European Linear Collider Workshop ECFA LC2013
BDS+MDI

IPBSM
Beam Size Measurement & Performance Evaluation

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DESY

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Introduction

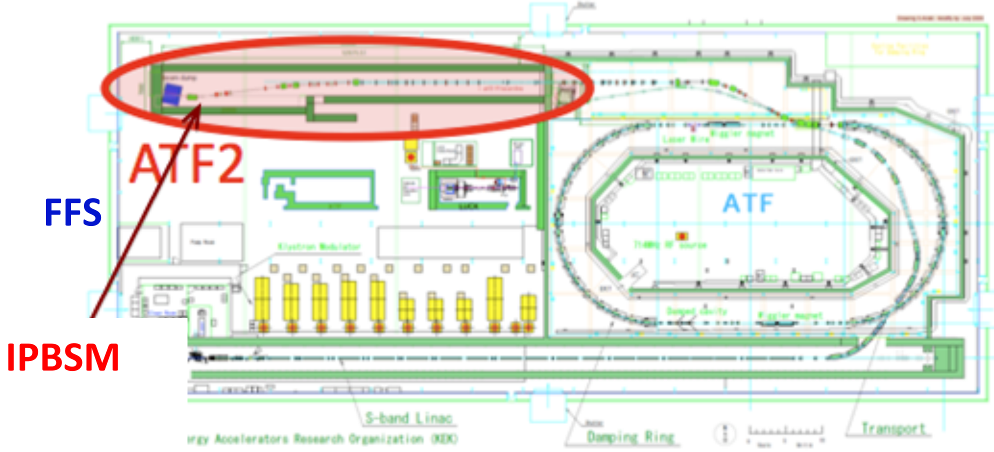
Measurement Scheme
Expected Performance
Role in Beam Tuning

Role of IPBSM (Shintake Monitor) at ATF2

$$L = \frac{n_b N^2 f_{rep}}{4\pi\sigma_x \boxed{\sigma_y}} H_D$$

Ultra-focused vertical beam size at IP !!
Crucial for **high luminosity**

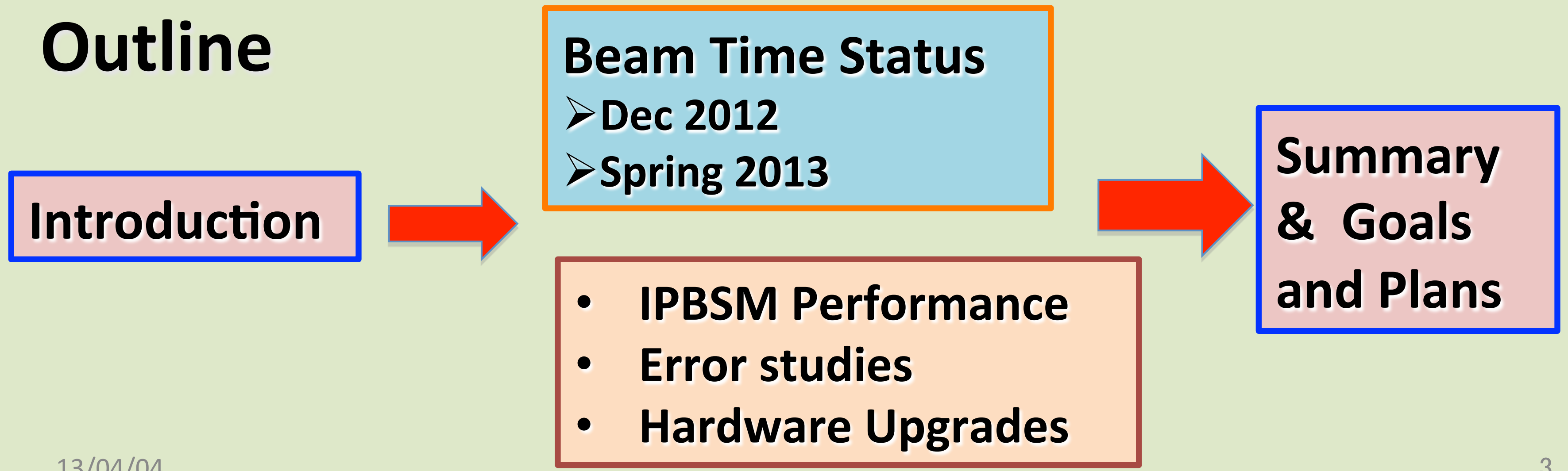
ATF2: Linear Collider FFS test facility@KEK



