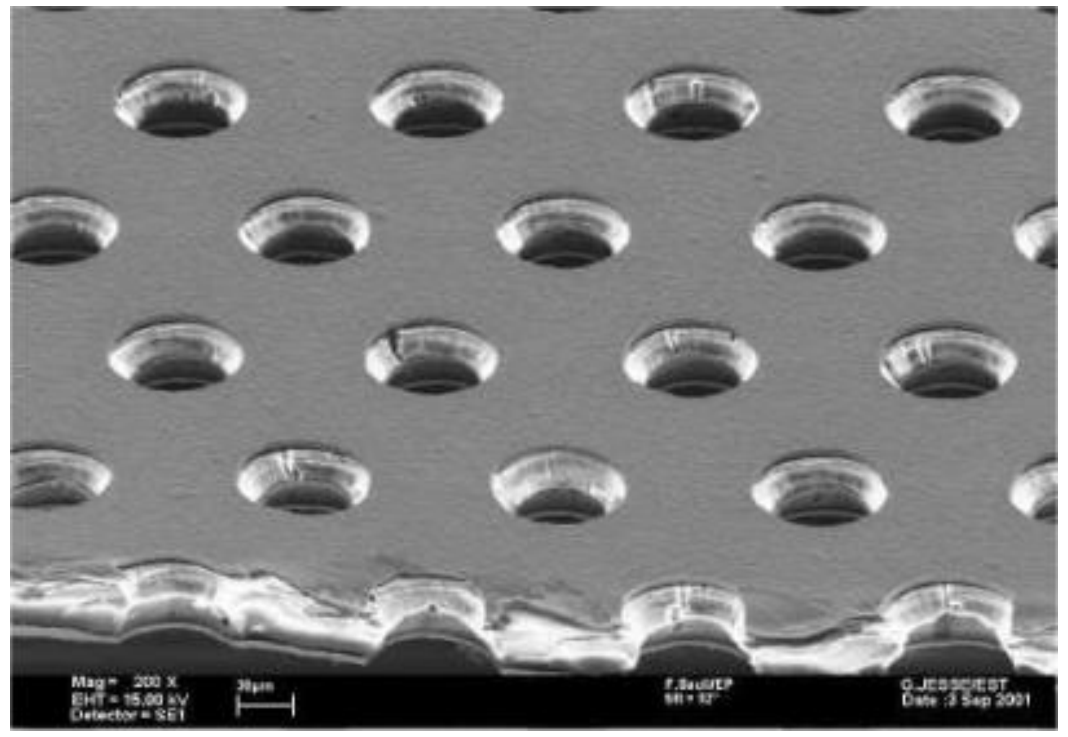
Example of gas-amplifier

GEM (Gas Electron Multiplier)

Gas amplification of ionised electrons
and read out as electrical signal

Introduction

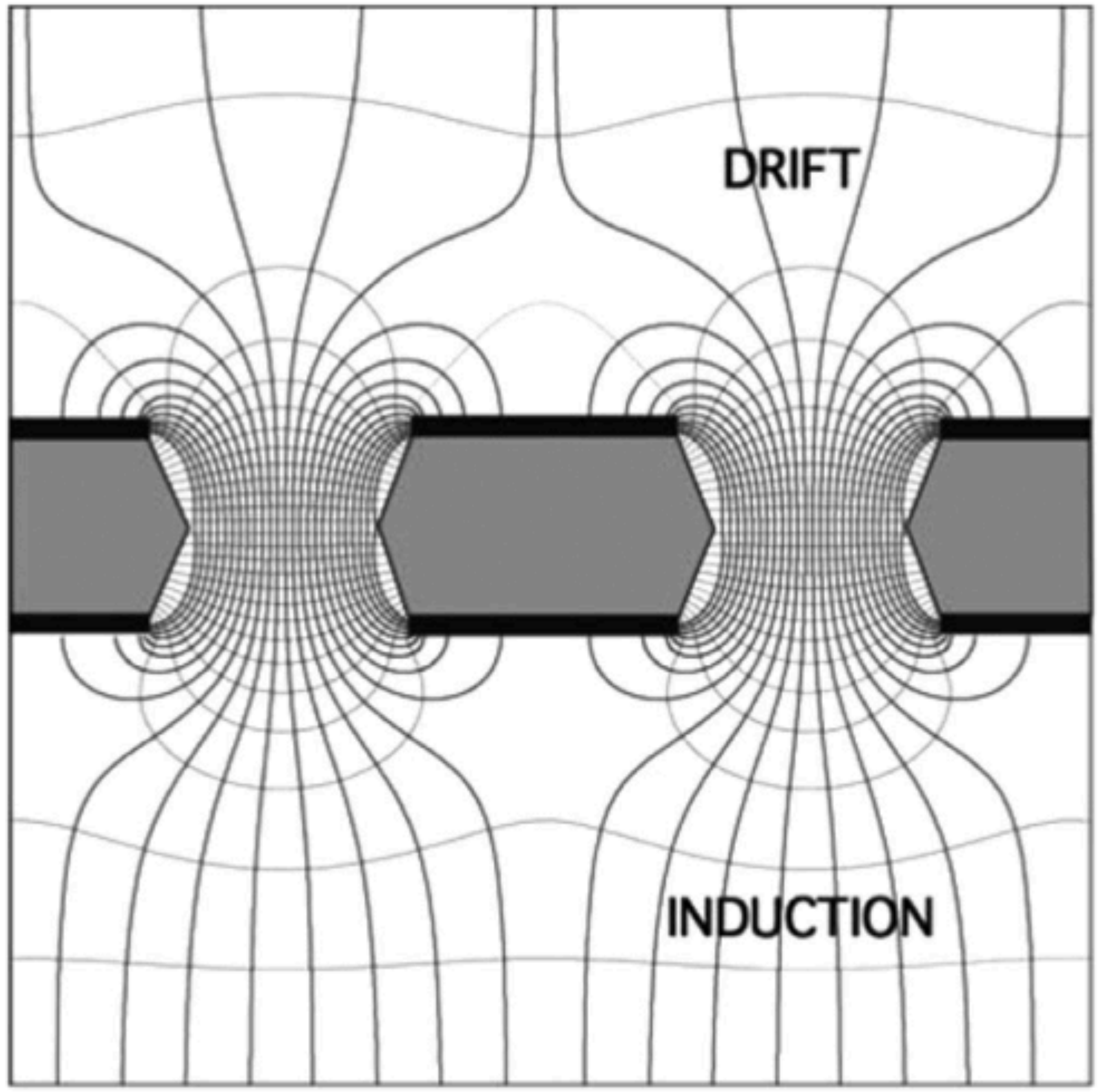
Gas Electron Multiplier (GEM)

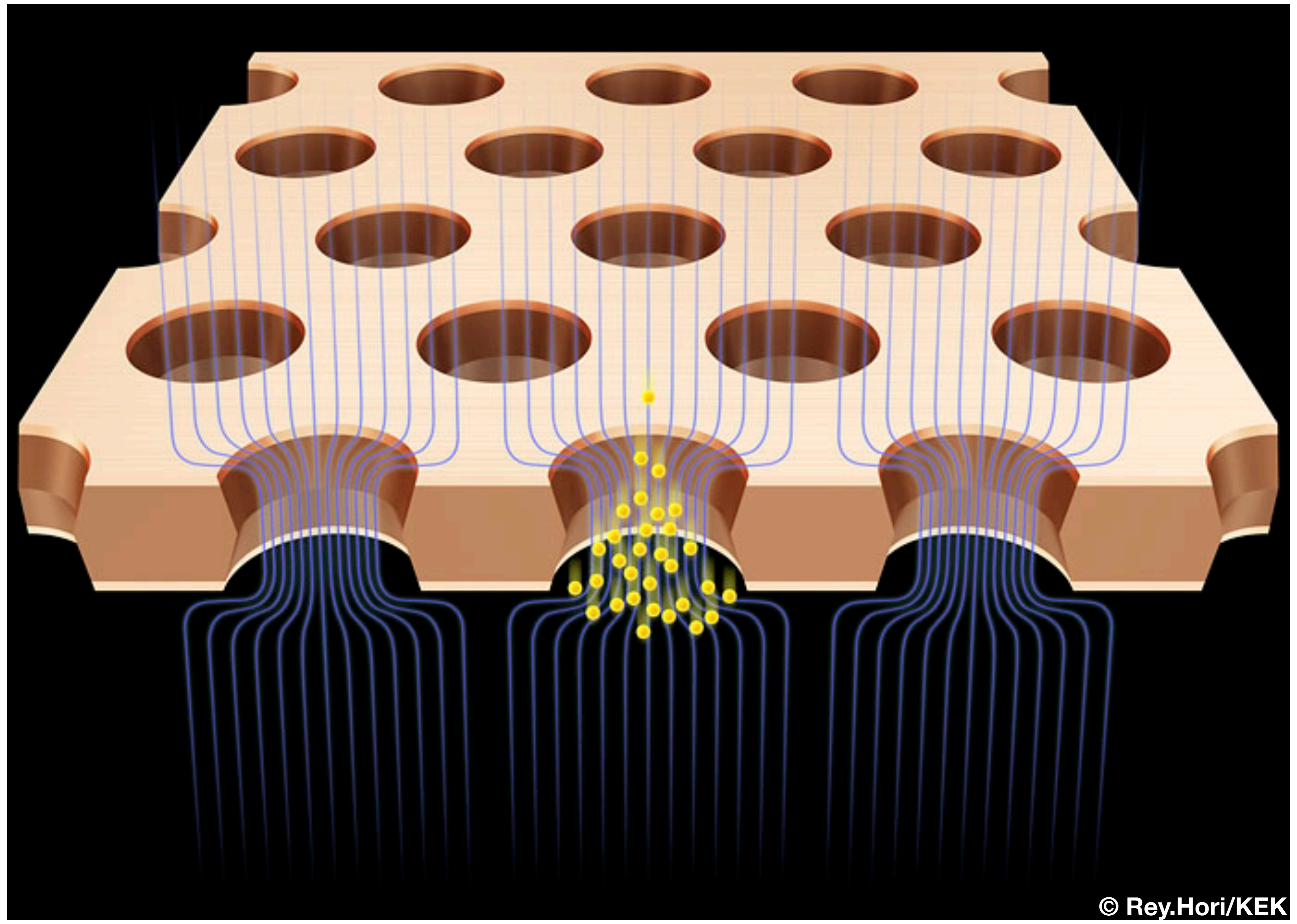


- large number of holes with a diameter of several tens to hundreds of μm
- plate-like structure with copper electrodes on both sides of an insulator { polyimide / liquid crystal polymer …

By applying a high voltage across a GEM
a high electric field is formed

Electric field in the region of the hole



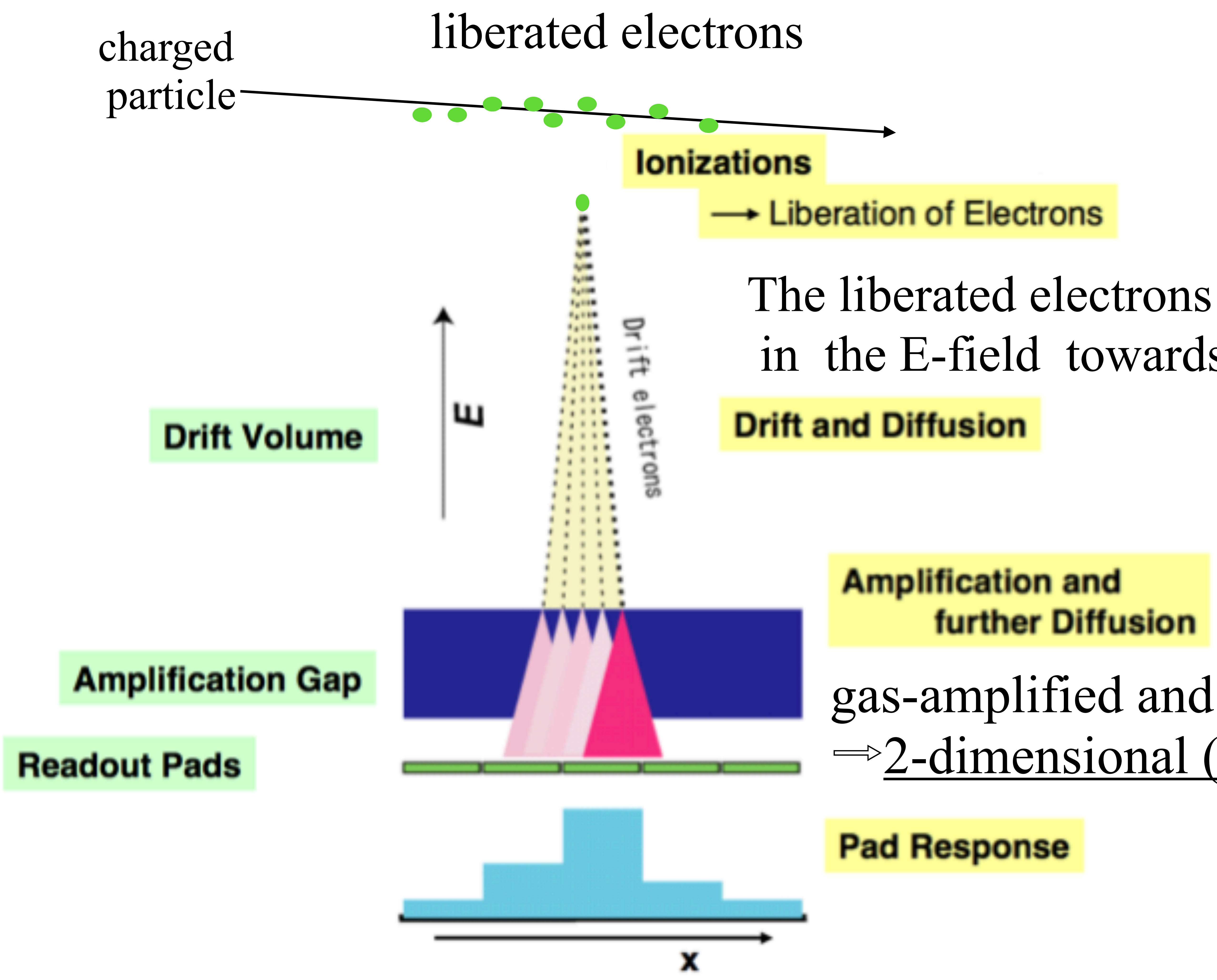


many ionised electrons are generated repeatedly by ionising collisions with the gas molecules

→ avalanche

Operating principle of TPC

A charged particle ionises the atoms of the gas mixture along its trajectory



The liberated electrons drift in the E-field towards the readout pad

gas-amplified and read as signal
⇒2-dimensional (x,y) information

z component is obtained from drift time ⇒3-dimensional (x, y, z) information

Content of Nakajima-san's talk

"Study of the spatial resolution of a GEM-based TPC"

and Aoki-san's talk

"The spatial resolution result of the first beam test of a ILD-TPC end-plane readout module with a gating foil for the ILC"

Content of my talk

Introduction

The aim of this study

Development of a high-performance GEM as a detector for LCTPC

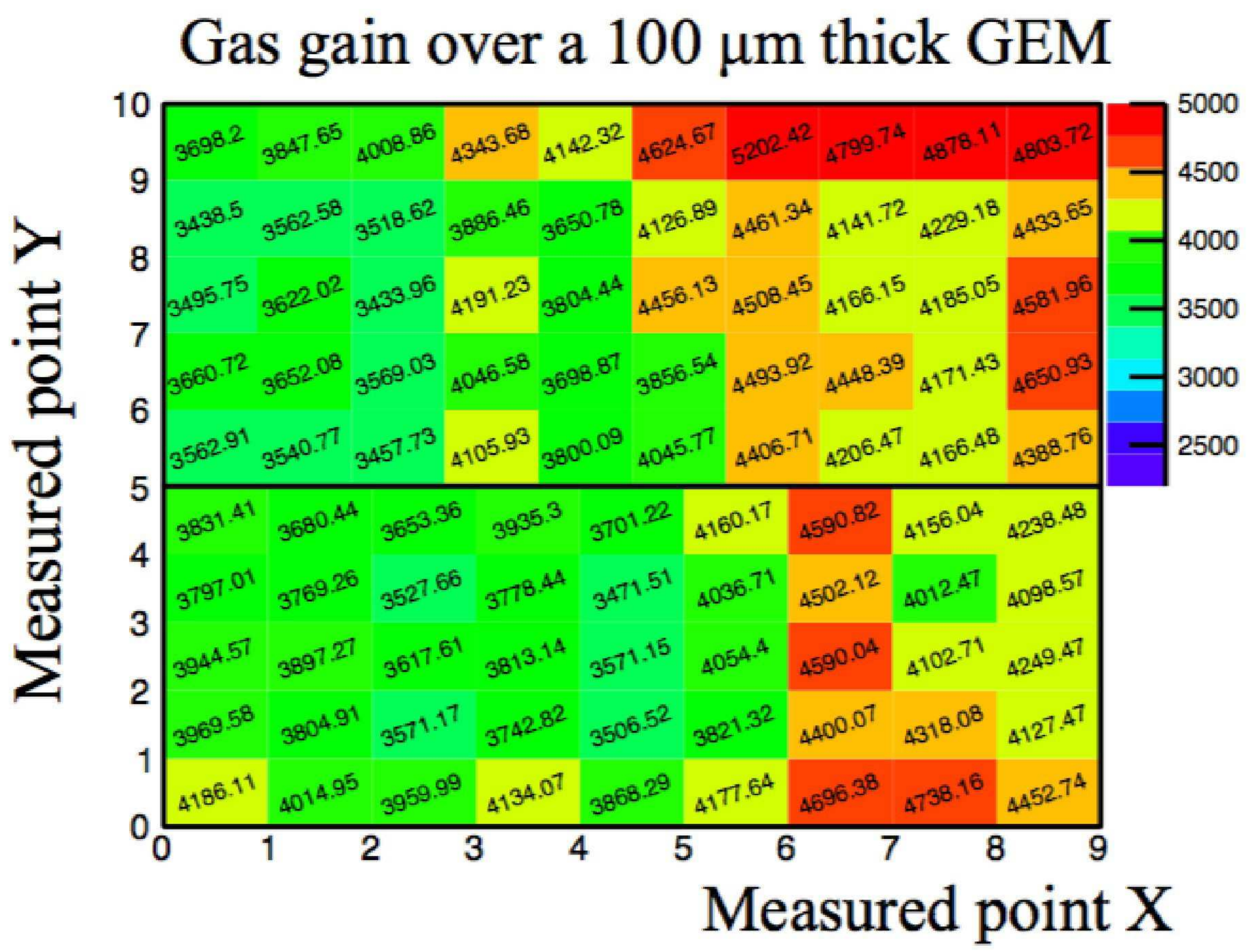
Our Asian-GEM has some problems

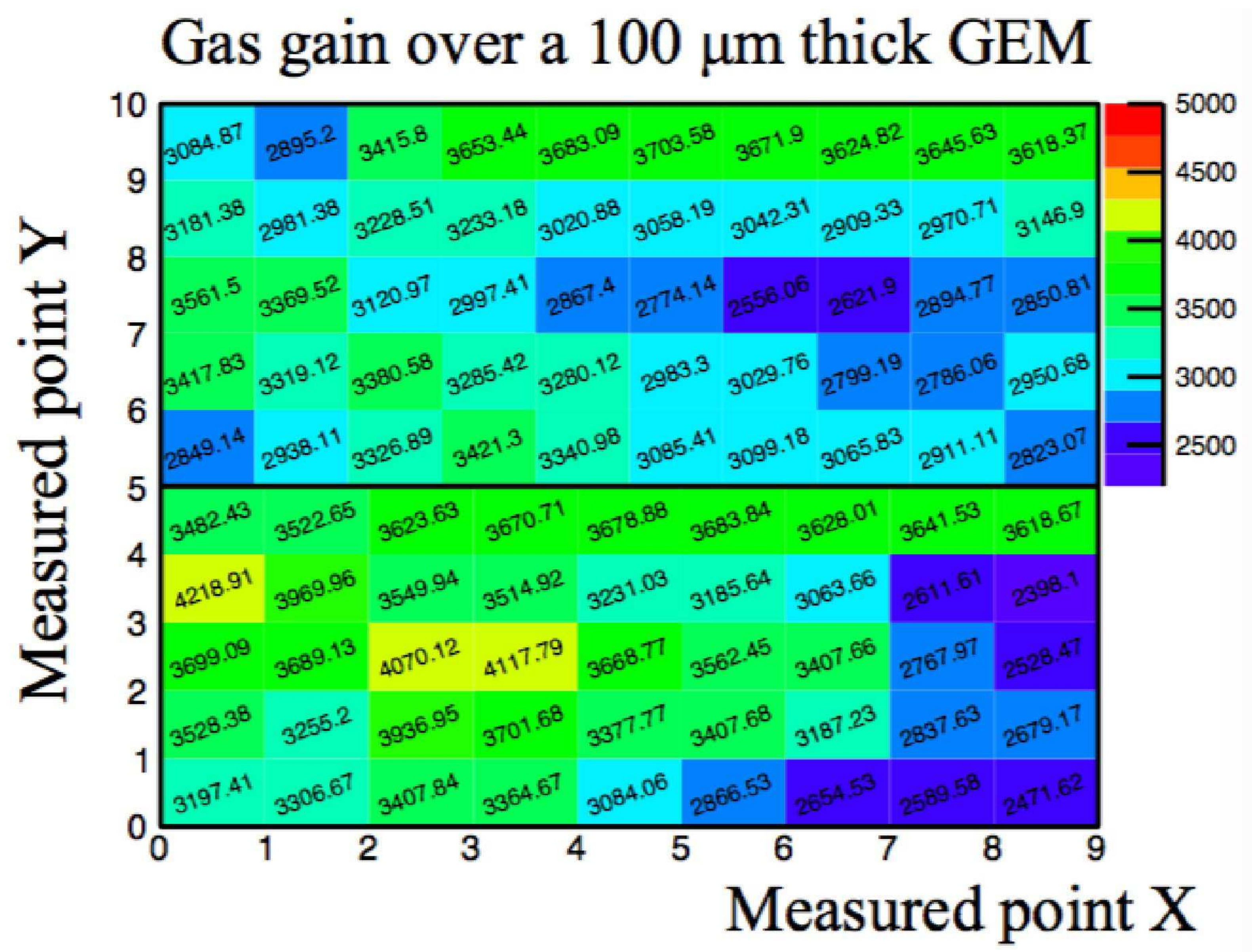
- discharge,
- need for support structure, and
- **gas gain non-uniformity**

GEM optimisation study by Theoretical approach

Thickness dependence of gain

From previous study, gas gain strongly depends on the thickness of GEM





Large gas gain non-uniformity

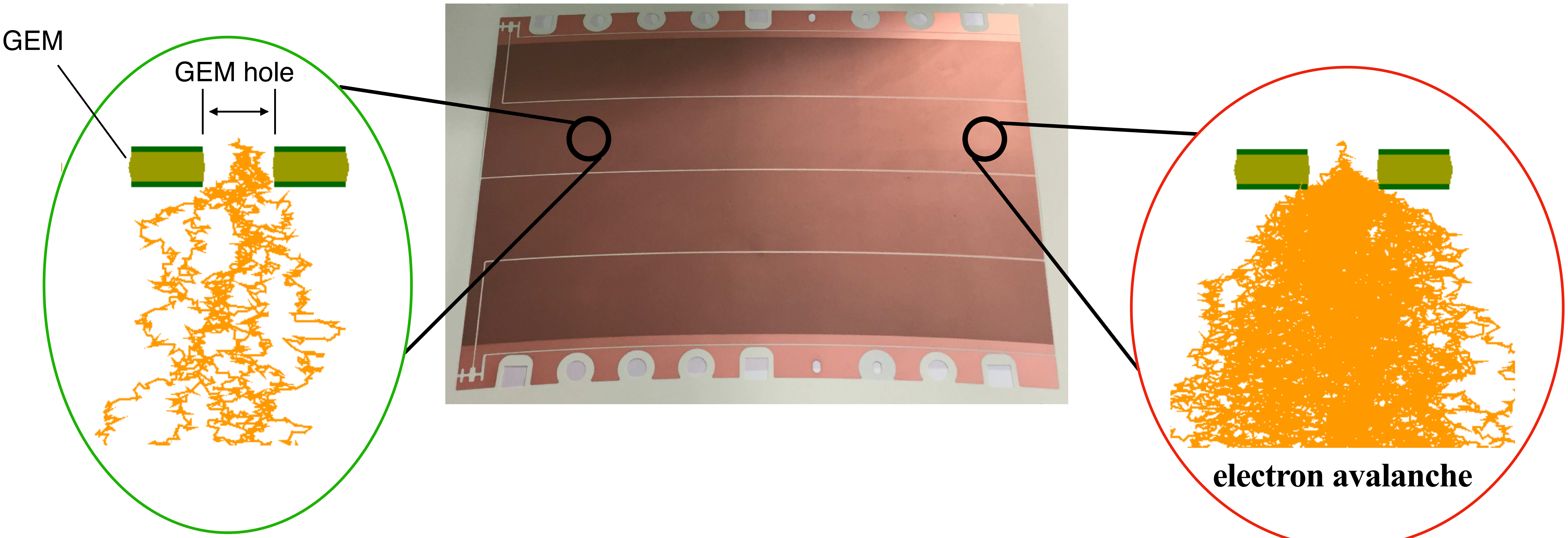
reached > 50 % difference

arxiv:1701.05421

→due only to effect of thickness?

Thickness dependence of gain

If the average gas gain is changed depending on the location
the very HIGH high-voltage must be applied to obtain a sufficiently large signal at a low gas gain region



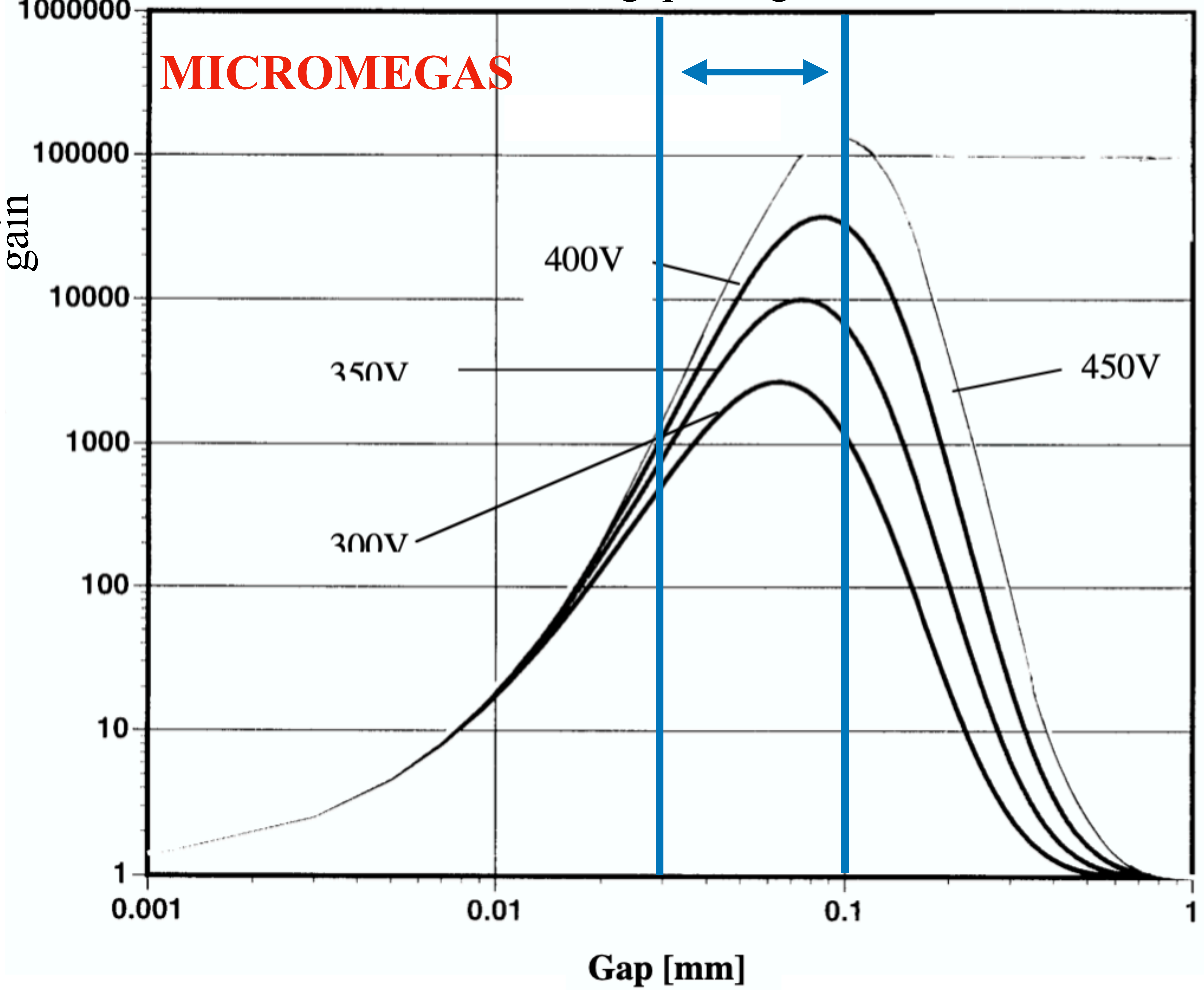
At locations with high gas gain, the avalanche will be too large
and the possibility of discharge will be too high

positive ions are also produced and
depends on gas gain

Need to reduce the thickness dependence of the gas gain on the location.

Motivation

$M = e^{\alpha d}$ as a function of the gap d gas: He + iC4H10 = 94:6



MICROMEGAS: results and prospects : I. Giomataris

Gas gain M is at maximum in the range of gaps between 30-100 μm

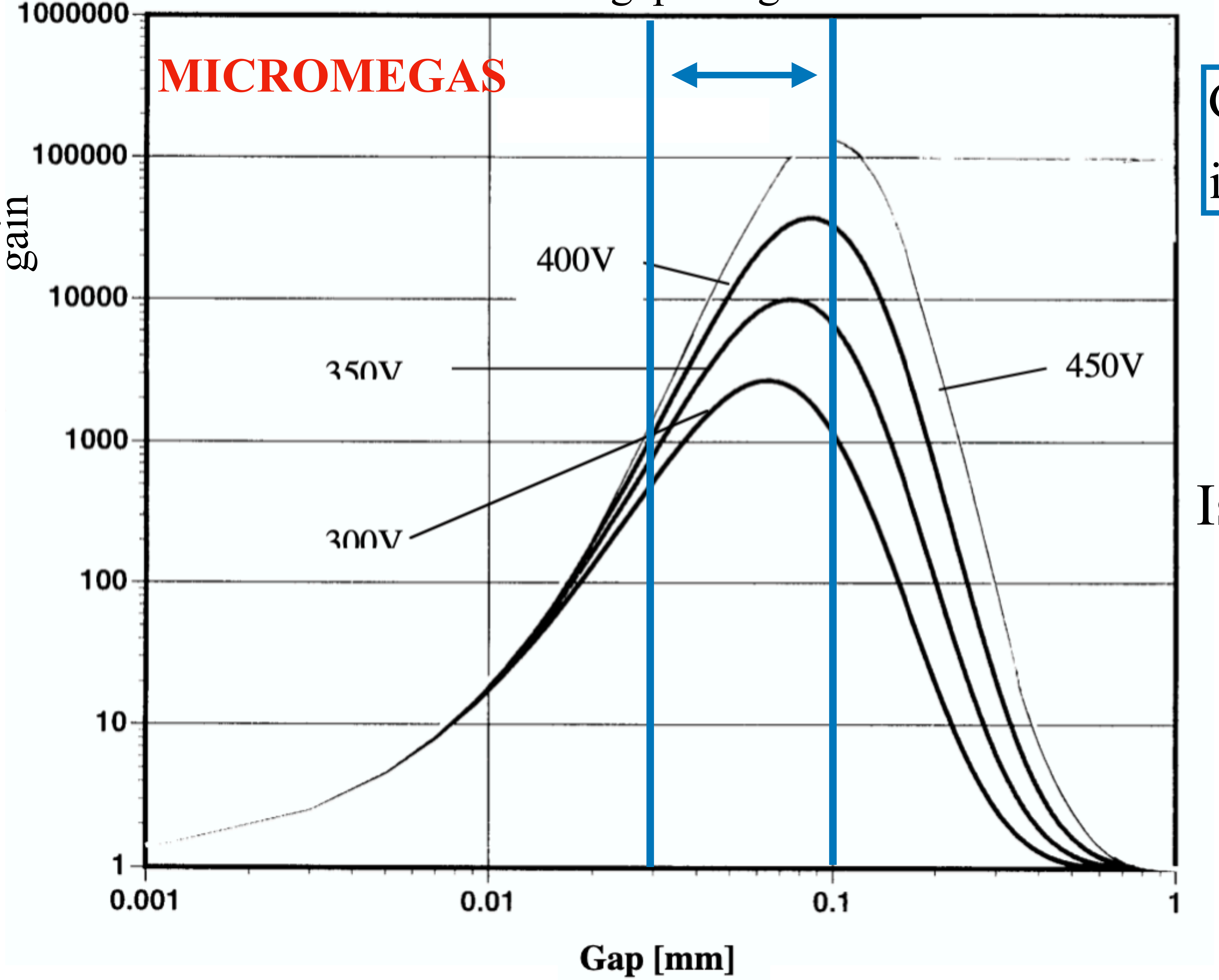
This is the range currently used by the MICROMEGAS detectors

and its fluctuations are canceled

Stability condition!!

Motivation

$M = e^{\alpha d}$ as a function of the gap d gas: He + iC4H10 = 94:6



MICROMEGAS: results and prospects : I. Giomataris

Gas gain M is at maximum
in the range of gaps between 30-100 μm

and its fluctuations are canceled

Is there a “Stability condition"
in the case of GEM?

Stability condition!!

Introduction

This study is performed to investigate the conditions under which the thickness dependence of the gas gain is constant.

Process

- Find the plateau in the thickness dependence of gas gain,
- Find the "Stability conditions", and
- Verify the theory by comparing with Garfield++.

- a toolkit for the simulation of gaseous detectors

Assumption

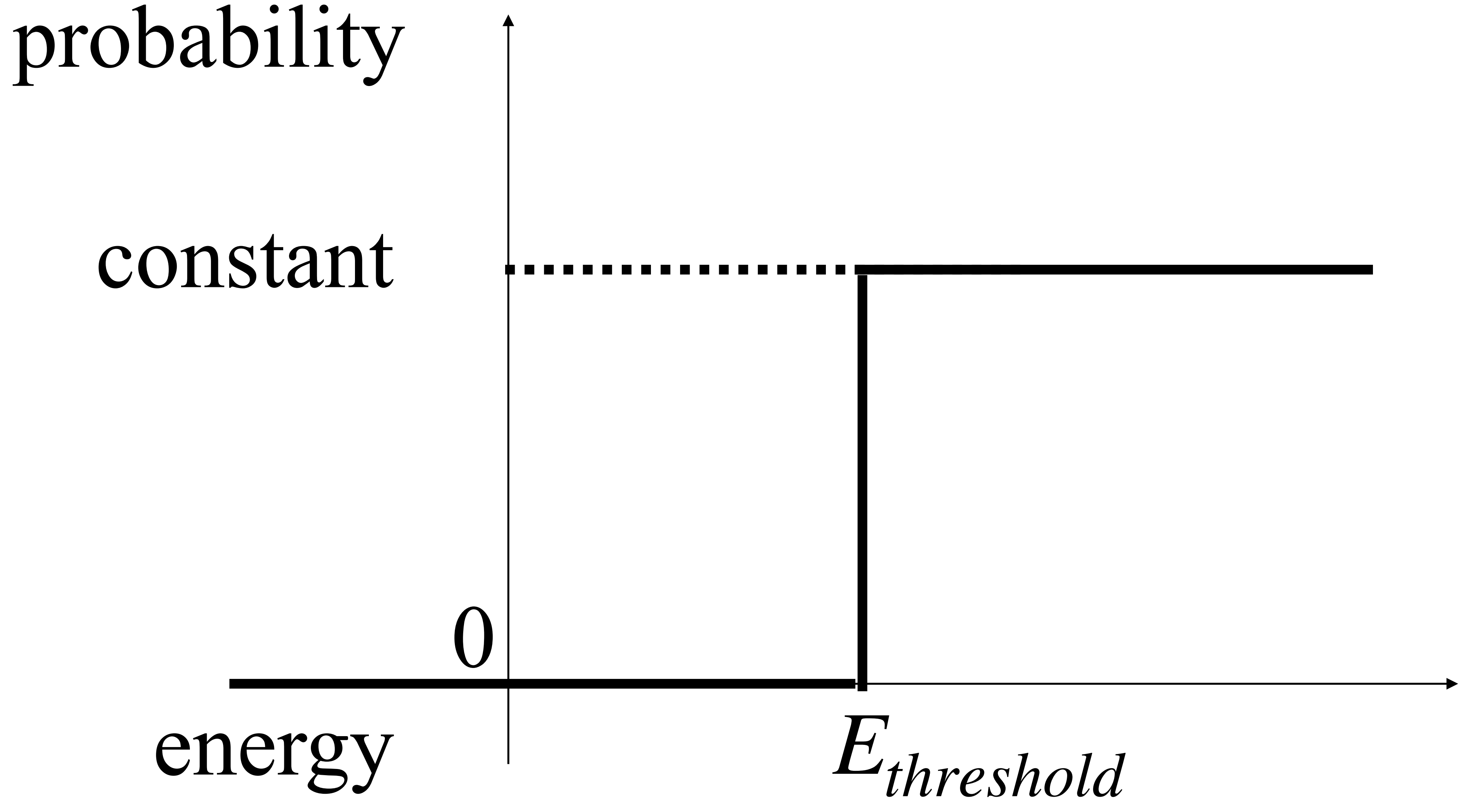
First, we assume that Legler's model[1] is correct

Legler's model have 2 assumptions

1. ionising collisions may occur <u>only after the electron flying over a minimum distance</u> so as to gain enough energy for ionisation from the E-field.

2. <u>the probability of ionising collision being constant</u>

   after the electron having reached the threshold energy like a step function



[1] STATISTICS OF ELECTRON AVALANCHES AND ULTIMATE RESOLUTION OF PROPORTIONAL COUNTERS

Theory

We have equation of gas gain variation $\frac{dG}{G}$

$$\frac{dG}{G} = \left(\frac{1}{1+\chi+\eta}\right)\left[1 - \frac{\epsilon}{\sigma_0}\left(\frac{\partial \sigma_0}{\partial \epsilon}\right)\right]\chi\,\delta\left(\frac{d\Delta}{\Delta}\right)$$

where

$$\epsilon = \frac{E}{n},\ E = \frac{V/\Delta}{n},\ \delta = \frac{V}{U_0},\ \eta = n\Delta\frac{U_0}{V}\sigma_0(\epsilon),\ \chi = \frac{\ln G}{\delta}$$

Δ : thickness of GEM

for stable operation,

$\frac{dG}{G} = 0$ is required

the coefficients can be deleted by choosing these parameters.

Therefore, we have the “Stability condition”

$$\frac{\partial \sigma_0}{\partial \epsilon} = \frac{\sigma_0}{\epsilon}$$

σ_0 :effective cross section

ϵ : scaling variable =E/n

Theory

We have equation of gas gain variation $\frac{dG}{G}$

$$\frac{dG}{G} = \left(\frac{1}{1+\chi+\eta}\right)\left[1-\frac{\epsilon}{\sigma_0}\left(\frac{\partial\sigma_0}{\partial\epsilon}\right)\right]\chi\,\delta\left(\frac{d\Delta}{\Delta}\right)$$

where $\epsilon = \frac{E}{n}$, $\underline{E = \frac{V}{\Delta}}$

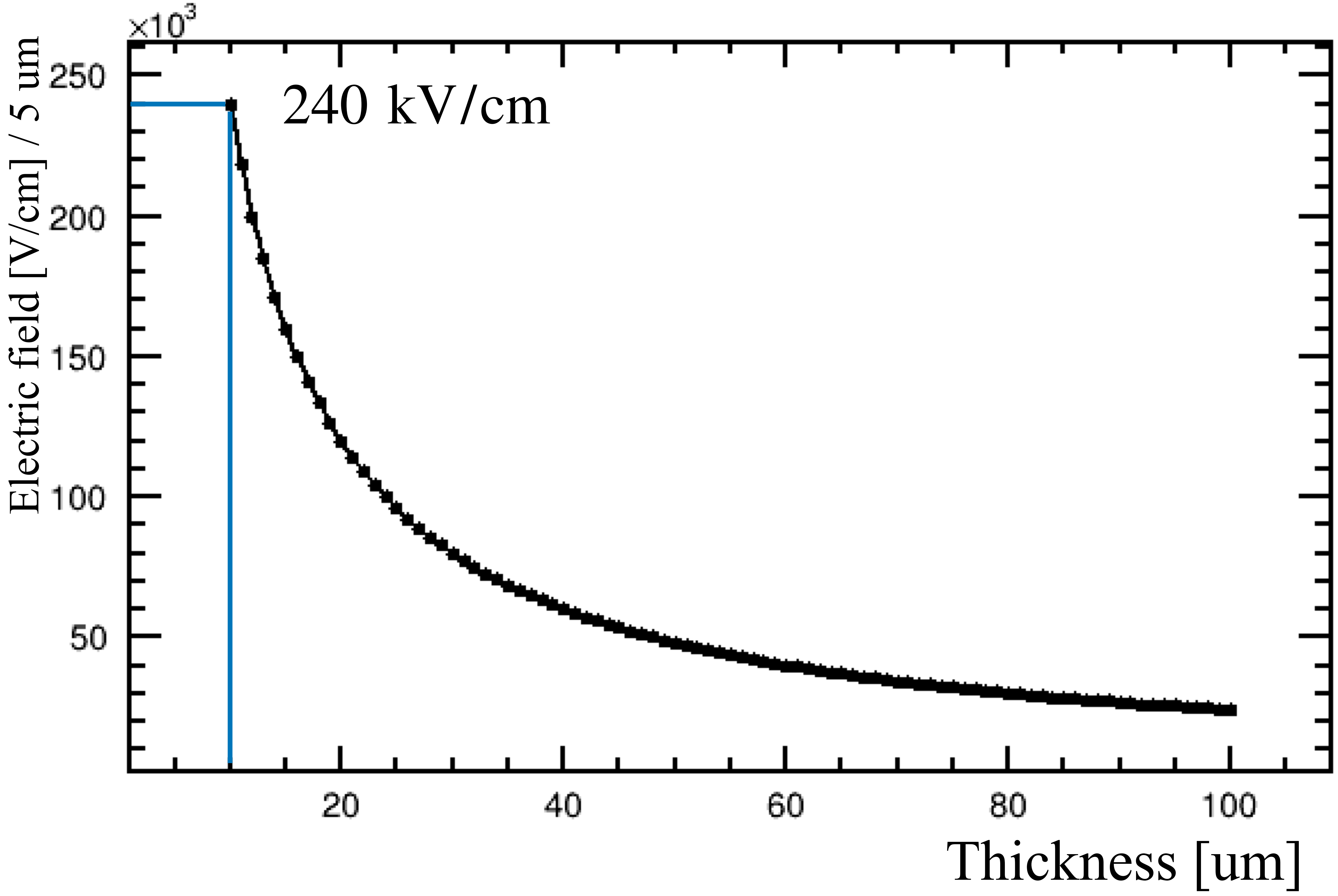
<u>uniform</u> electric field

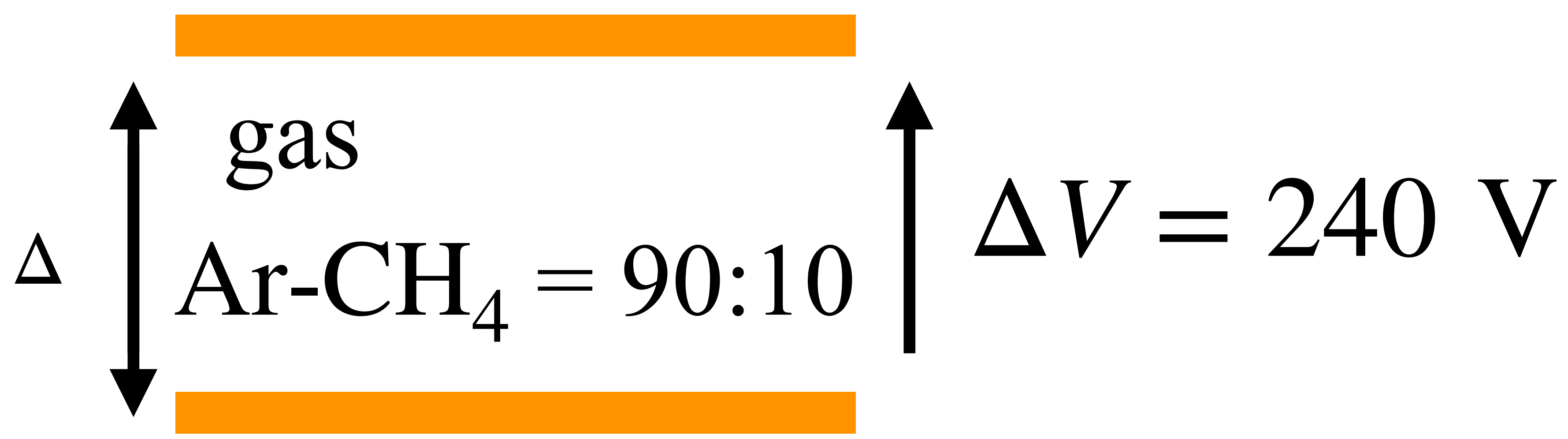
We need to check whether our theory is only correct in the <u>uniform</u> electric field or not.