ATF: 1.28 GeV LINAC , DR
→high quality e- beam with extremely small normalized vertical emittance $\gamma\varepsilon_y$

IPBSM is crucial for achieving ATF2 's Goal 1 !!
focus **σ_y to design 37 nm**
➔verify Local Chromaticity Correction

ATF2 Goal 2: O(nm) beam trajectory stabilization



Outline

Introduction

Beam Time Status
- **Dec 2012**
- **Spring 2013**

- **IPBSM Performance**
- **Error studies**
- **Hardware Upgrades**

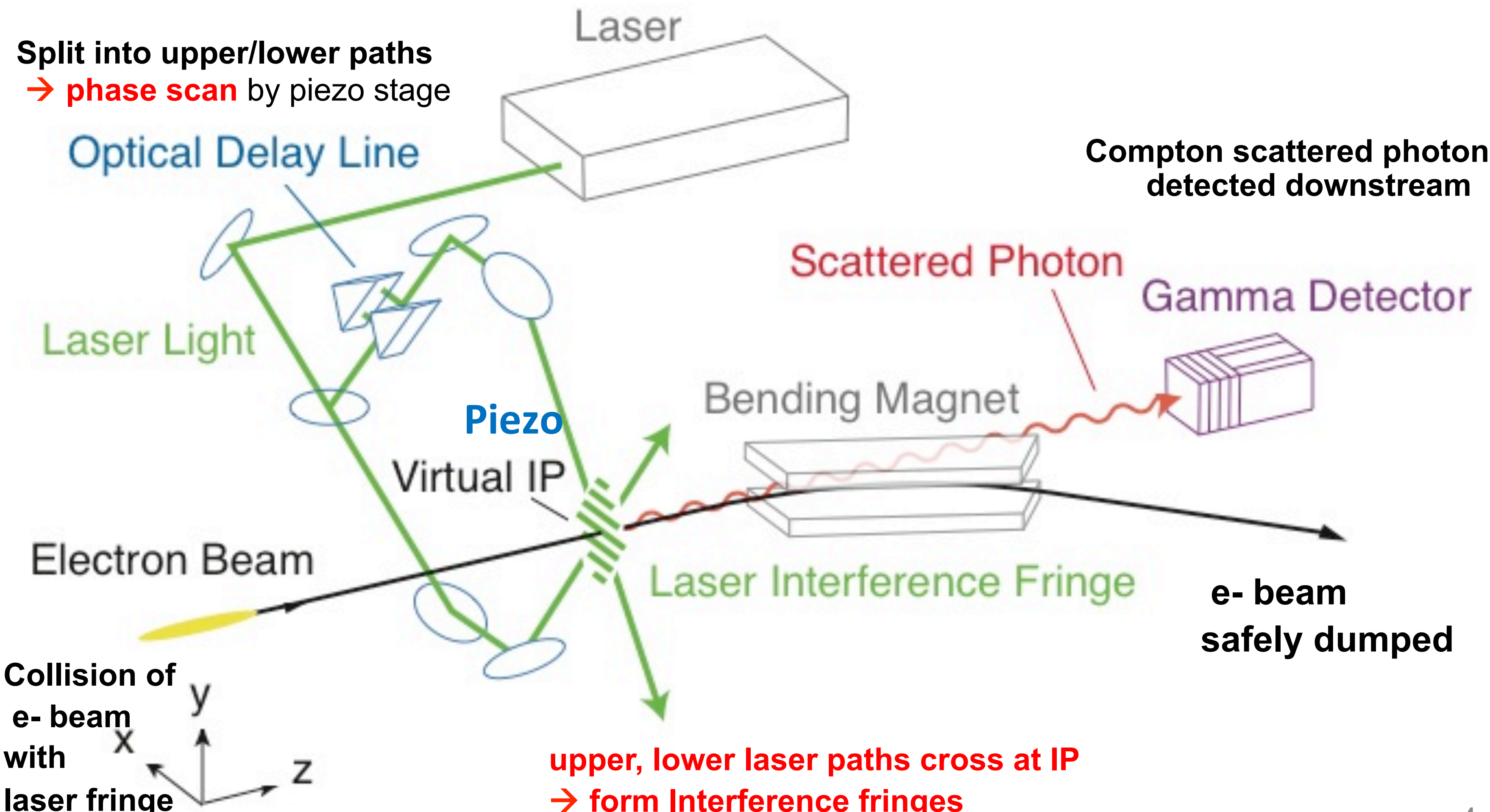
Summary & Goals and Plans

Measurement Scheme

- use **laser interference fringes** as target for e- beam

Only device able to measure σ_y < 100 nm !!

- Crucial for ATF2 beam tuning and realization of ILC



Split into upper/lower paths
→ **phase scan** by piezo stage

Piezo

Compton scattered photons detected downstream

e- beam safely dumped

Collision of e- beam with laser fringe

upper, lower laser paths cross at IP
→ form Interference fringes

Detector measures

signal **Modulation Depth "M"**

$$M = \frac{N_+ - N_-}{N_+ + N_-} = \left|\cos(\theta)\exp\left(-2(k_y\sigma_y)^2\right)\right|$$

$$\Rightarrow \quad \sigma_y = \frac{d}{2\pi}\sqrt{2\ln\left(\frac{|\cos(\theta)|}{M}\right)}$$