⟹Parallel plate geometry

Thickness vs Electric Field

Parallel plate geometry





The gap of parallel plates

Δ : 10 μm ~ 70 μm

$$E = \frac{V}{\Delta}$$

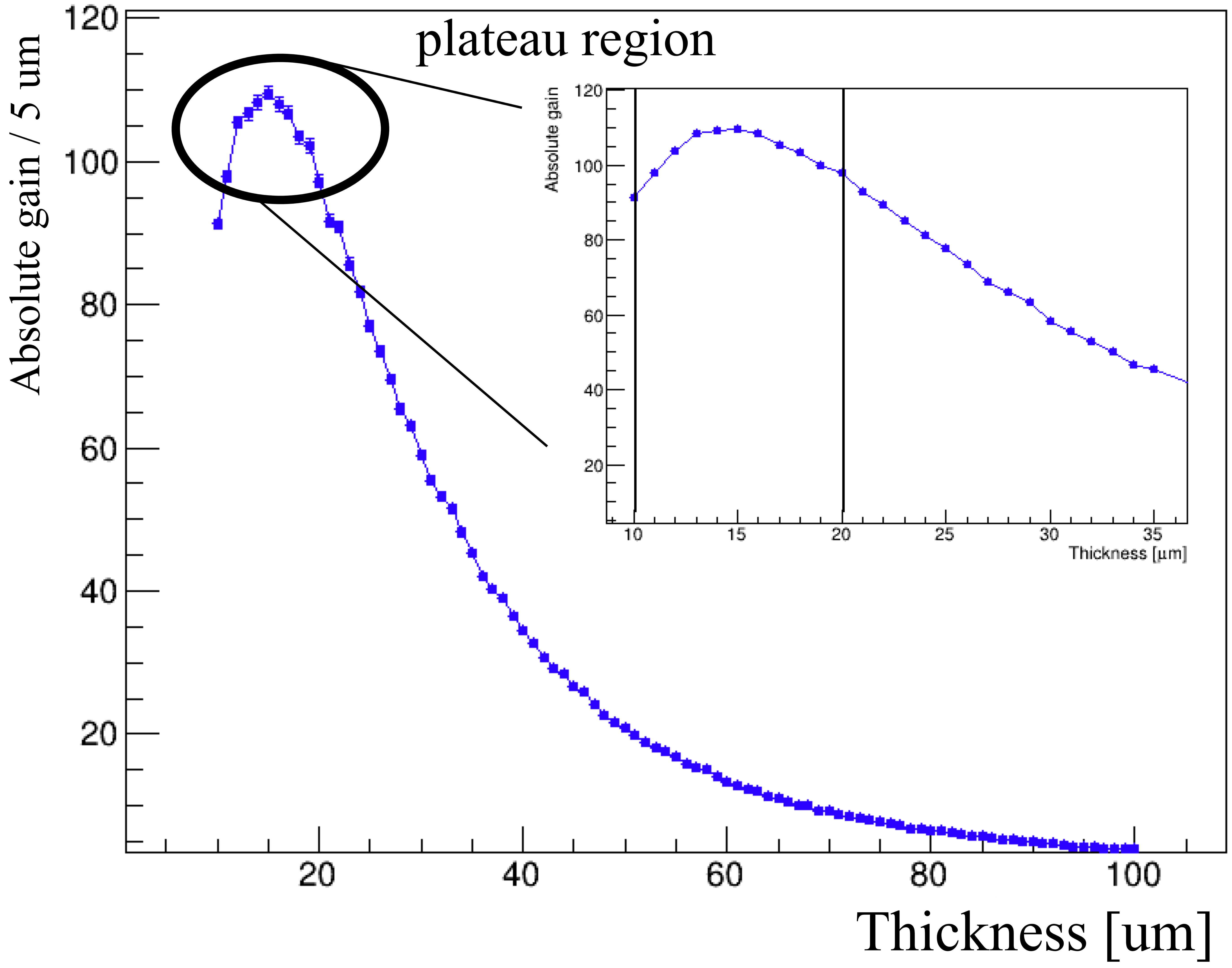
$$\Delta = 10\ \mu m: \quad E = \frac{240}{10 \times 10^{-4}}\ \frac{\mathrm{V}}{\mathrm{cm}}$$

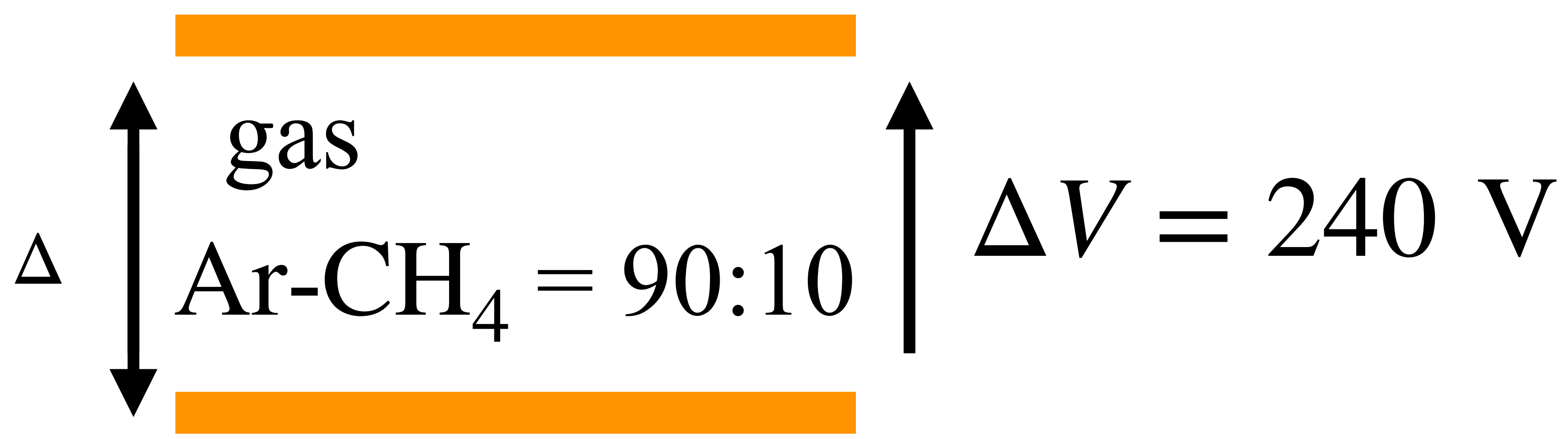
$$= 240\ \mathrm{kV/cm}$$

uniform electric field

Thickness dependence of gas gain

Parallel plate geometry



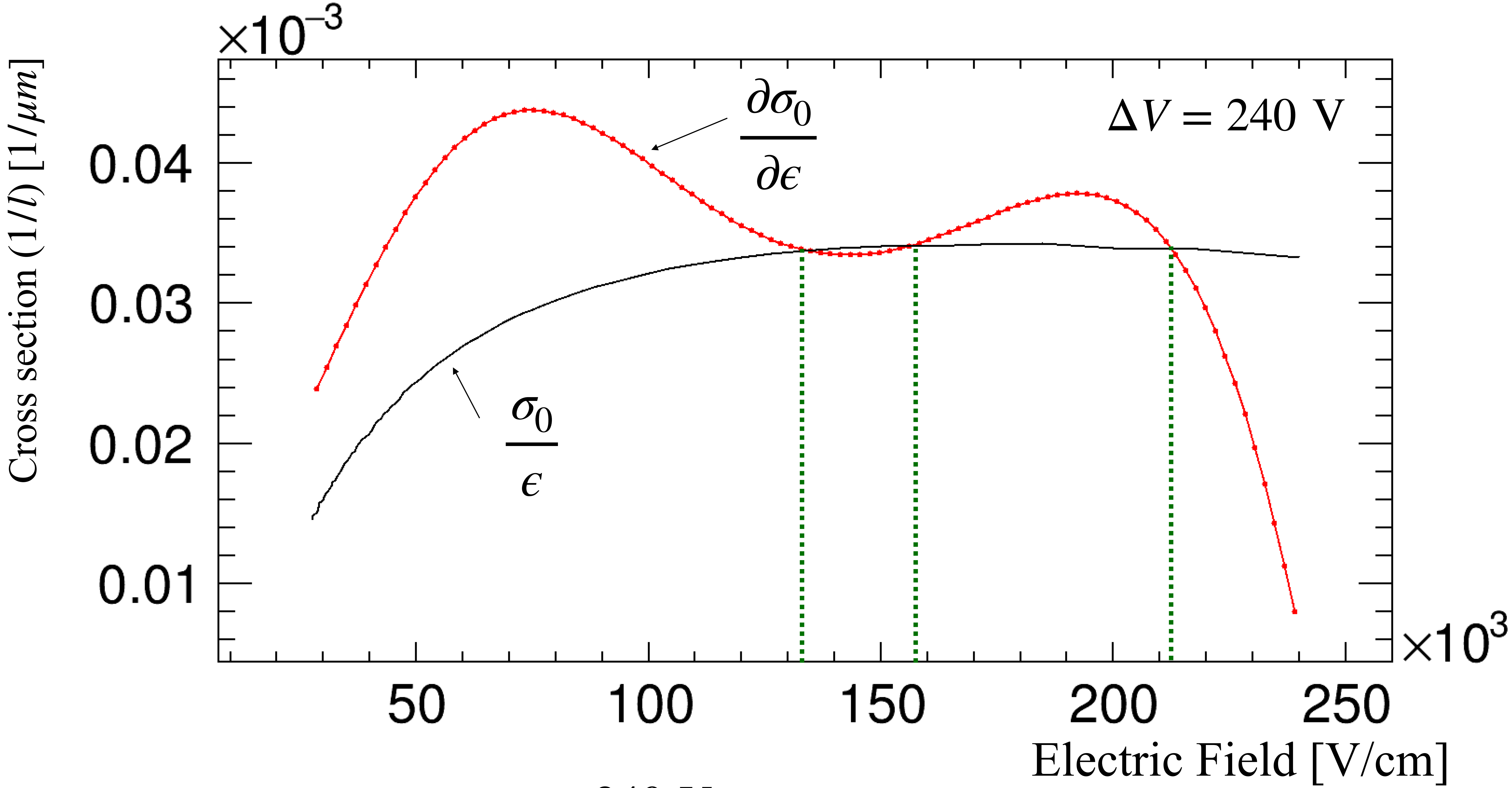


The gap of parallel plates

Δ : 10 μm ~ 70 μm

The plateau area was found in the range of 10 μm ~ 20 μm

Result



The plateau area was found in the range of 10 μm ~ 20 μm

$$\frac{240\ \text{V}}{130\ \text{kV/cm}} \sim 18\ \mu m$$

this intersection point correspond to the thickness

Stability condition is satisfied!!

$$\frac{\partial\sigma_0}{\partial\epsilon} = \frac{\sigma_0}{\epsilon}$$

Asian GEM geometry

In the case of parallel plate, “Stability condition” was satisfied

$$\frac{\partial \sigma_0}{\partial \epsilon} = \frac{\sigma_0}{\epsilon}$$

in the plateau region

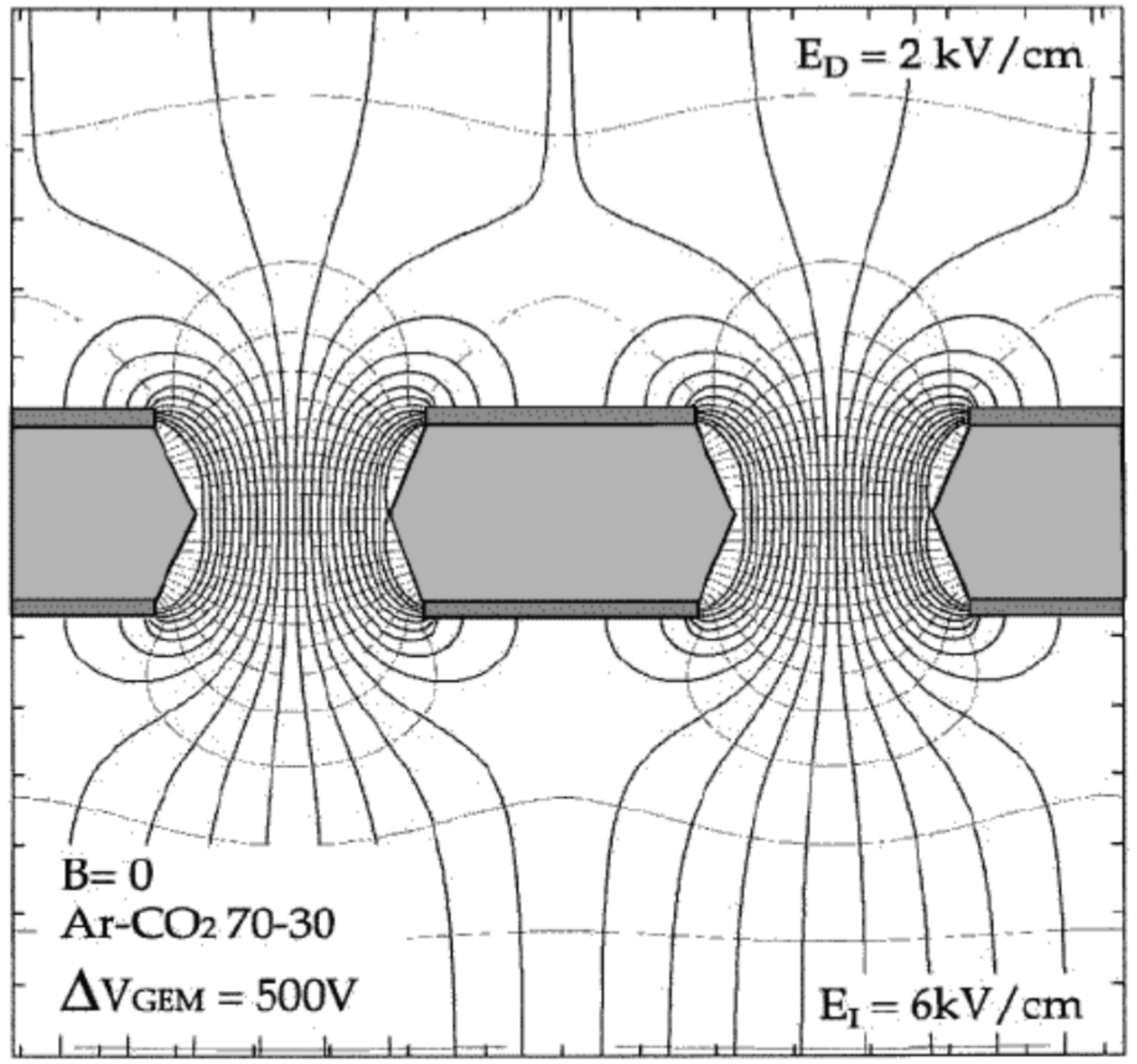
Next Step

In the case of GEM

we found the equation

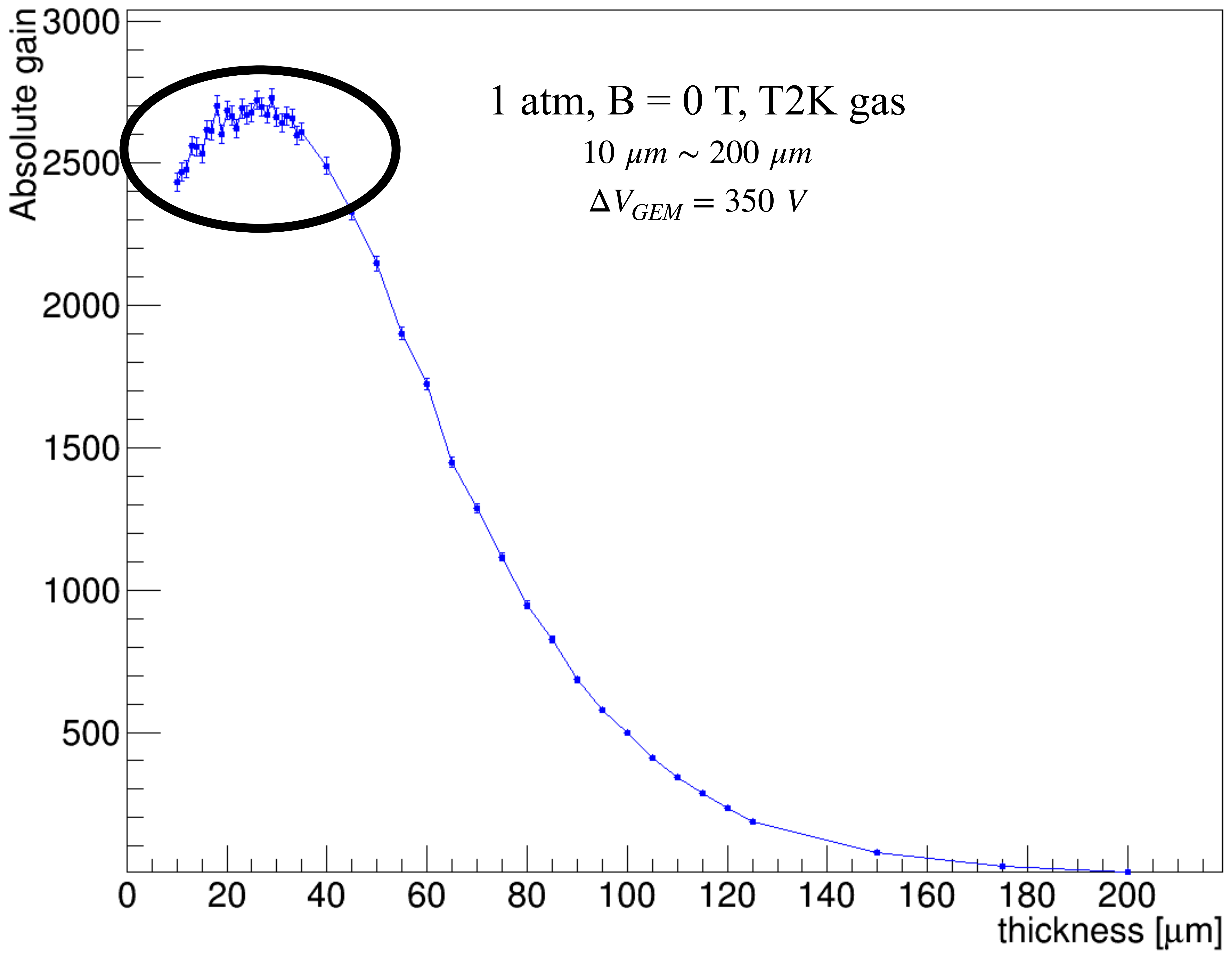
$$\frac{dG}{G} = \left(\frac{1}{1+\chi+\eta}\right)\left[1 - \frac{\epsilon}{\sigma_0}\left(\frac{\partial \sigma_0}{\partial \epsilon}\right)\right]\chi\,\delta\left(\frac{d\Delta}{\Delta}\right)$$

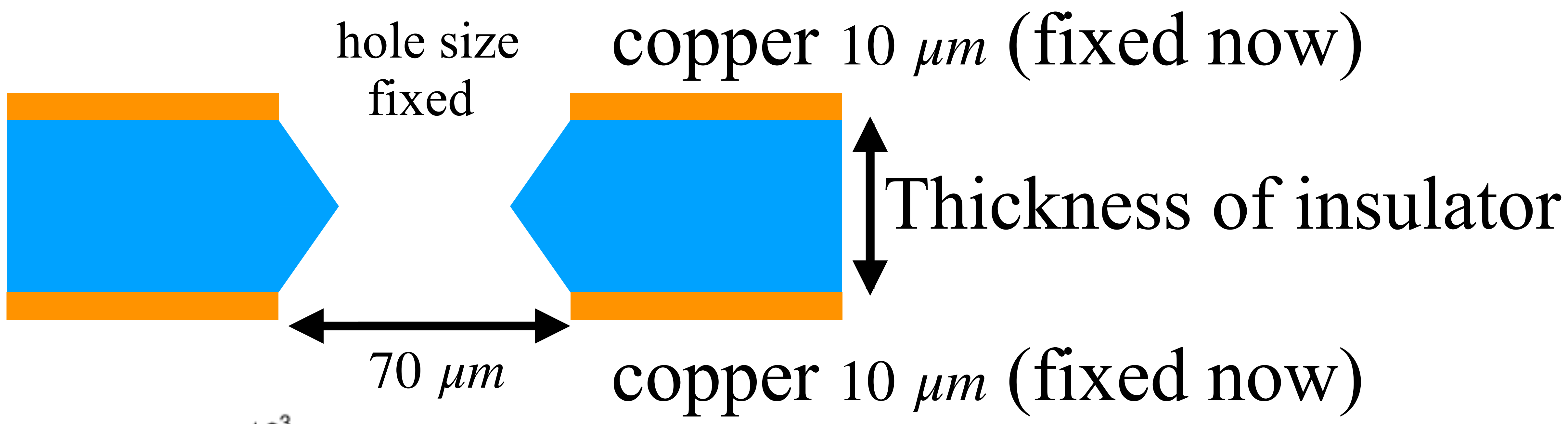
by assuming the <u>uniform</u> electric field

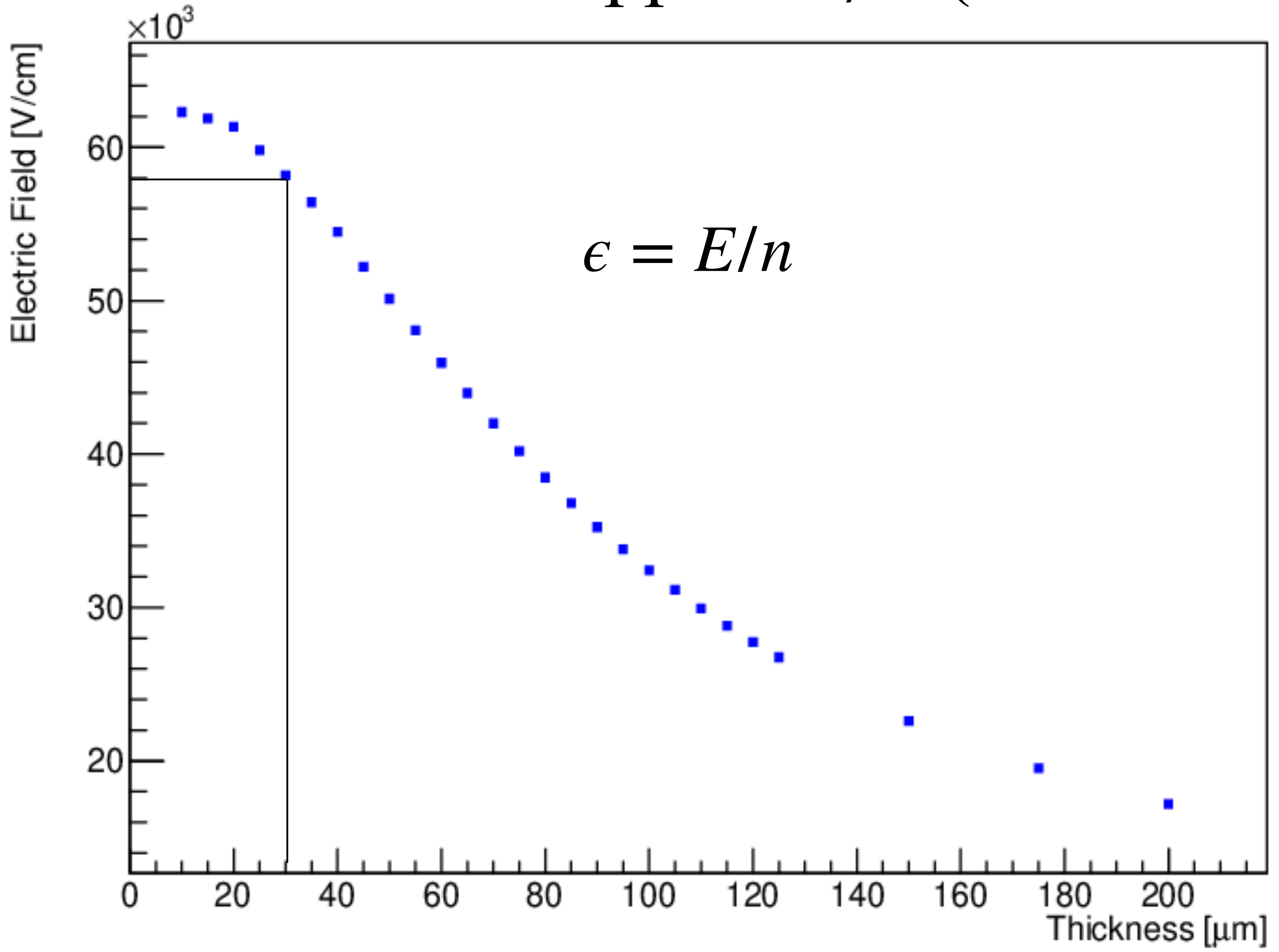


Question: How well does uniform E-Field approximation work inside the GEM hole?

Thickness dependence of gain:Asian GEM





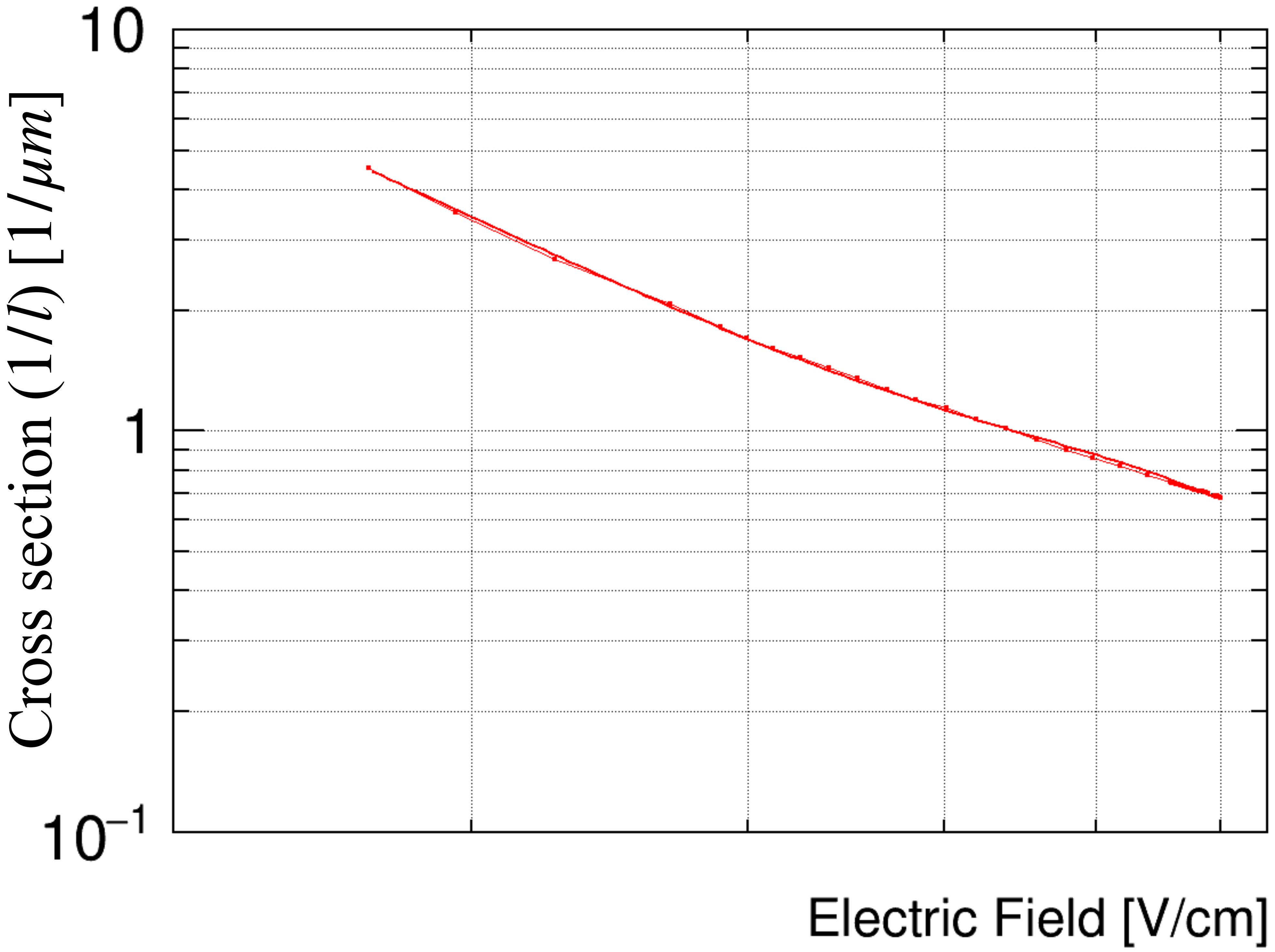


The plateau area was found in the range of $10\ \mu m \sim 40\ \mu m$

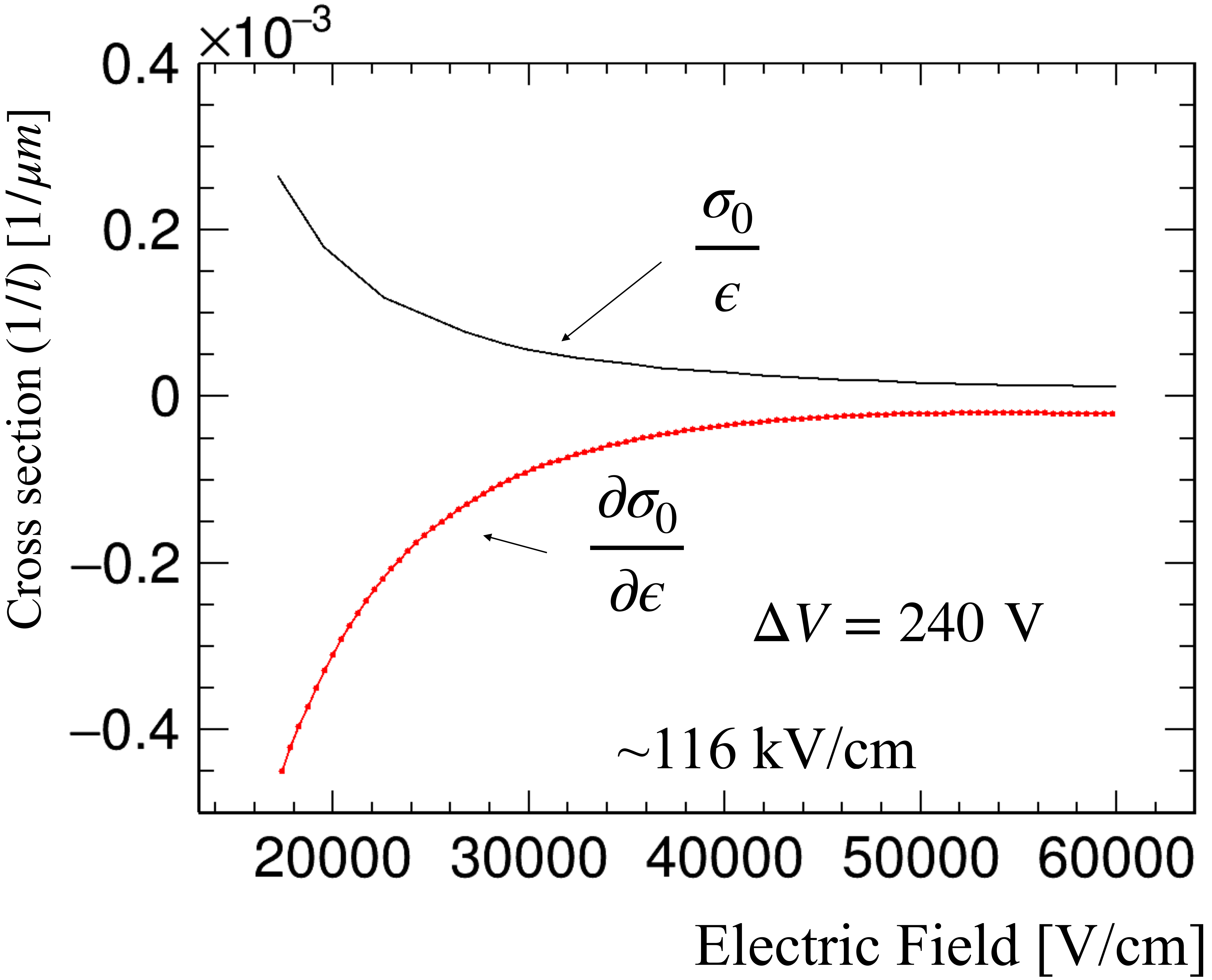
cf. CERN GEM:thickness 50 μm ($E \sim 60$ kV/cm)

e.g. VCI2019, Francesco Fallavollita

Result



The plateau area was found
in the range of 10 $\mu m \sim$ 40 μm



NO intersection point was found

→ Is uniform Electric field assumption not correct?

Alkhazov's Theory

more generally $$J(1) = \int_0^\infty dl\ p_i(l)\ e^{-\alpha l} = \frac{1}{2}$$

Once $p_i(l)$ is decided, α will be decided. (α is functional of $p_i(l)$)

└── model-dependent

- Legler's model : $p_i(l) = a_i\ e^{-a_i(l-x_0)}\ \theta(l - x_0)$
- Snyder's model: $p_i(l) = \alpha\ e^{-\alpha l}$

p_i depends on l and other variable β

$$G = \exp(\Delta \cdot \alpha[p_i(l, \beta)])$$

where

Δ: thickness of GEM

Calculation of Townsend coefficient α

the probability for the 1st ionizing collision

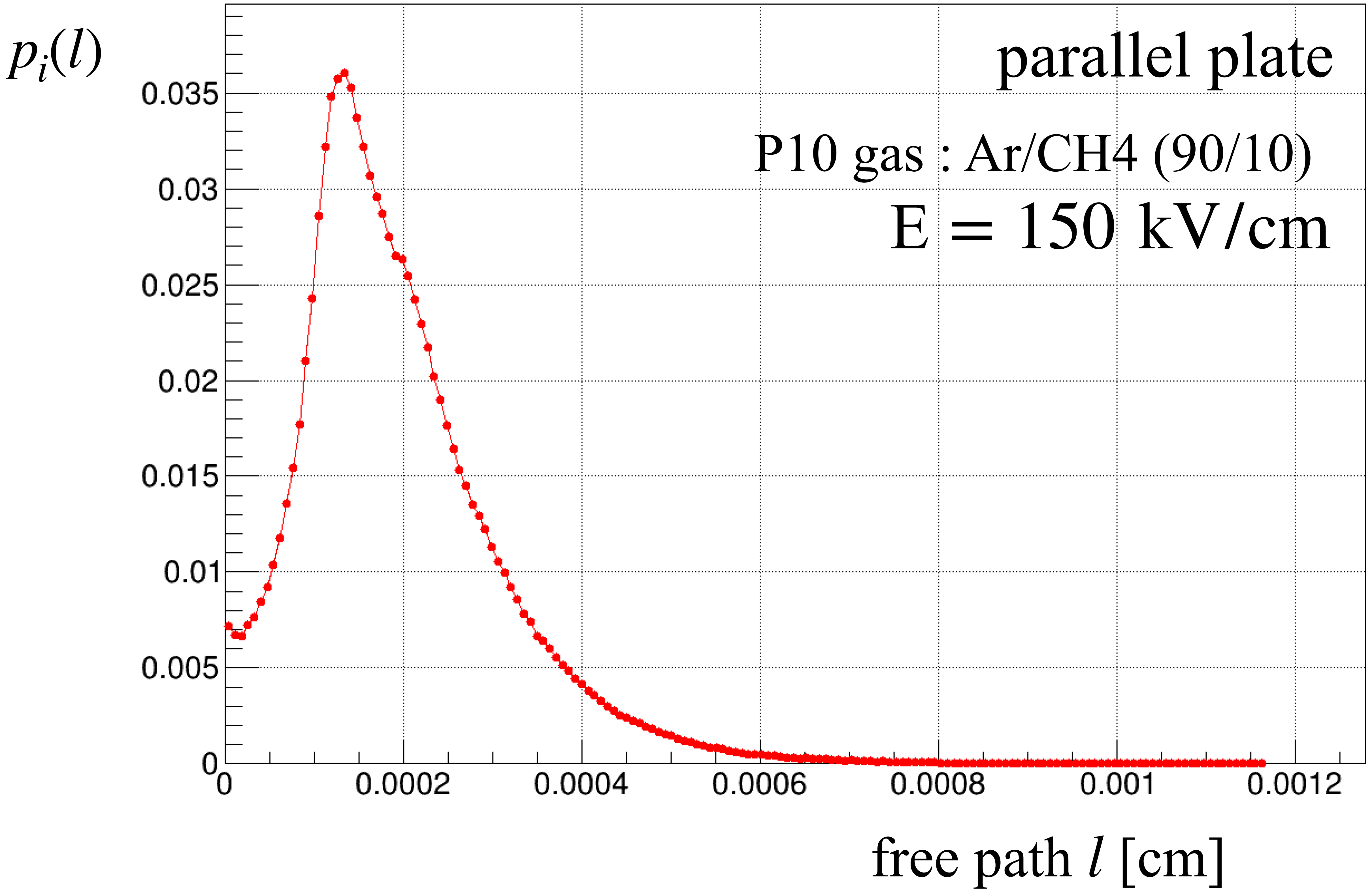
$$J(1) = \int_0^\infty dl\ p_i(l)\ \exp^{-\alpha l} = \frac{1}{2}$$

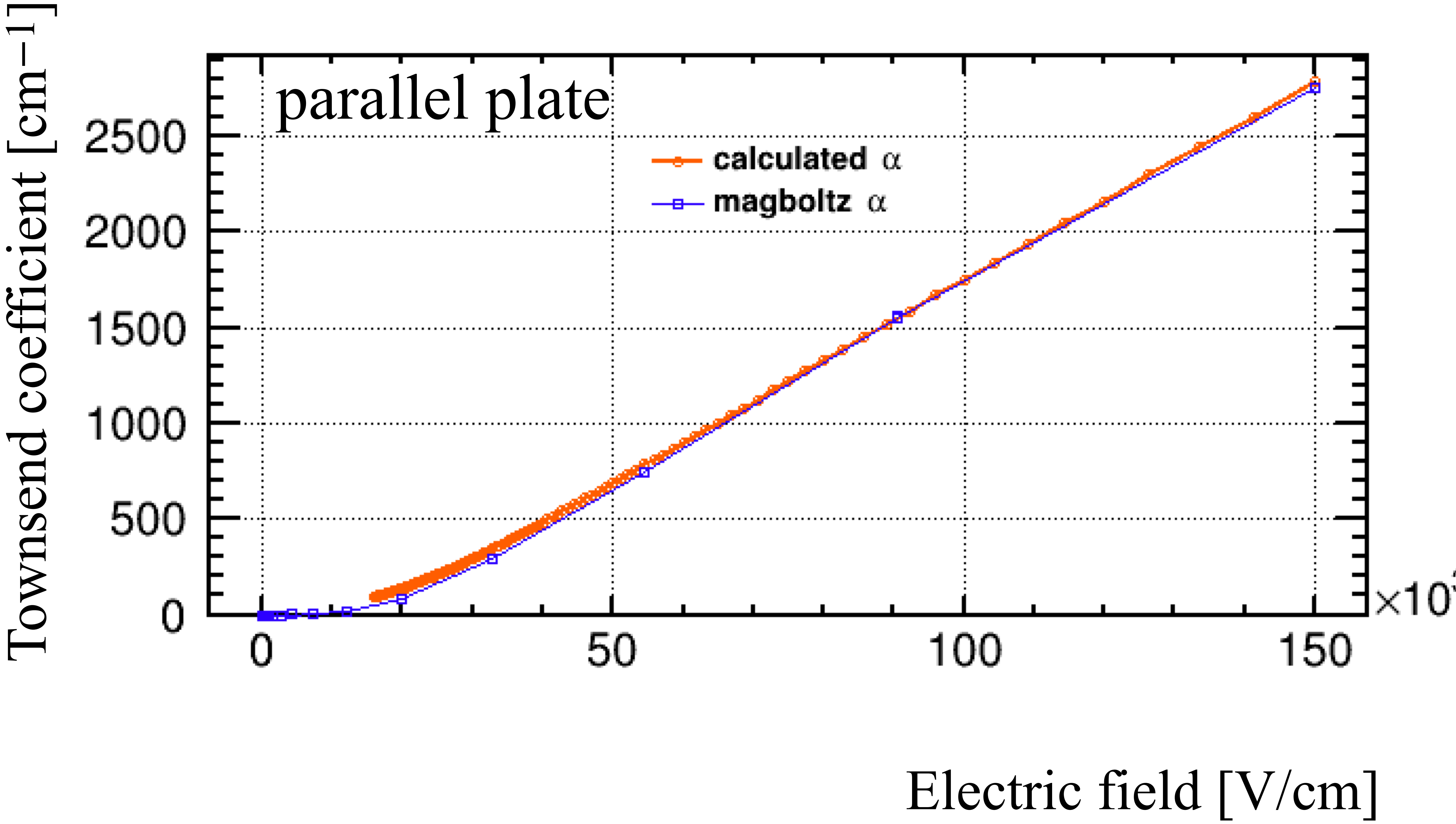
α : Townsend coefficient
l : Free path

$\alpha, l, p_i(l)$ depend on electric field

The value of α which makes the integral $\frac{1}{2}$ is the α for that electric field strength.

$p_i(l)$: the probability of 1st ionising collision taking place at the distance l from the origin of the seed electron.





Analytically calculated result is reasonable agreement with magboltz result!