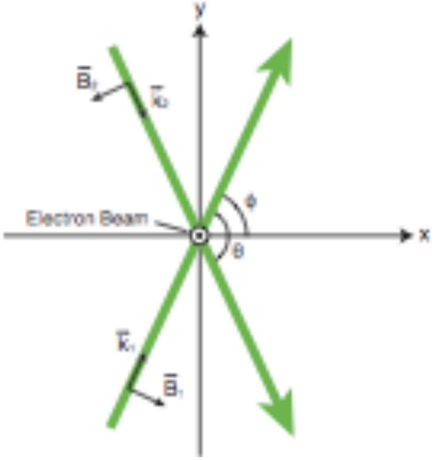
measurable range

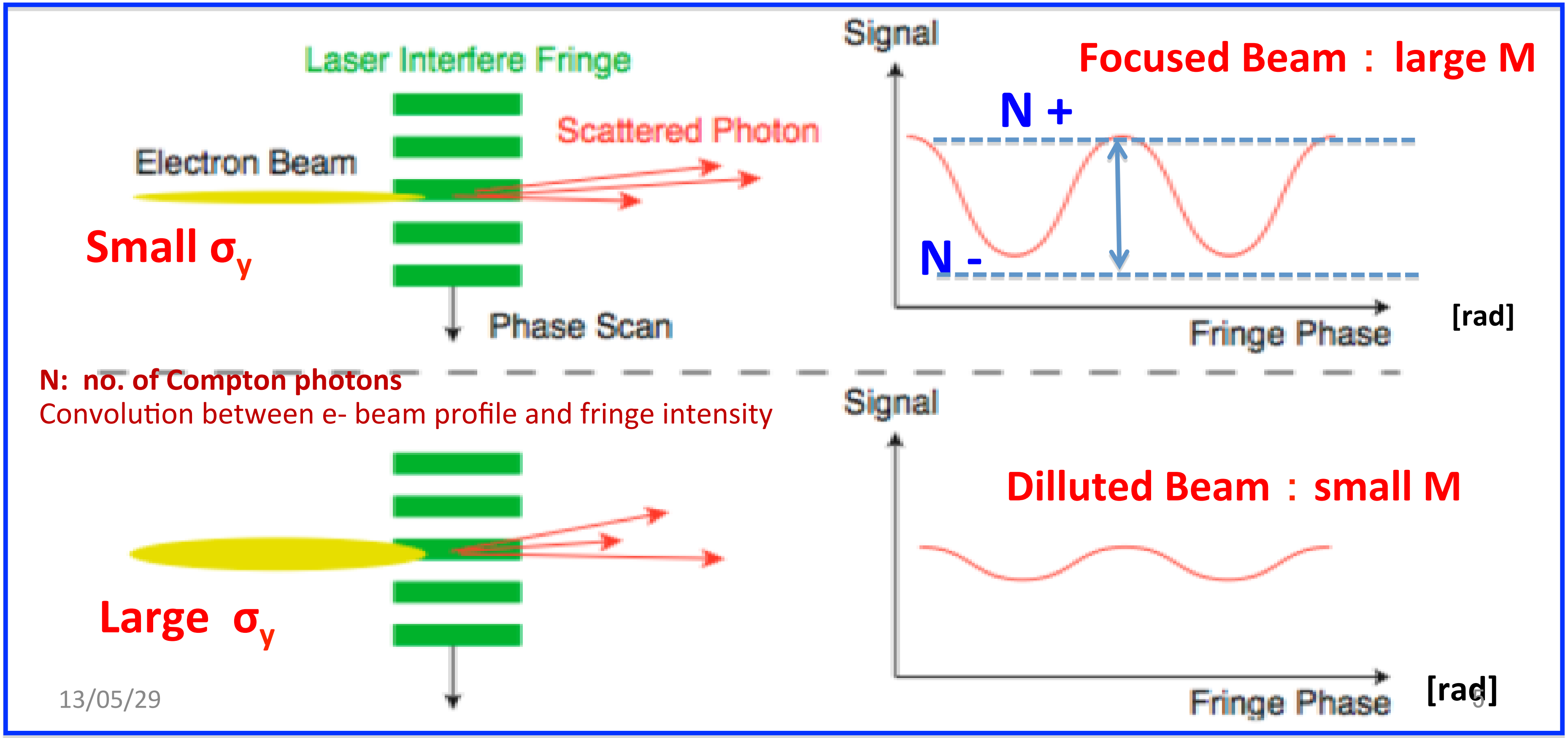
determined by **fringe pitch**

$$d = \frac{\pi}{k_y} = \frac{\lambda}{2\sin(\theta/2)}$$

depend on

crossing angle θ (and λ)





Focused Beam : large M

Small σ_y

N: no. of Compton photons

Convolution between e- beam profile and fringe intensity

Dilluted Beam : small M

Large σ_y

Crossing angle θ	174°	30°	8°	2°
Fringe pitch $d = \frac{\pi}{k_y} = \frac{\lambda}{2\sin(\theta/2)}$	266 nm	1.03 μm	3.81 μm	15.2 μm
Lower limit	20 nm	80 nm	350 nm	1.2 μm
Upper limit	110 nm	400 nm	1.4 μm	6 μm

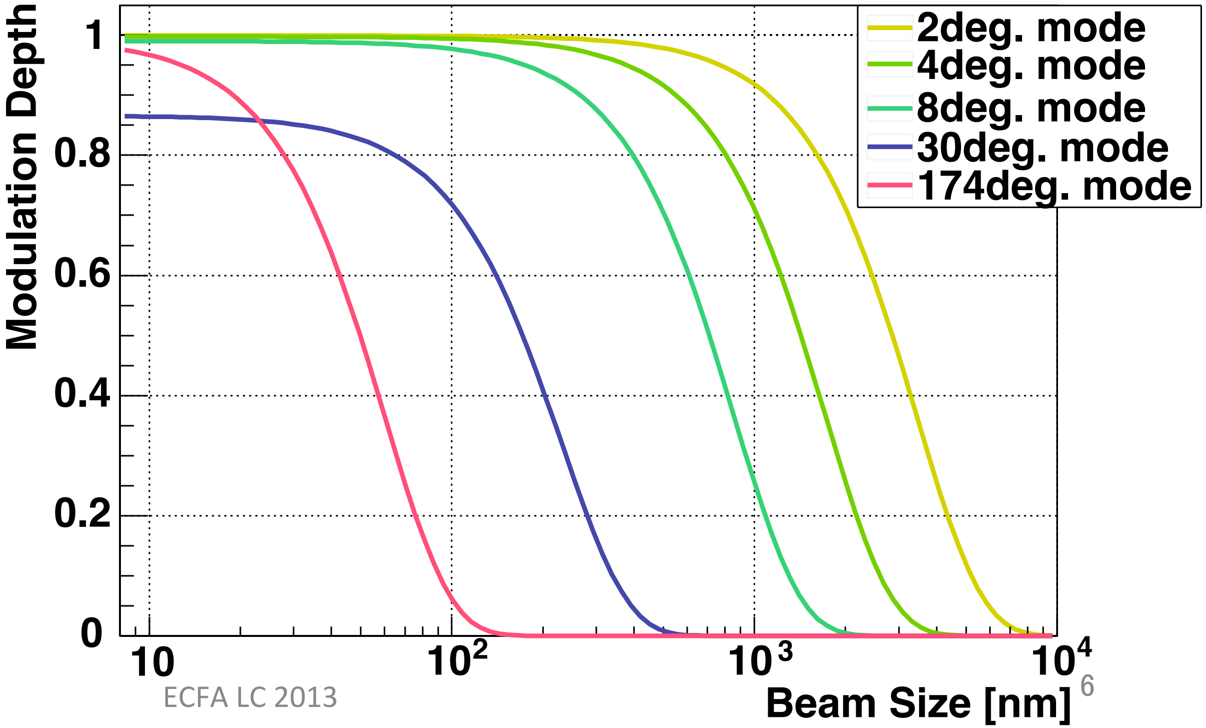
Expected Performance

Measures
σ_y^* = 20 nm ∼few μm
with < **10% resolution**

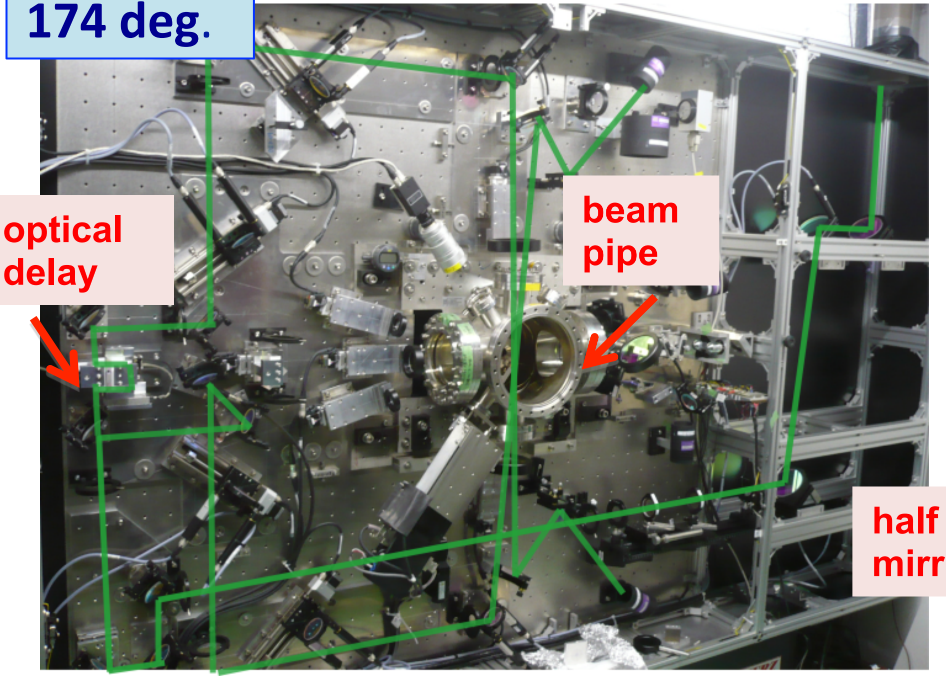
$$\sigma_y = \frac{d}{2\pi}\sqrt{2\ln\left(\frac{|\cos(\theta)|}{M}\right)}$$