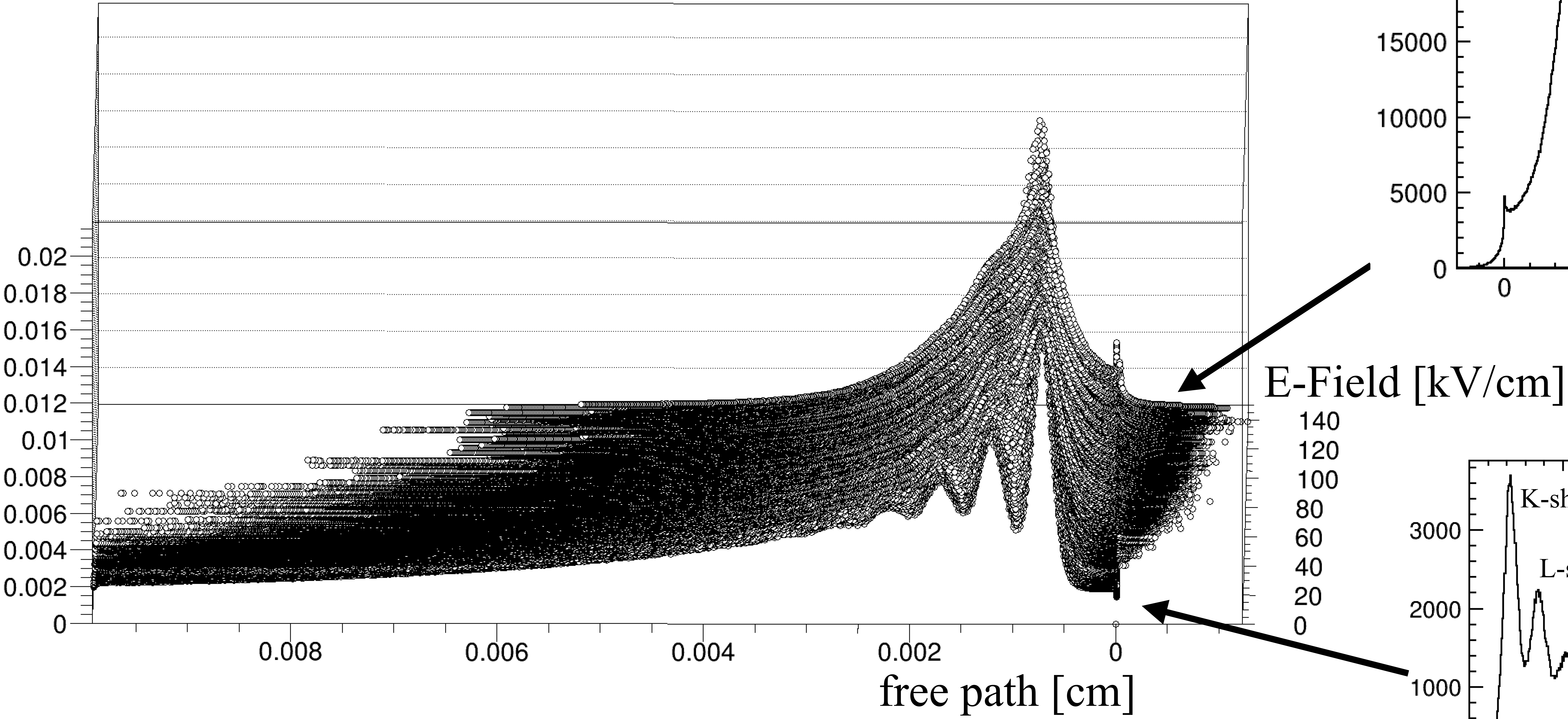
Data set for GEM

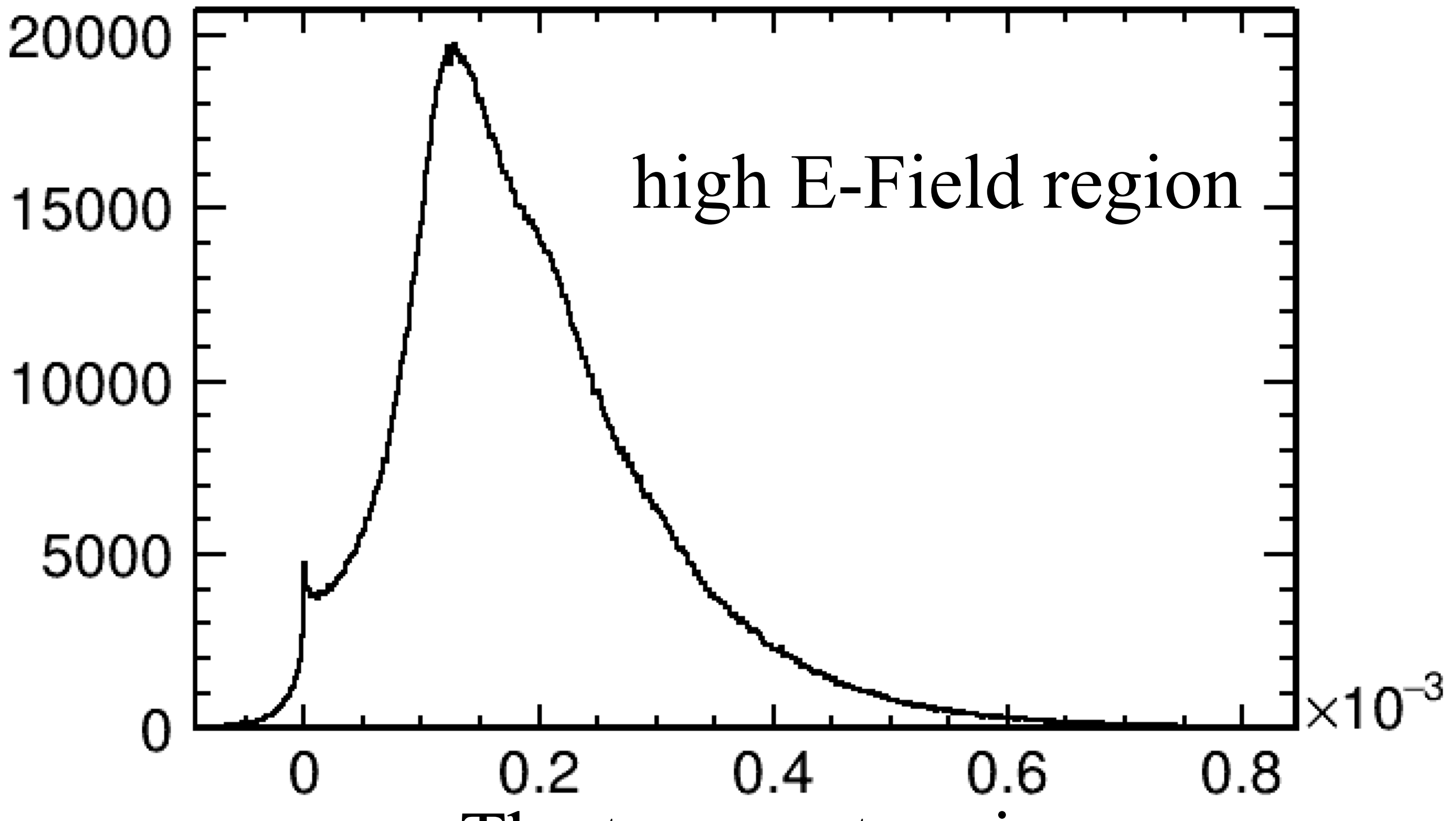
The data set was prepared using a parallel plate with different gaps from 16 um to 130 um

applied voltage V = 240 V

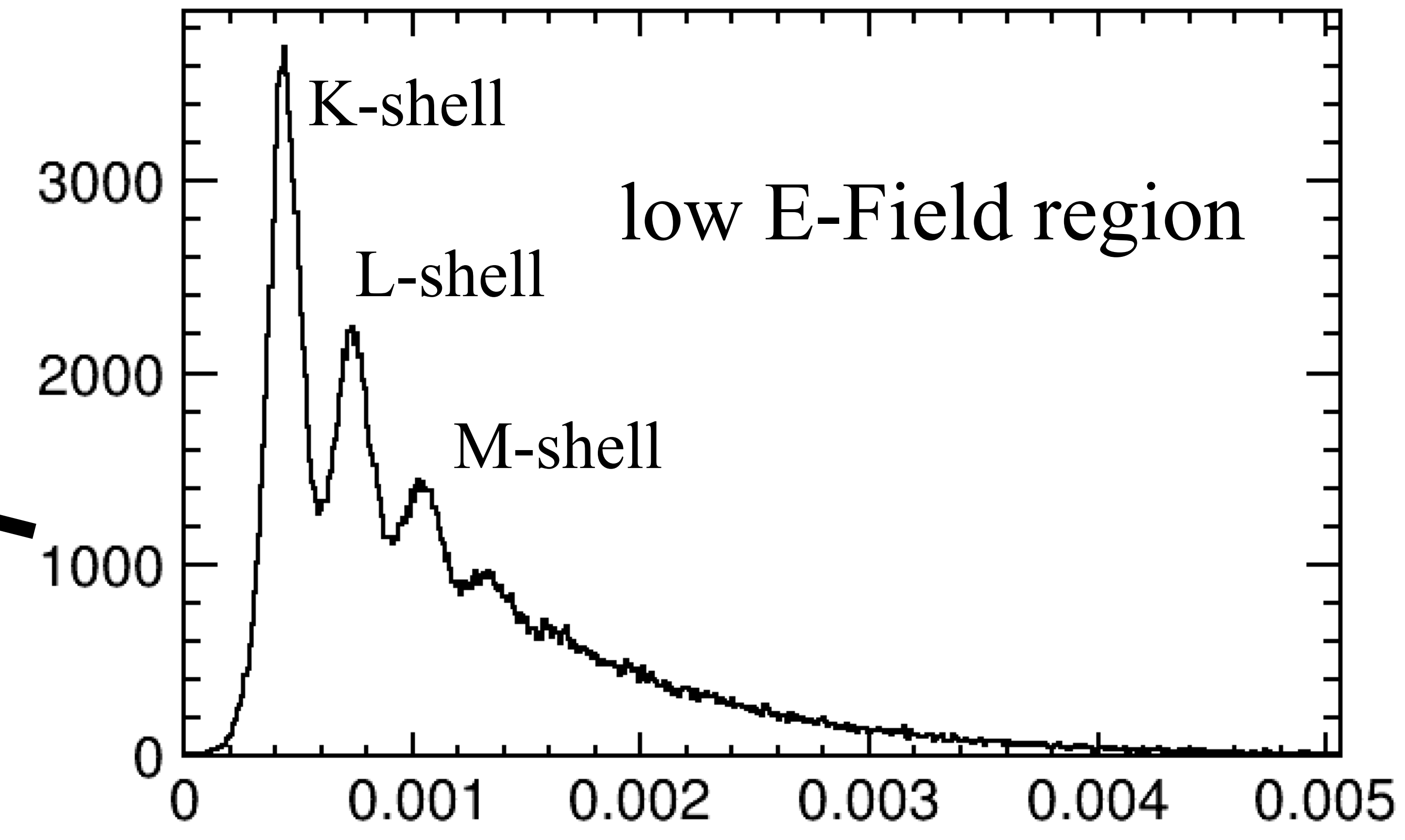
P10 gas : Ar/CH_4 (90/10)

z axis: $p_i(l)$the probability for the 1st ionizing collision



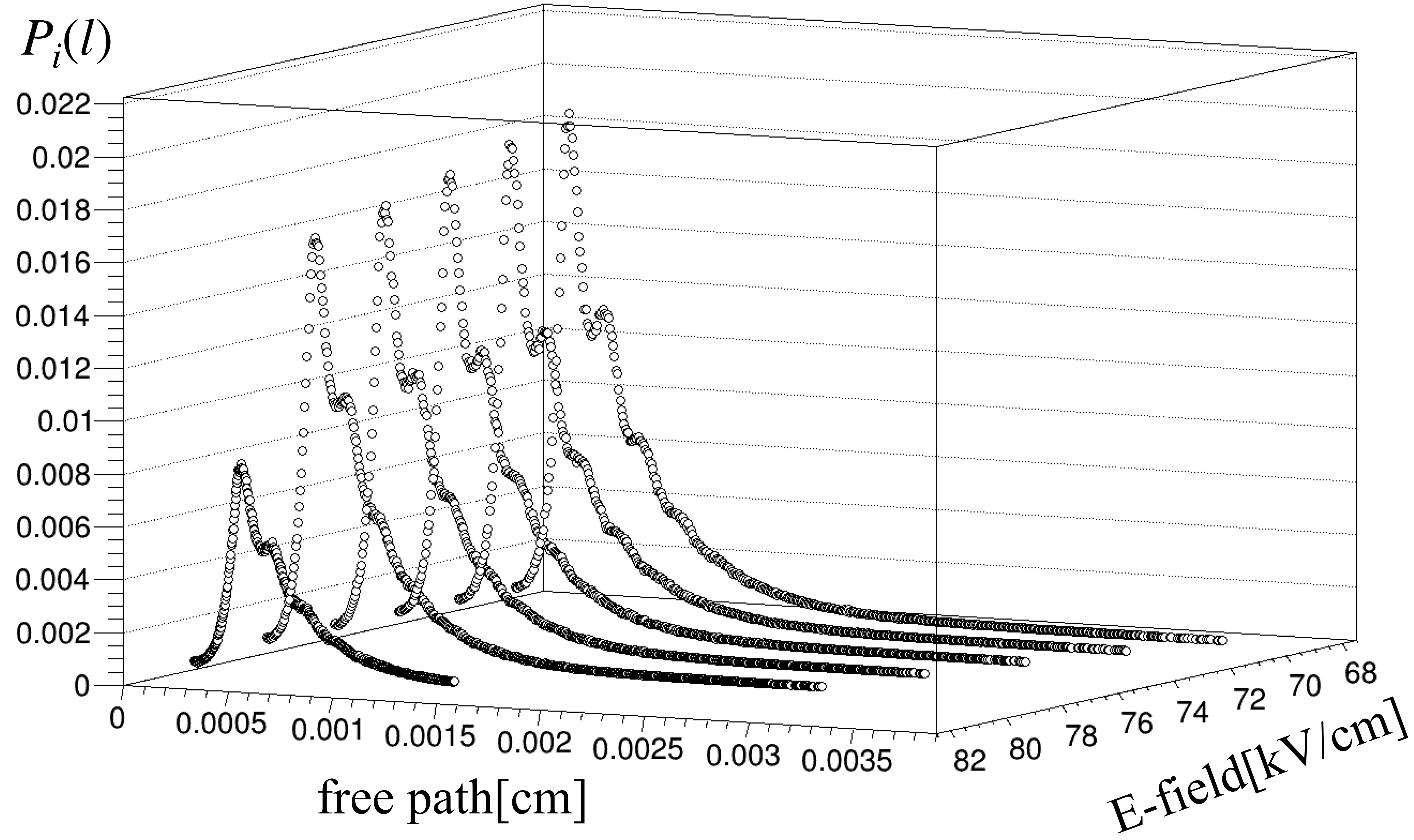


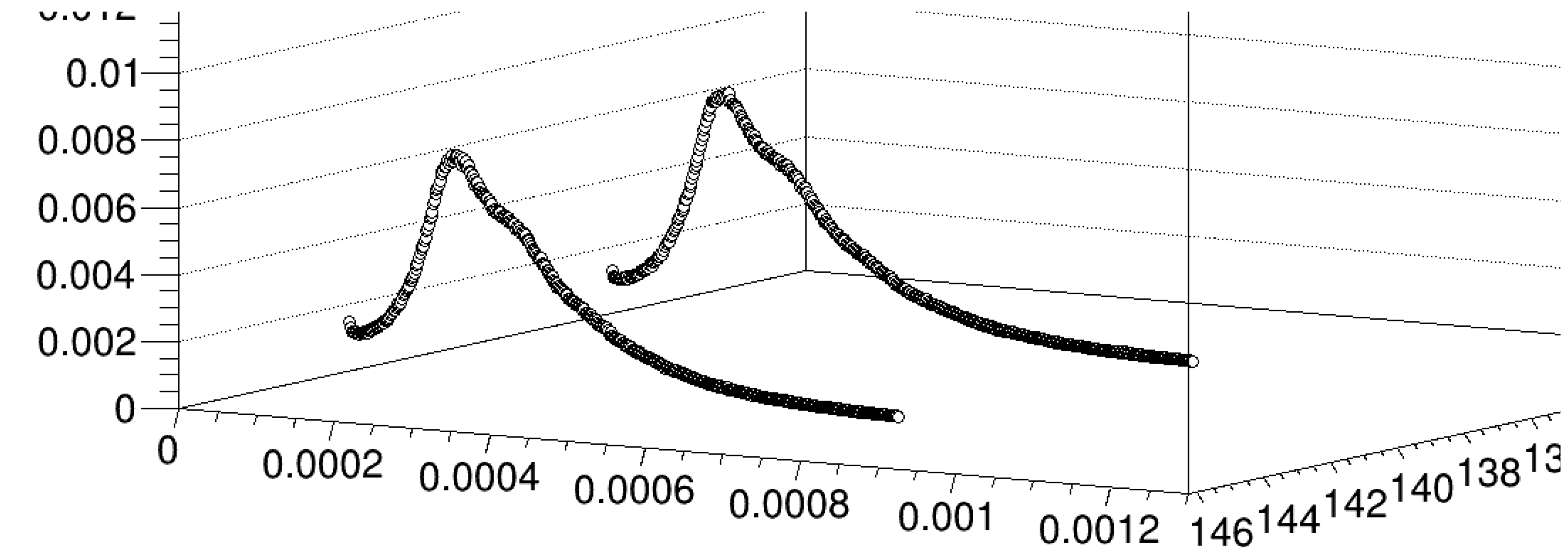
The temperature increases
and the distribution is smeared



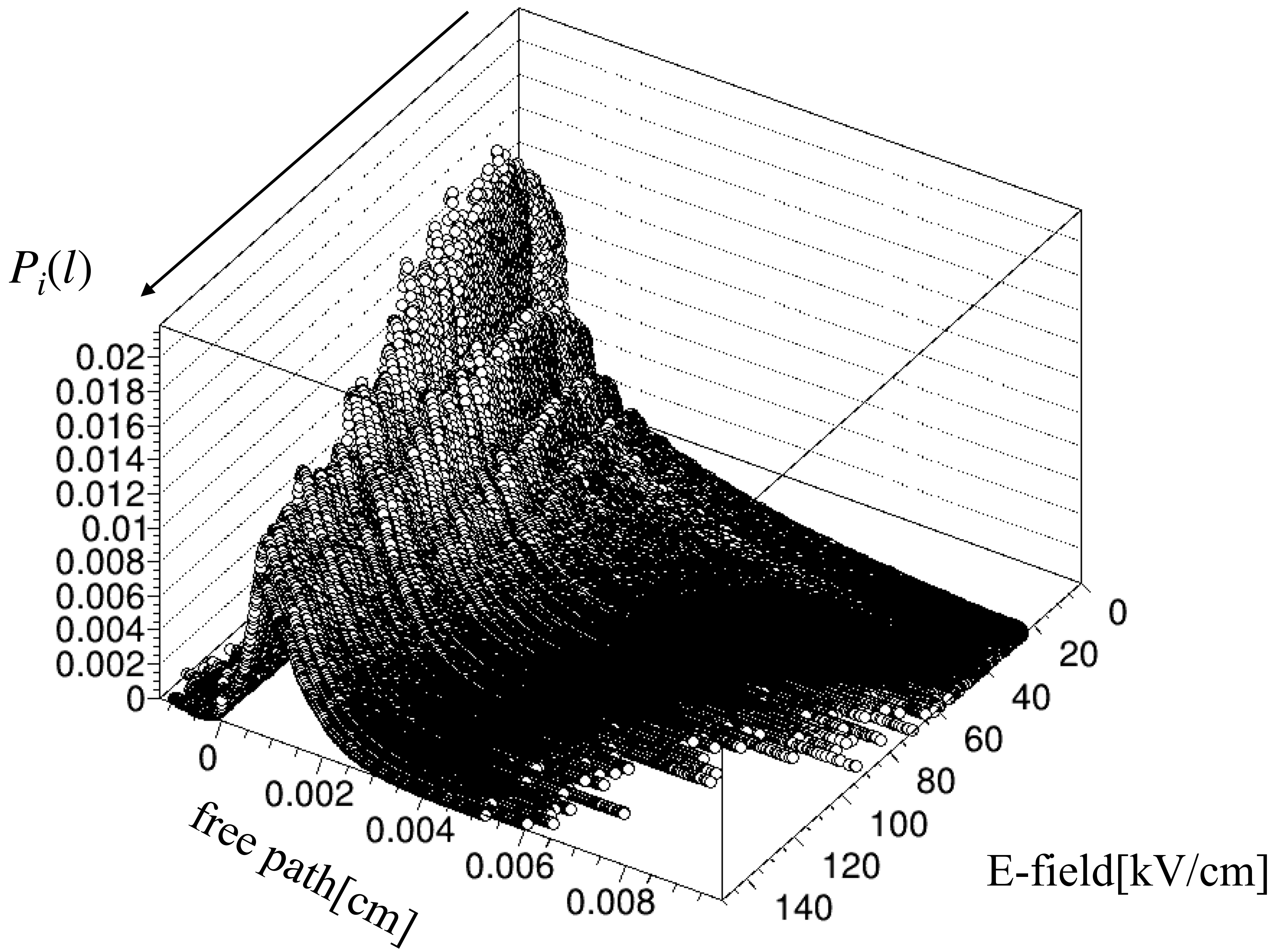
$p_i(l)$ as a function of E, l

$p_i(l)$: the probability of 1st ionising collision taking place at the distance l from the origin of the seed electron.