σ_y and M for each θ mode

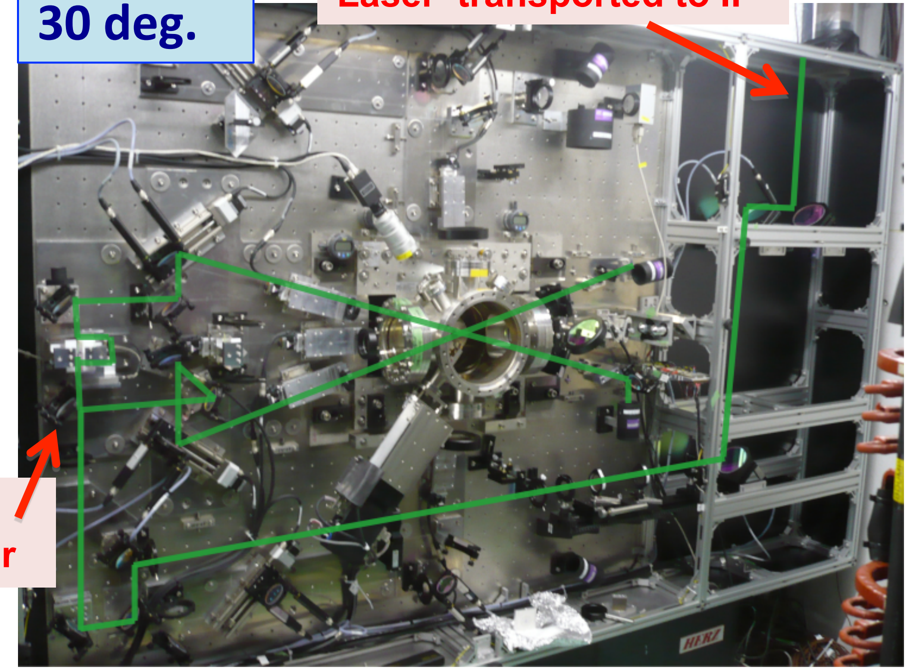
select appropriate mode according to beam focusing



174 deg.

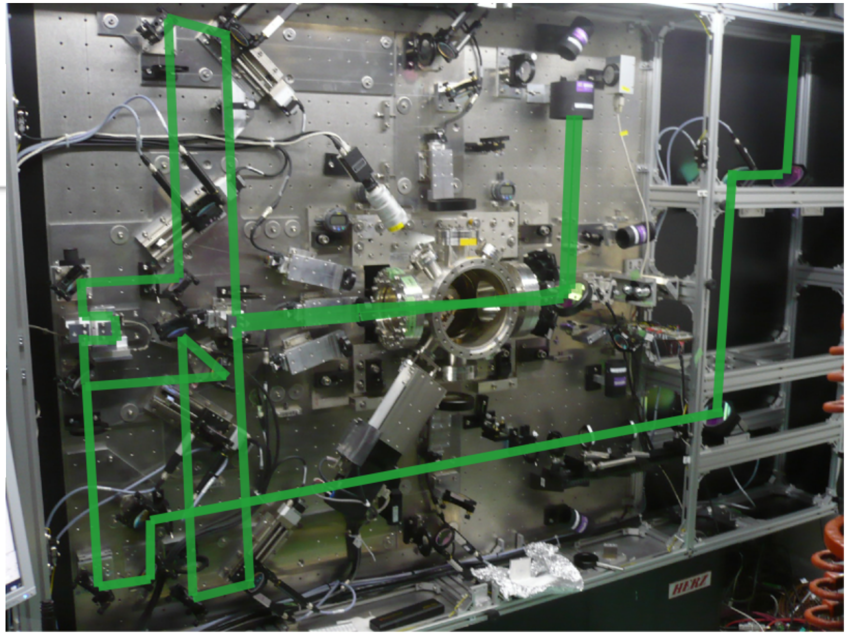


30 deg.



2 - 8 deg

Crossing angle continuously adjustable by prism



Vertical table

1.7 (H) x 1.6 (V) m

- **Interferometer**
- **Phase control (piezo stage)**

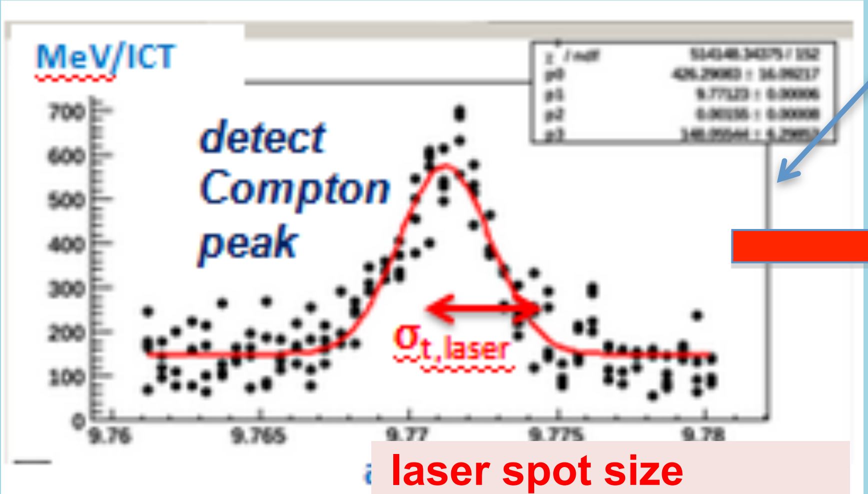
path for each θ mode
(auto-stages + mirror actuators)

Role of IPBSM in Beam Tuning

beforehand ....
Construct & confirm laser paths, timing alignment

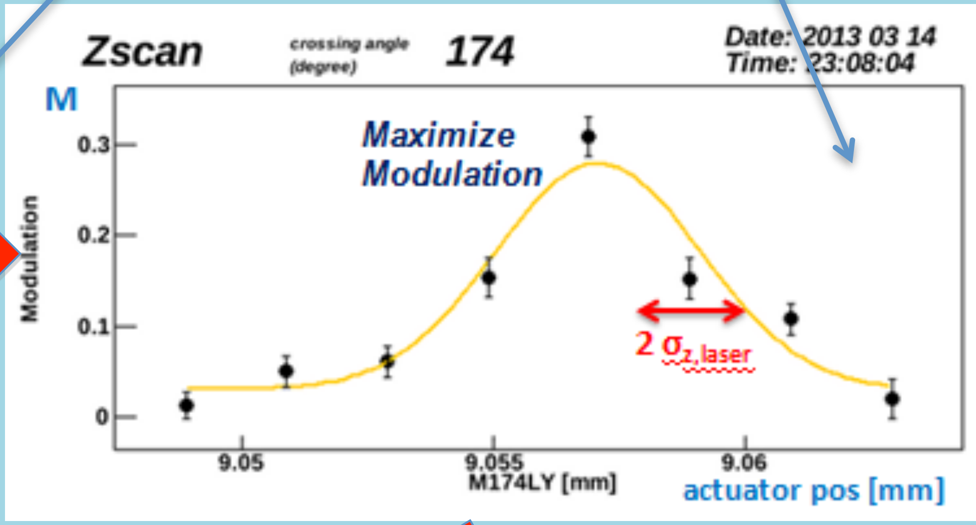
precise position alignment by remote control