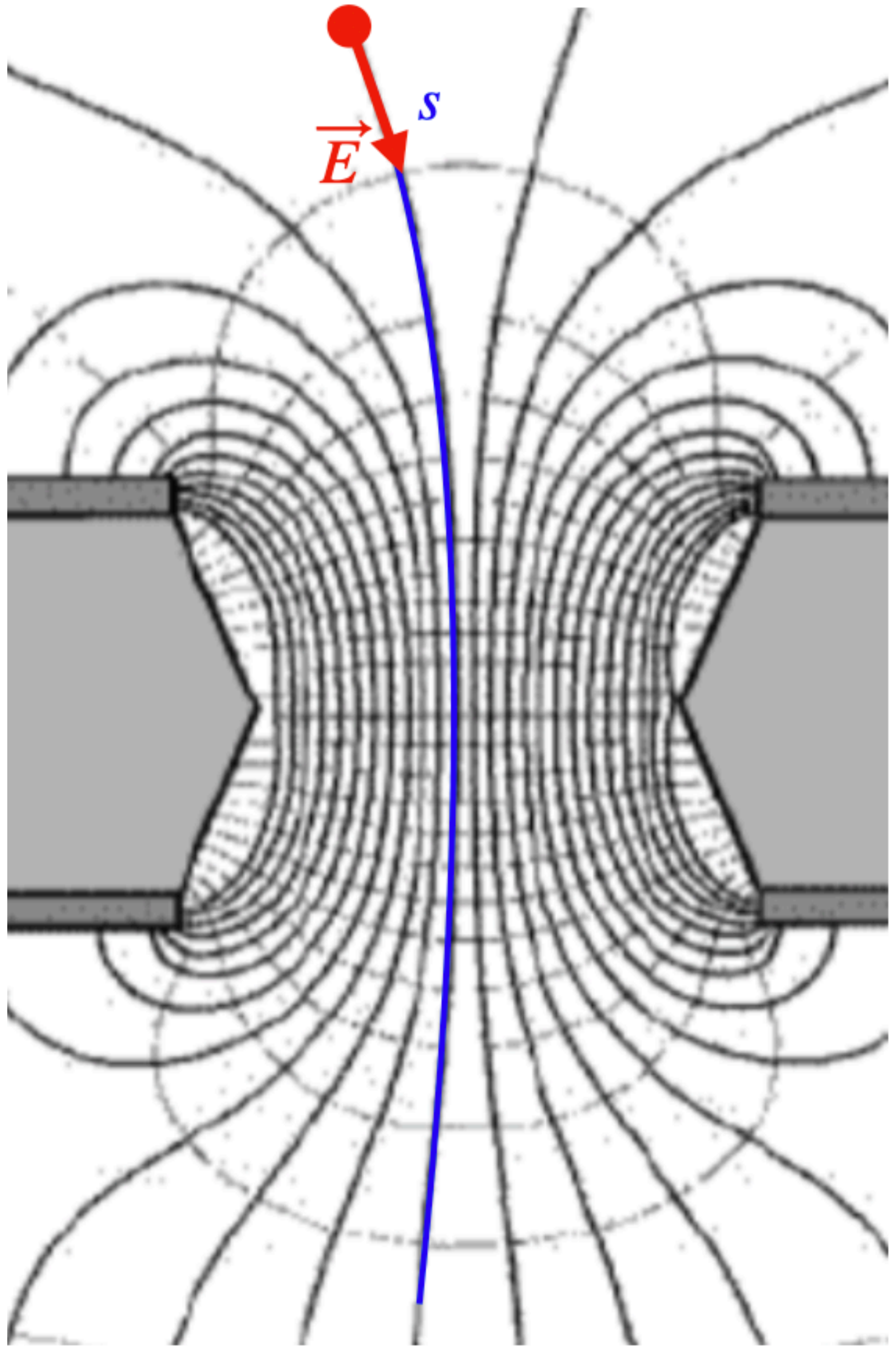
As the E-field increases,
the range of free path and $p_i(l)$ decreases



The strong electric field heats up the electrons,
making it difficult to see their structure.

Theory: GEM geometry

Inside Garfield++, the gas gain is calculated as $G = \exp \int_0^{\infty} \alpha(E)\, ds$ along path of electron



①Calculate the electric field $E(s, x_0, y_0)$

②Get the value of α

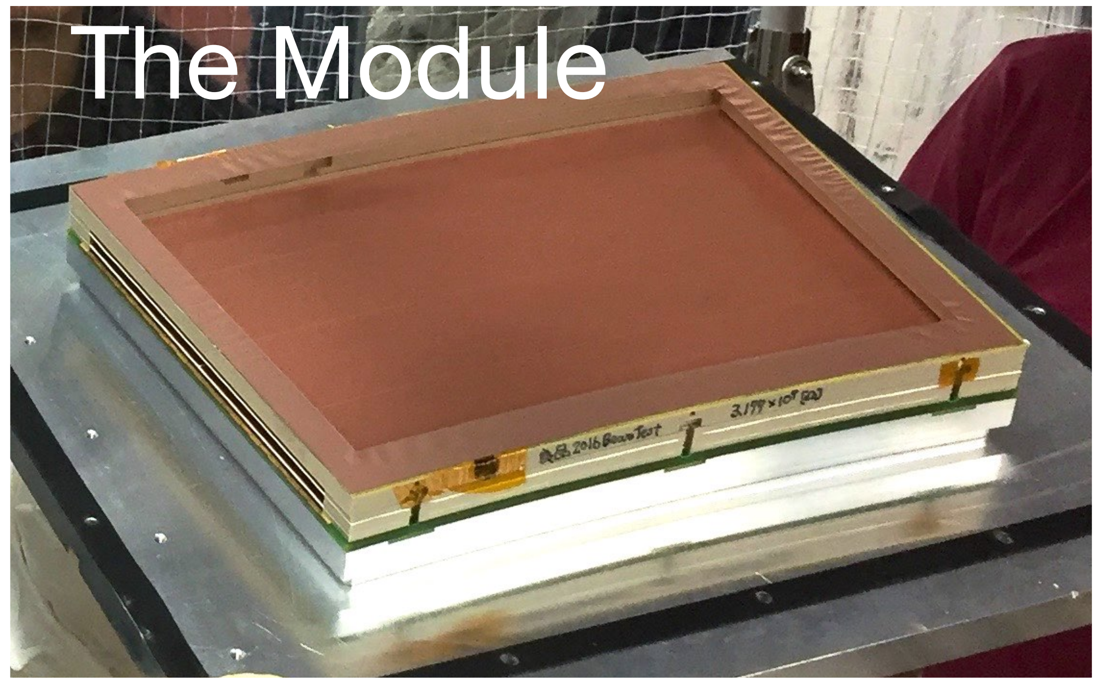
③Calculate $\int_0^{\infty} ds\ \alpha\left(E(s, x_0, y_0)\right)$ along the electric field lines s

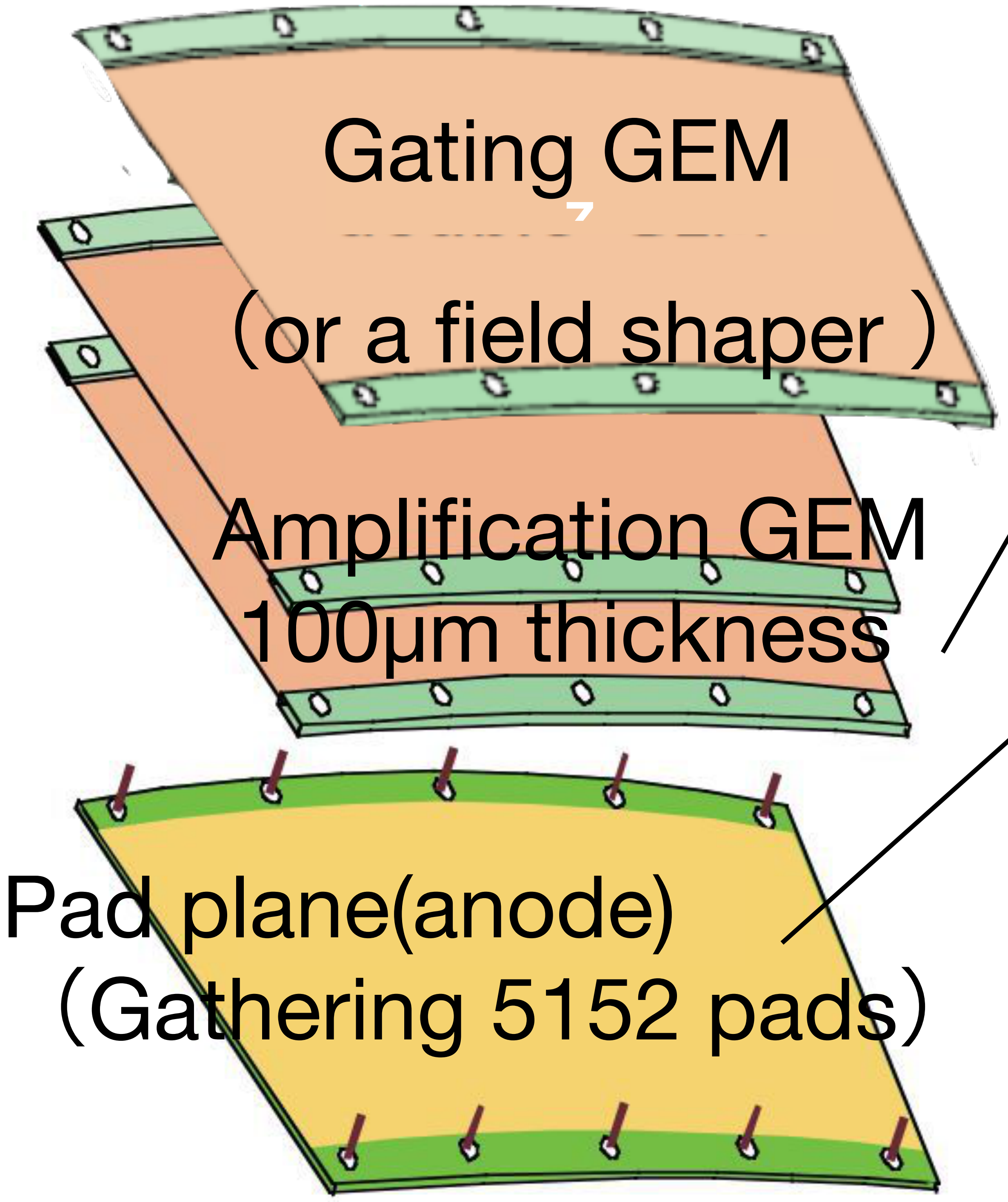
If we use the α for all possible values of the electric field inside the GEM hole, <u>the value of the gas gain should match the Garfield++ output</u>

On going

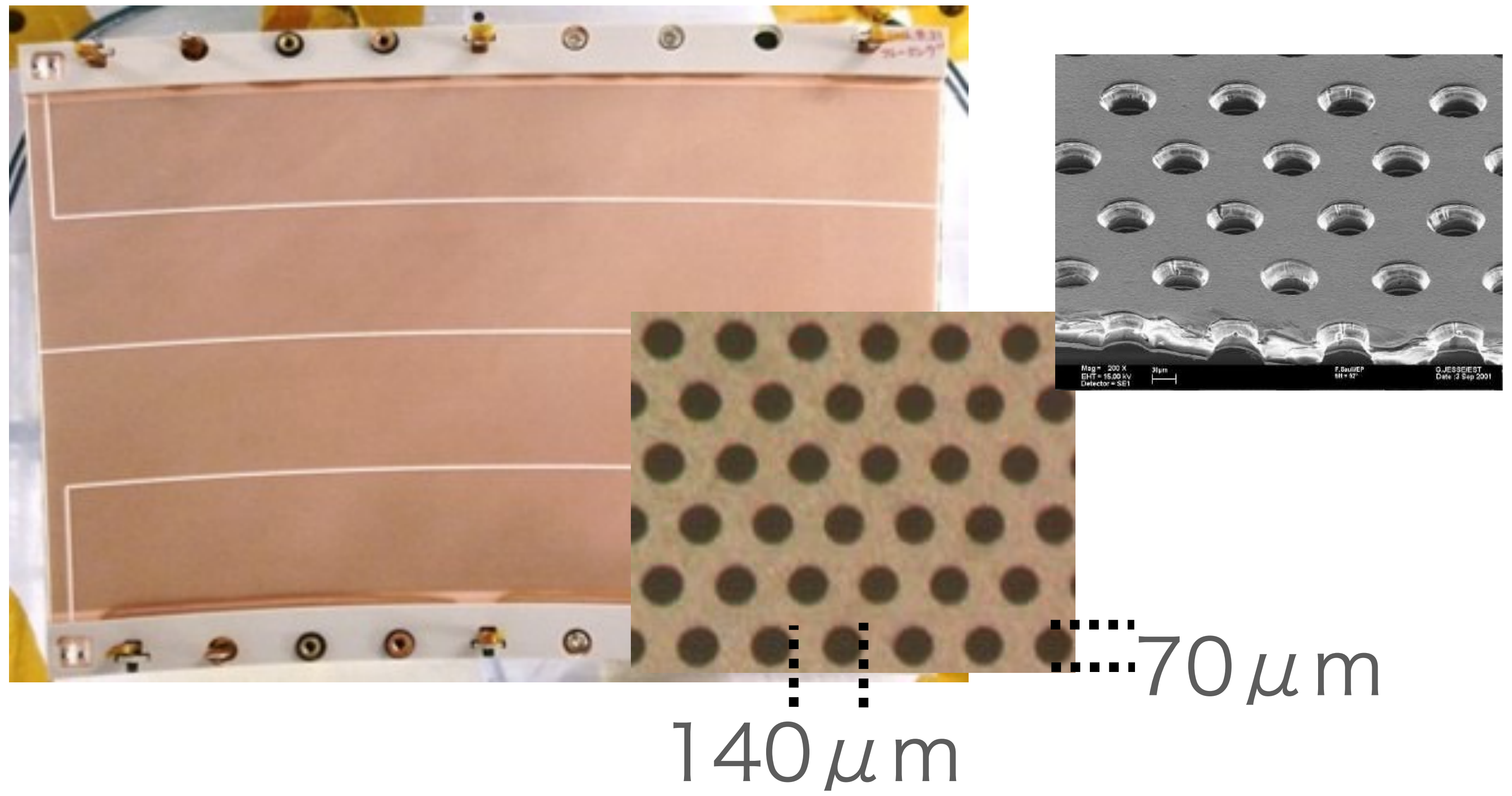
Summary and plan

- Derived the equation of gas gain variation and found the "Stability conditions"
- Found the plateau in the thickness dependence of gas gain
- As a functional of $p_i(l)$, the value of Townsend coefficient α when the integral is 1/2 was found to be in reasonable agreement with the result of magboltz.
- Calculate the gain analytically by using these results,.



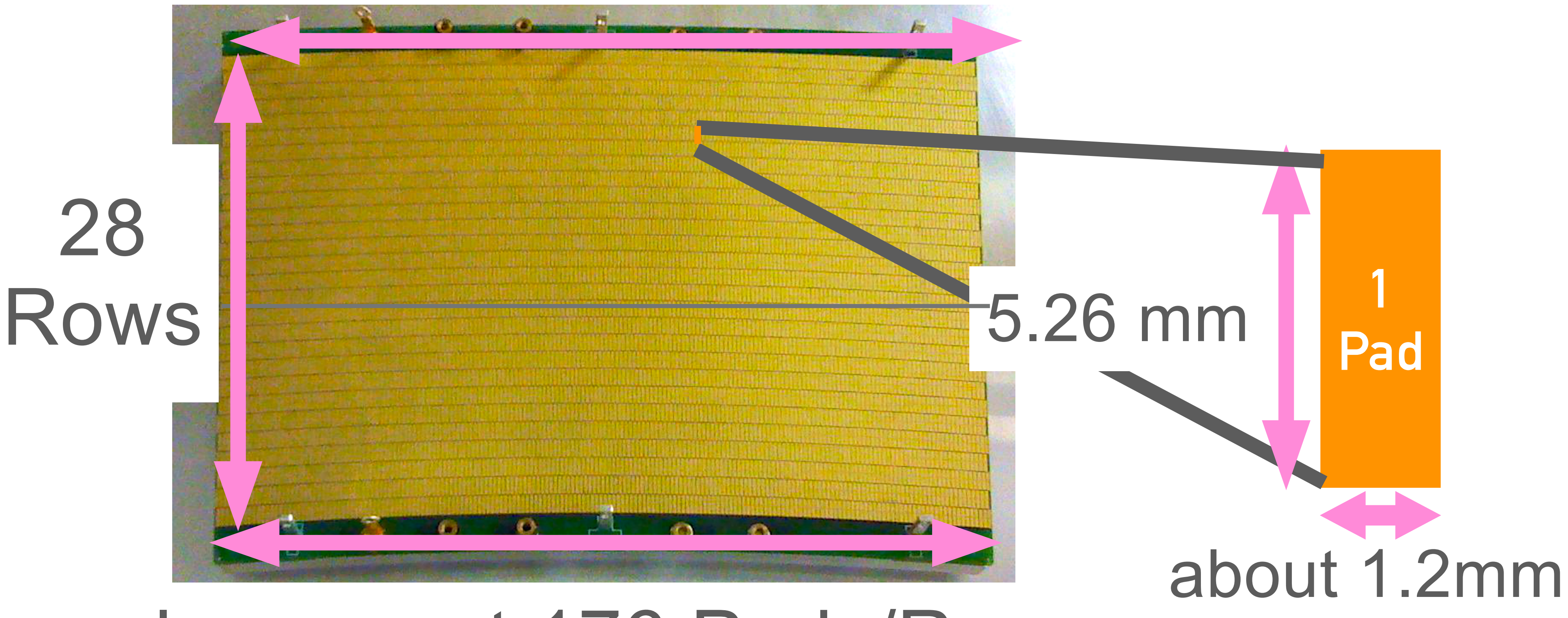


Gas Electron Multiplier



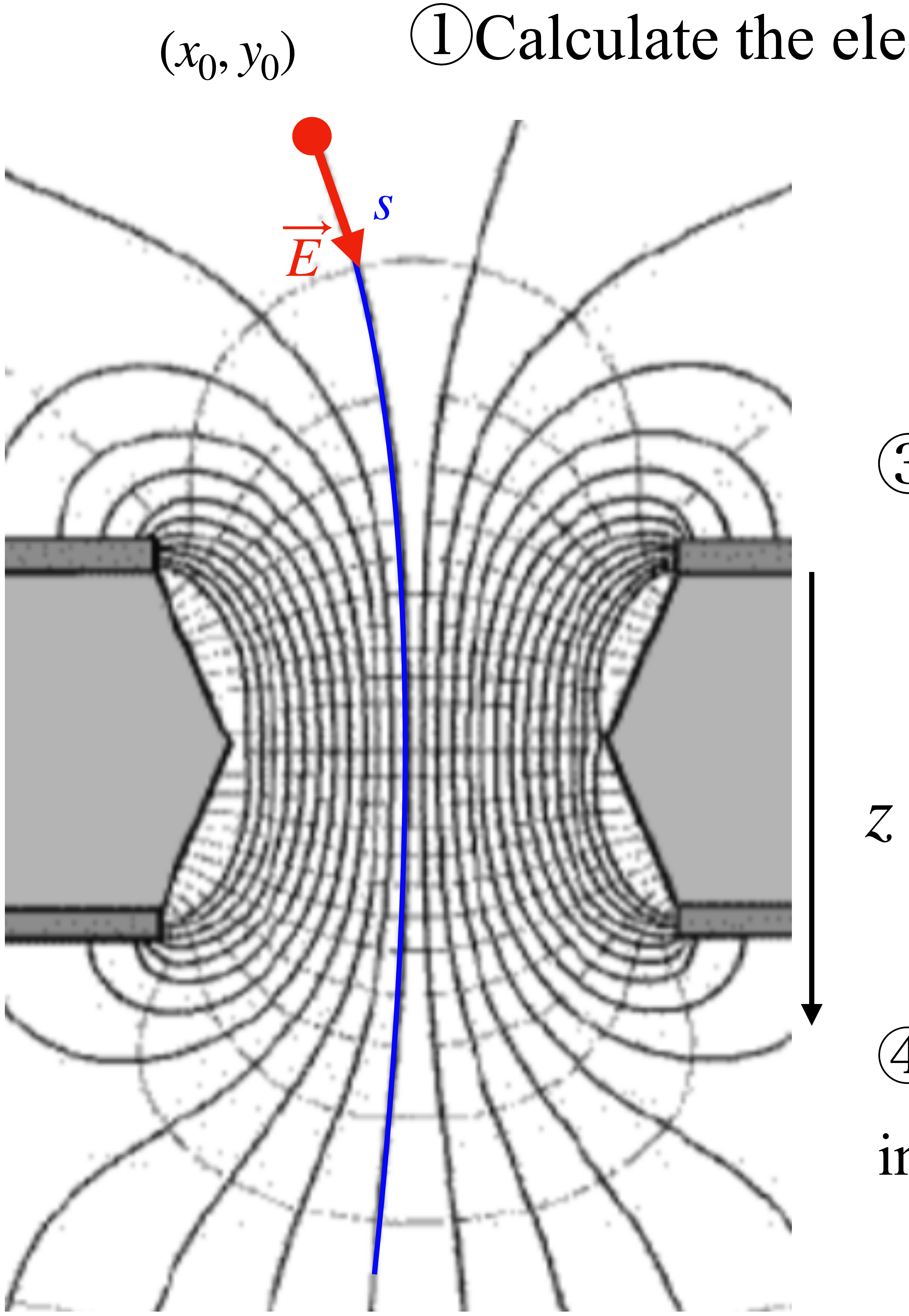
Pad Plane (anode)

Upper part :192 Pads/Rows



lower part:176 Pads/Rows

T2K gas Ar : CF_4 : iC_4H_{10} = 95 : 3 : 2



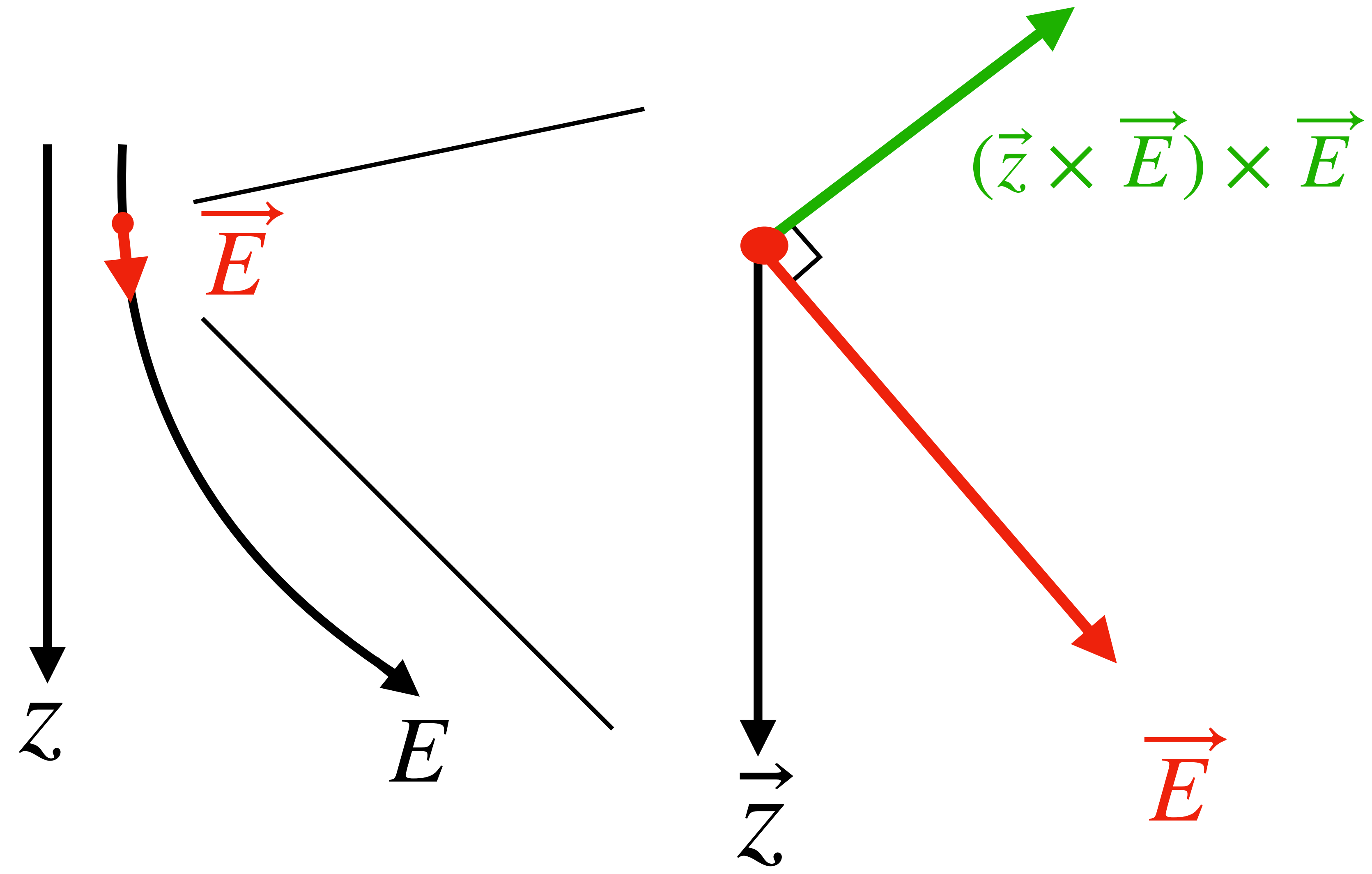
①Calculate the electric field $E(s, x_0, y_0)$

②Get the value@E

- Mean free path l
- Transverse diffusion D_T
- Townsend coefficient α

$$\sigma = D_T\sqrt{l}$$

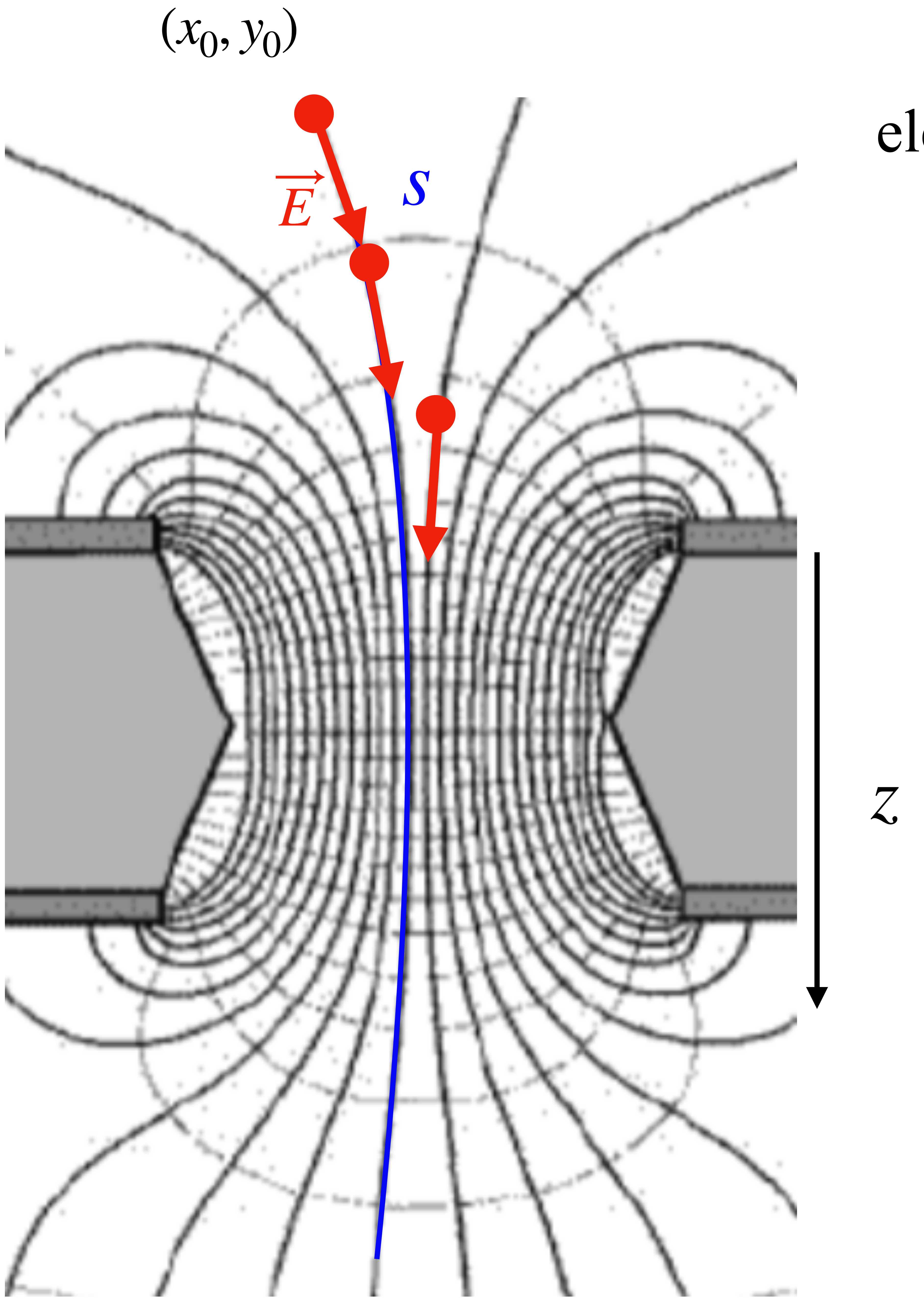
③Calculate the standard deviation of electron diffusion(?) σ



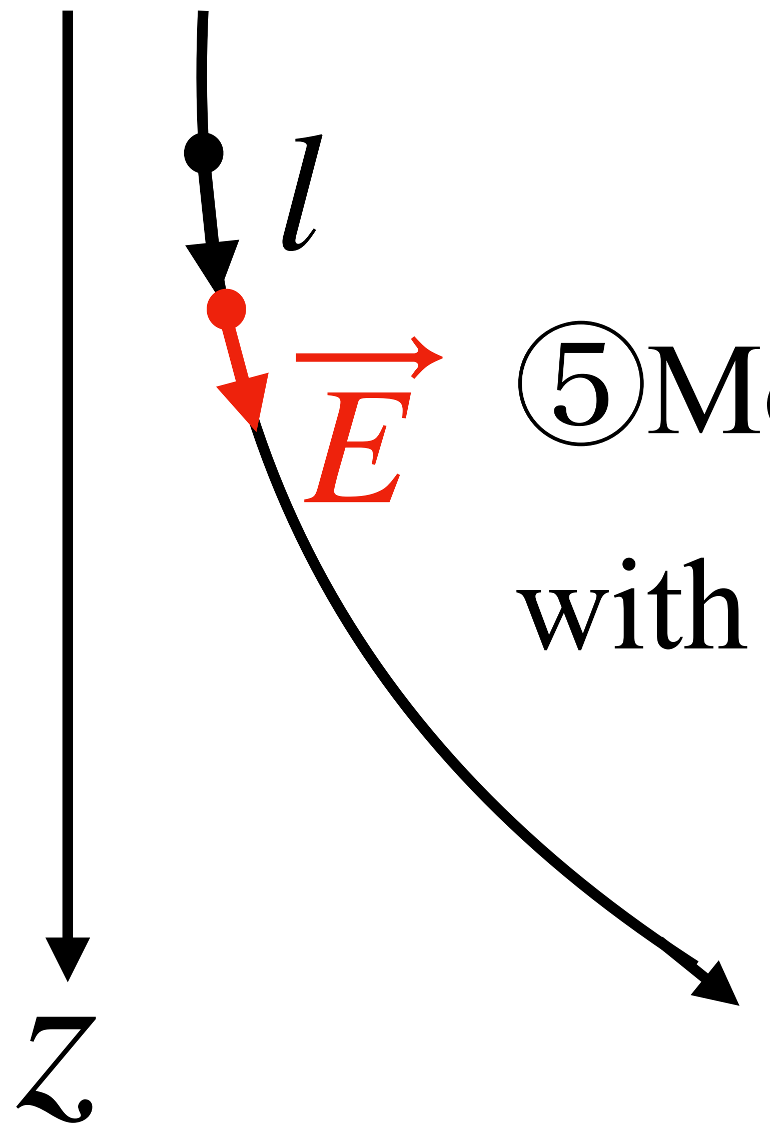
④Gaussian-smear drift electrons with σ

in the plane perpendicular to the electric field vector

In this way, we can include the process of electrons jumping to neighboring electric field lines by diffusion



electrons can jump to neighboring electric field lines by diffusion



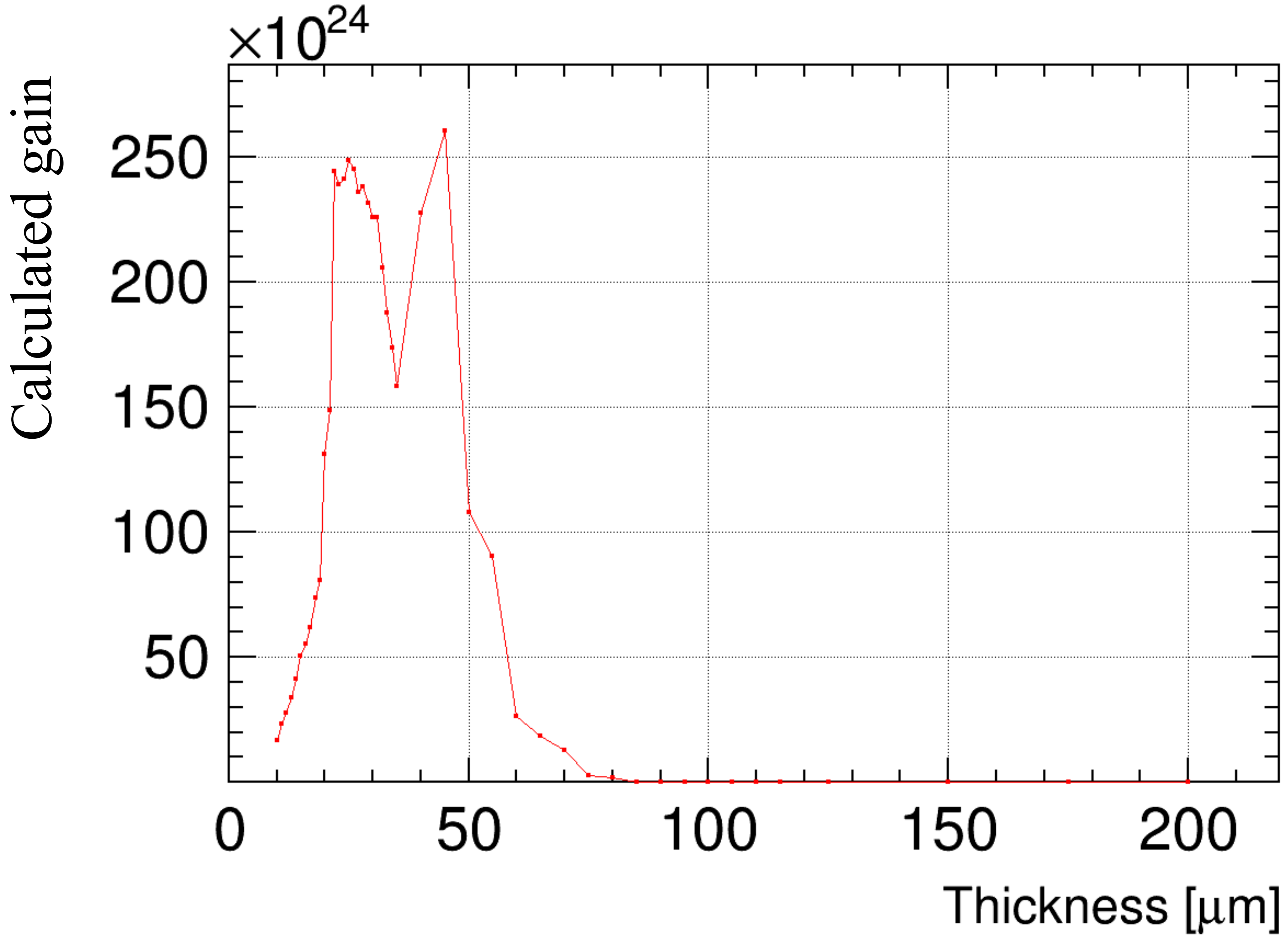
⑤Move electrons along electric field line with mean free path l

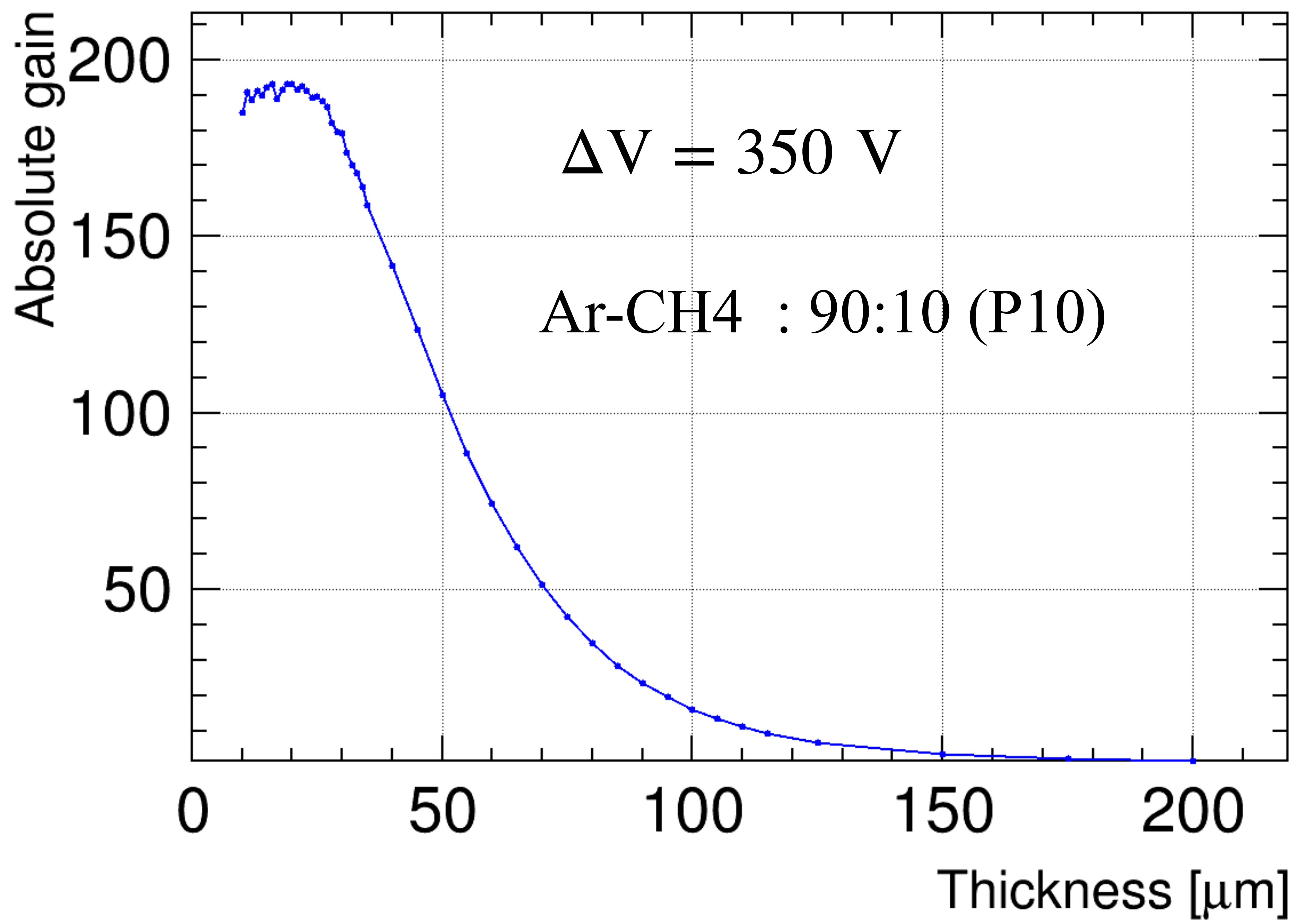
Repeat the process ①~⑤

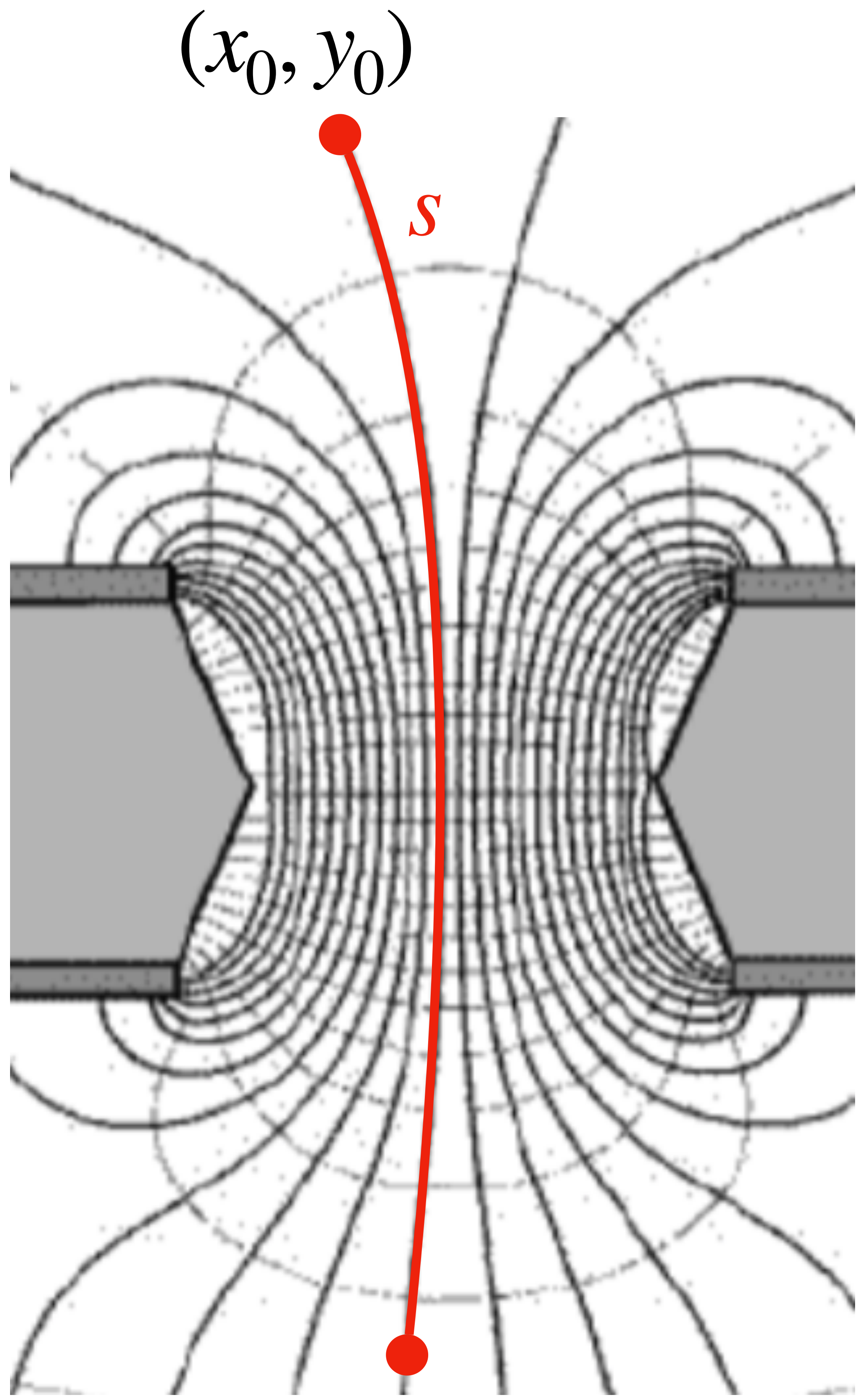
and calculate $\int_0^\infty ds\ \alpha\left(E(s, x_0, y_0)\right)$

along the electric field lines s

Compare gas gain







Calculated gain is too high $\sim \mathcal{O}(10^{24})$

due to the path that pass through the center of the GEM hole?

Source of failure

In order to get Cross section,

looked at free path distribution after each collision

this does not involve the probability of ionising collision

always cause avalanche every step

→increases like an “avalanche” literally

Proper way to calculate gas gain

we need to consider the probability of ionising collision
whether each electrons cause avalanche or not for each step

Correct way

To include the probability of encountering the ionisation collision for each step

we have to use Townsend coefficient α instead of the mean free path l

$$G(x_0 y_0) = exp\left[\int_0^\infty ds\ \rho\sigma\left(\frac{E(s, x_0, y_0)}{\rho}\right)\right] \rightarrow G(x_0 y_0) = exp\left[\int_0^\infty ds\ \alpha\left(E(s, x_0, y_0)\right)\right]$$

with $\rho\sigma = \frac{1}{l}$

with $\alpha = a_i(-1 + 2\mathrm{e}^{-\alpha \mathrm{x}_0})$

$$x_0 = \frac{U_0}{E}$$

U_0 : ionisation potential

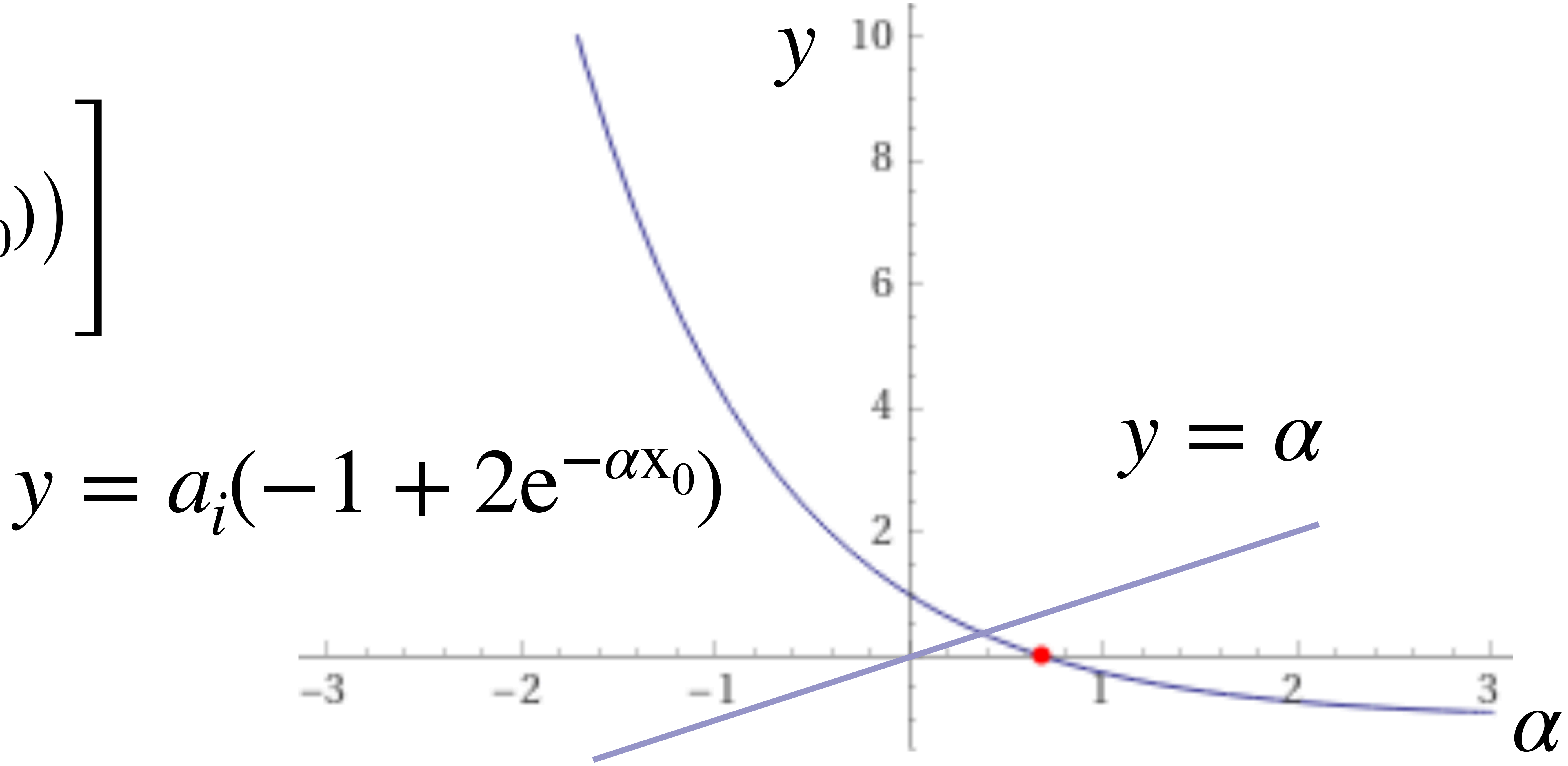
$$a_i = \rho\sigma$$

To calculate gas gain

$$G(x_0 y_0) = exp\left[\int_0^\infty ds\ \alpha\left(E(s, x_0, y_0)\right)\right]$$

need to solve for α

$$\alpha = a_i(-1 + 2e^{-\alpha x_0})$$



however, this cannot be solved analytically

solved by using Newton's method

townsend coefficient

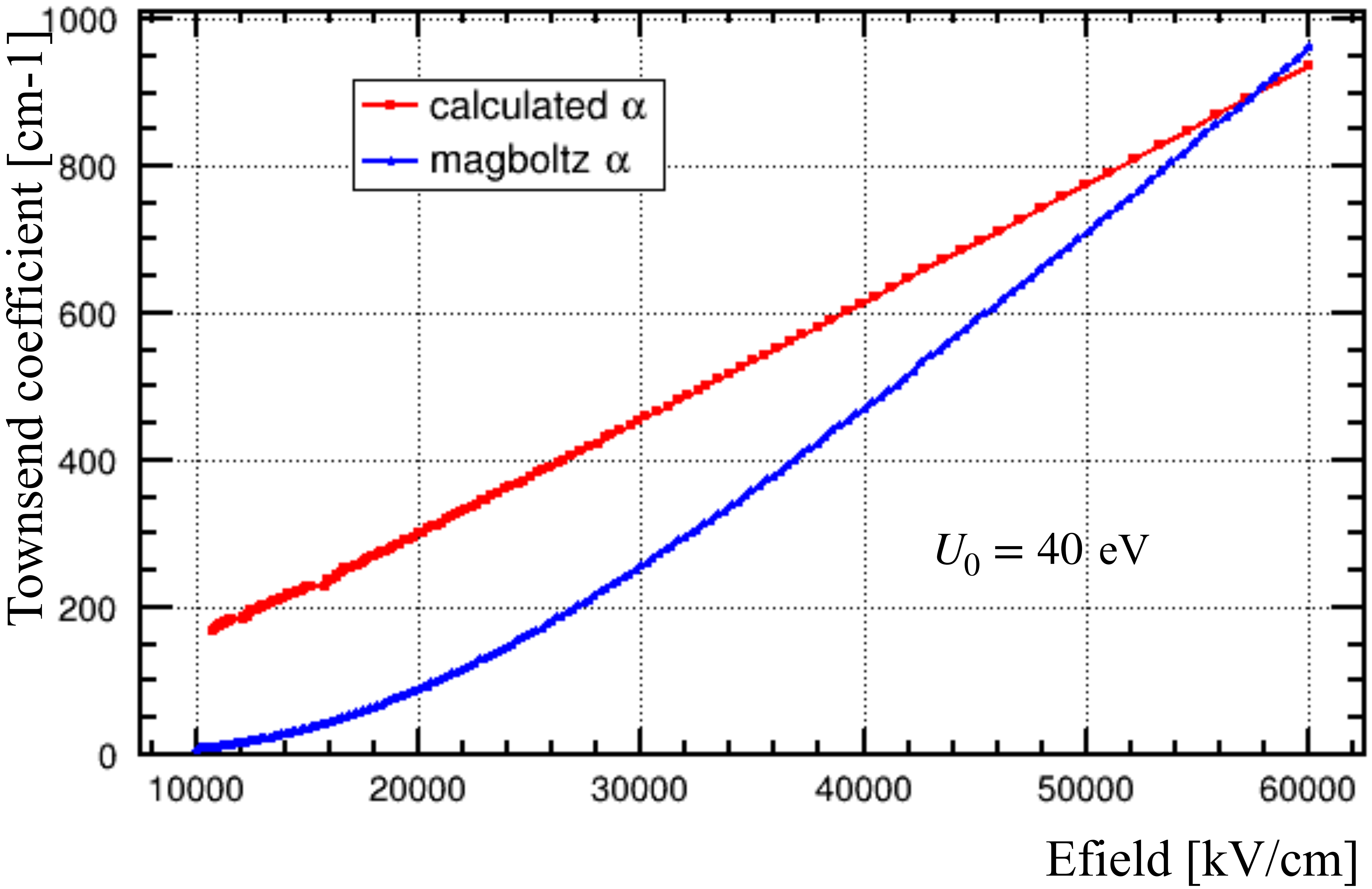
$$\alpha = a_i(-1 + 2e^{-\alpha x_0})$$

$$a_i = \rho\sigma \ [\mathrm{cm}^{-1}] \quad X_0 = \frac{U_0}{E} \ [\mathrm{cm}]$$

U_0 : first ionisation energy (tune parameter)

Legler has shown, that with an appropriate choice of the model parameter U0 (Uo ≈ Ui where Ui is the first ionisation potential) the calculated distributions are in good agreement with experimental spectra.

G.D. ALKHAZOV 1970



first ionisation potential in argon

IP = 127109.842(4) cm^{-1} (15.759610 eV)

from Precision VUV spectroscopy of Ar I at 105 nm

I Velchev et al 1999 J. Phys. B

Behaviour of Calculated α is different from the result of magboltz output

also the value of tuned U_0 @E=60 kV/cm is too high

we conclude that the assumption "Legler's model was correct" was wrong after some calculation, we obtain

$$\frac{\partial}{\partial \Delta} \ln G = \alpha[p_i(l,\beta)] + \Delta \cdot \frac{\int_0^\infty dl \left(\frac{\partial}{\partial \Delta} p_i(l,\epsilon)\right) e^{-\alpha[p_i(l,\epsilon)]l}}{\int_0^\infty dl\, p_i(l,\epsilon) l e^{-\alpha[p_i(l,\epsilon)]l}}$$

$$= \alpha[p_i(l,\beta)] + \Delta \cdot \frac{\int_0^\infty dl \left(\frac{\partial}{\partial \Delta} \exp\left(-\int_{x_0}^l dl' \rho\sigma(El')\right) \rho\sigma(El)\right) e^{-\alpha[p_i(l,\epsilon)]l}}{\int_0^\infty dl\, \exp\left(-\int_{x_0}^l dl' \rho\sigma(El')\right) \rho\sigma(El) l e^{-\alpha[p_i(l,\epsilon)]l}}$$

need to calculate them to find “Stability condition”

$$\int_0^\infty dl\, \exp\left(-\int_{x_0}^l dl' \rho\sigma(El')\right) \rho\sigma(El) l e^{-\alpha[p_i(l,\epsilon)]l}$$

$$\int_0^\infty dl \left(\frac{\partial}{\partial \Delta} \exp\left(-\int_{x_0}^l dl' \rho\sigma(El')\right) \rho\sigma(El)\right) e^{-\alpha[p_i(l,\epsilon)]l}$$

$$\frac{\partial}{\partial \Delta} \ln G = \alpha[p_i(l,\beta)] + \Delta \cdot \frac{\partial \alpha[p_i(l,\beta)]}{\partial \Delta}$$

$$\frac{\partial \alpha[p_i(l,\beta)]}{\partial \Delta} \equiv \frac{\int_0^\infty dl \left(\frac{\partial}{\partial \Delta} p_i(l,\epsilon) \right) e^{-\alpha[p_i(l,\epsilon)]l}}{\int_0^\infty dl\; p_i(l,\epsilon) l e^{-\alpha[p_i(l,\epsilon)]l}}$$

with the condition

$$\frac{\partial}{\partial \Delta} \ln G = 0$$

we find the general form of "Stability Condition"

$$\frac{\partial \alpha[p_i(l,\beta)]}{\partial \Delta} = \frac{\alpha[p_i(l,\beta)]}{\Delta} \quad \text{and} \quad \frac{\partial \alpha[p_i(l,\beta)]}{\partial \epsilon} = \frac{\alpha[p_i(l,\beta)]}{\epsilon}$$

similar form with "old" one $\quad \dfrac{\partial \sigma_0}{\partial \epsilon} = \dfrac{\sigma_0}{\epsilon}$

$$J(1) = \int_0^\infty dl\ p_i(l)\ e^{-\alpha l} = \frac{1}{2}$$

Once $p_i(l)$ is decided, α is will be decided. (α is functional of $p_i(l)$)

└── model-dependent

- Legler's model : $p_i(l) = a_i\ e^{-a_i(l-x_0)}\ \theta(l - x_0)$
- Snyder's model: $p_i(l) = \alpha\ e^{-\alpha l}$

p_i depends on l and other variable β

$$G = \exp(\Delta \cdot \alpha[p_i(l, \beta)])$$

where

Δ: thickness of GEM

$$\frac{\partial}{\partial \beta}\ \ln G = \frac{\partial}{\partial \beta}(\Delta \cdot \alpha[p_i(l, \beta)])$$

$$= \frac{\partial \Delta}{\partial \beta} \cdot \alpha[p_i(l, \beta)] + \Delta \cdot \frac{\partial \alpha[p_i(l, \beta)]}{\partial \beta}$$

for thickness dependence of gas gain

$\beta = \Delta,\ V$:fixed

⟶

$$\frac{\partial}{\partial \Delta}\ \ln G = \frac{\partial \Delta}{\partial \Delta} \cdot \alpha[p_i(l, \beta)] + \Delta \cdot \frac{\partial \alpha[p_i(l, \beta)]}{\partial \Delta}$$

for thickness dependence of gas gain

$$\frac{\partial}{\partial \Delta} \ln G = \frac{\partial \Delta}{\partial \Delta} \cdot \alpha[p_i(l,\beta)] + \Delta \cdot \frac{\partial \alpha[p_i(l,\beta)]}{\partial \Delta}$$

we need to calculate this term

$$\int_0^\infty dl \; p_i(l,\epsilon) \; e^{-\alpha[p_i(l,\epsilon)]l} = \frac{1}{2} \qquad \text{where} \qquad \epsilon = \frac{V}{\Delta\rho} = \frac{E}{\rho} \qquad E = \frac{V}{\Delta}$$

$$\frac{d}{d\beta}\int_0^\infty dl \; p_i(l,\epsilon) \; e^{-\alpha[p_i(l,\epsilon)]l} = 0$$

$$\int_0^\infty dl \; \left(\frac{\partial}{\partial\beta} p_i(l,\epsilon)\right) \; e^{-\alpha[p_i(l,\epsilon)]l} + \int_0^\infty dl \; p_i(l,\epsilon)\left(\frac{\partial}{\partial\beta e^{-\alpha[p_i(l,\epsilon)]l}}\right) = 0$$

$$\int_0^\infty dl \; \left(\frac{\partial}{\partial\beta} p_i(l,\epsilon)\right) \; e^{-\alpha[p_i(l,\epsilon)]l} = -\int_0^\infty dl \; p_i(l,\epsilon)\left(\frac{\partial}{\partial\beta e^{-\alpha[p_i(l,\epsilon)]l}}\right)$$

$$= \int_0^\infty dl \; p_i(l,\epsilon) l e^{-\alpha[p_i(l,\epsilon)]l} \frac{\partial \alpha[p_i(l,\epsilon)]}{\partial\beta}$$

$$\int_0^\infty dl \ \left(\frac{\partial}{\partial\beta}p_i(l,\epsilon)\right) \ e^{-\alpha[p_i(l,\epsilon)]l} = -\int_0^\infty dl \ p_i(l,\epsilon)\left(\frac{\partial}{\partial\beta e^{-\alpha[p_i(l,\epsilon)]l}}\right)$$

$$= \int_0^\infty dl \ p_i(l,\epsilon) l e^{-\alpha[p_i(l,\epsilon)]l} \frac{\partial\alpha[p_i(l,\epsilon)]}{\partial\beta}$$

$$\frac{\partial\alpha[p_i(l,\epsilon)]}{\partial\beta} = \frac{\int_0^\infty dl \ \left(\frac{\partial}{\partial\beta}p_i(l,\epsilon)\right) \ e^{-\alpha[p_i(l,\epsilon)]l}}{\int_0^\infty dl \ p_i(l,\epsilon) l e^{-\alpha[p_i(l,\epsilon)]l}}$$

for thickness dependence of gas gain

$\beta = \Delta,\ V$:fixed

$$\frac{\partial\alpha[p_i(l,\epsilon)]}{\partial\Delta} = \frac{\int_0^\infty dl \ \left(\frac{\partial}{\partial\Delta}p_i(l,\epsilon)\right) \ e^{-\alpha[p_i(l,\epsilon)]l}}{\int_0^\infty dl \ p_i(l,\epsilon) l e^{-\alpha[p_i(l,\epsilon)]l}}$$

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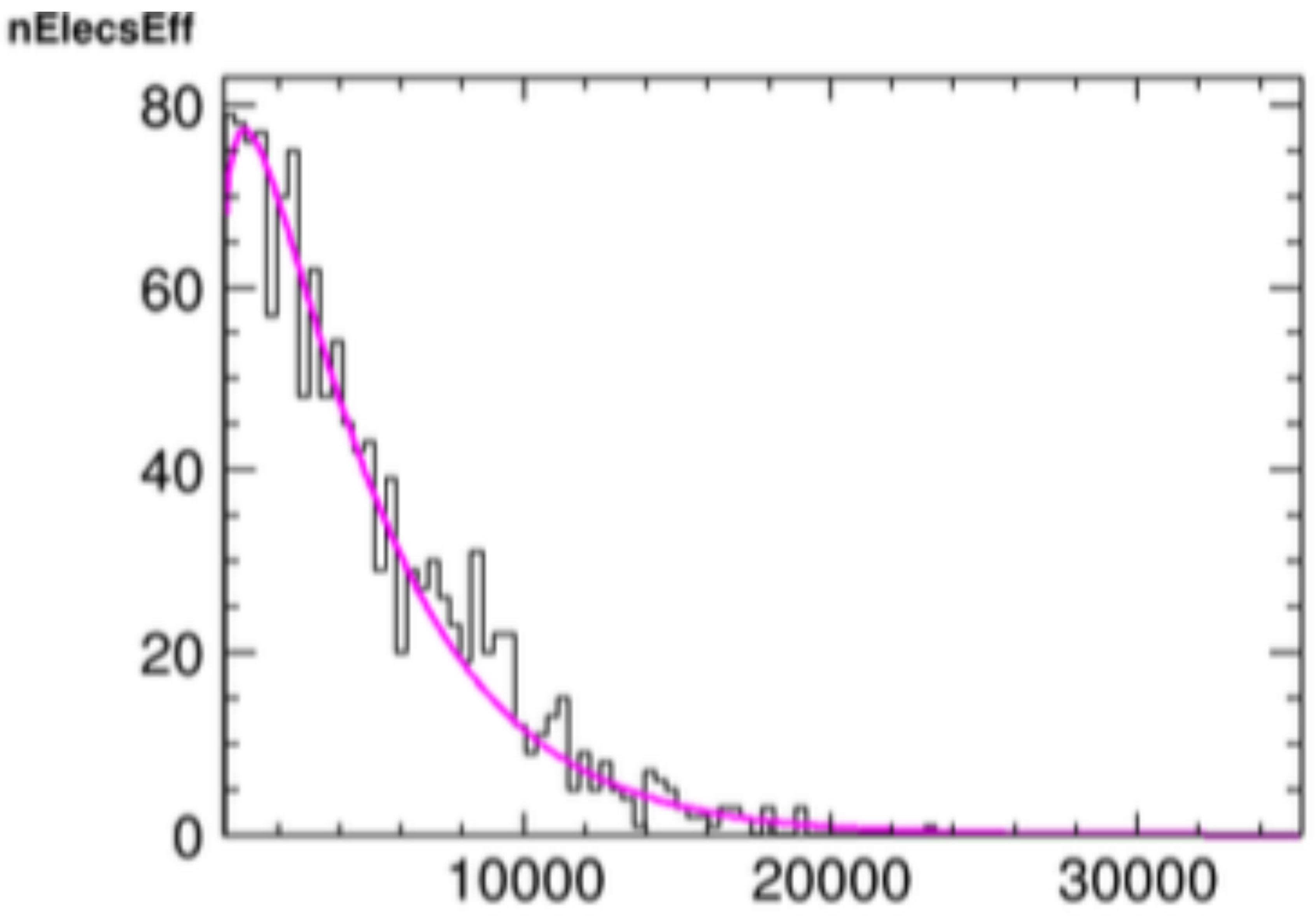
$$\frac{\partial}{\partial \Delta}\ \ln G = \frac{\partial \Delta}{\partial \Delta} \cdot \alpha[p_i(l,\beta)] + \Delta \cdot \frac{\partial \alpha[p_i(l,\beta)]}{\partial \Delta}$$

$$= \alpha[p_i(l,\beta)] + \Delta \cdot \frac{\int_0^\infty dl\ \left(\frac{\partial}{\partial \Delta} p_i(l,\epsilon)\right)\ e^{-\alpha[p_i(l,\epsilon)]l}}{\int_0^\infty dl\ p_i(l,\epsilon) l e^{-\alpha[p_i(l,\epsilon)]l}}$$

For the general discussion, we need to find a model-independent form of $p_i(l,\epsilon)$

In the first place,
the self-consistent equation $p(z)$ denotes the probability distribution function of gas gain

$$p(z) = \frac{1}{\alpha z}\int_z^\infty dz' \int_0^{z'} dz'' p(z'')\ p(z'-z'')\ p_i(\frac{1}{\alpha}\ln\frac{z'}{z})$$



model-independent form of $p_i(l,\epsilon)$

$$p_i(l,\epsilon) = \exp\left(-\int_{x_0}^{l} dl'\rho\sigma(El')\right)\rho\sigma(El)$$

$$J(1) = \int_0^\infty dl\ p_i(l)\ e^{-\alpha l} = \frac{1}{2}$$

$$\frac{\partial}{\partial \Delta} \ln G = \alpha[p_i(l,\beta)] + \Delta \cdot \frac{\int_0^\infty dl \ \left(\frac{\partial}{\partial \Delta} p_i(l,\epsilon)\right) \ e^{-\alpha[p_i(l,\epsilon)]l}}{\int_0^\infty dl \ p_i(l,\epsilon) l e^{-\alpha[p_i(l,\epsilon)]l}}$$

$$= \alpha[p_i(l,\beta)] + \Delta \cdot \frac{\int_0^\infty dl \ \left(\frac{\partial}{\partial \Delta} \exp\left(-\int_{x_0}^l dl' \rho\sigma(El')\right) \rho\sigma(El)\right) \ e^{-\alpha[p_i(l,\epsilon)]l}}{\int_0^\infty dl \ \exp\left(-\int_{x_0}^l dl' \rho\sigma(El')\right) \rho\sigma(El) l e^{-\alpha[p_i(l,\epsilon)]l}}$$

need to calculate them to find “Stability condition”

$$\int_0^\infty dl \ \exp\left(-\int_{x_0}^l dl' \rho\sigma(El')\right) \rho\sigma(El) l e^{-\alpha[p_i(l,\epsilon)]l}$$

$$\int_0^\infty dl \ \left(\frac{\partial}{\partial \Delta} \exp\left(-\int_{x_0}^l dl' \rho\sigma(El')\right) \rho\sigma(El)\right) \ e^{-\alpha[p_i(l,\epsilon)]l}$$