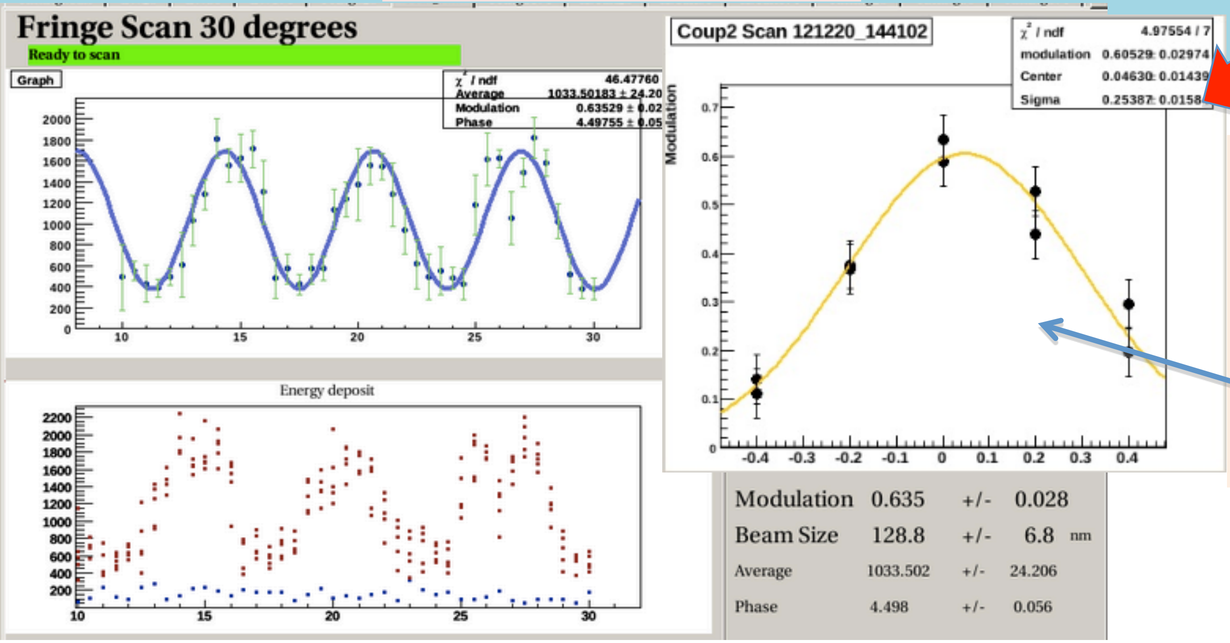
transverse : laser wire scan



laser spot size
$\sigma_{t,laser}$ = 15 – 20 μm

Longitudinal : z scan





Fringe Scan 30 degrees

Ready to scan

χ^2 / ndf	46.47760
Average	1033.50183 ± 24.20
Modulation	0.63529 ± 0.02
Phase	4.49755 ± 0.05

Coup2 Scan 121220_144102

χ^2 / ndf	4.97554 / 7
modulation	0.60529 ± 0.02974
Center	0.04630 ± 0.01439
Sigma	0.25387 ± 0.0158

Modulation	0.635	+/-	0.028	
Beam Size	128.8	+/-	6.8	nm
Average	1033.502	+/-	24.206	
Phase	4.498	+/-	0.056	

After all preparations ……….

continuously measure σ_y using fringe scans

→ Feed back to multi-knob tuning

Beam Time Status

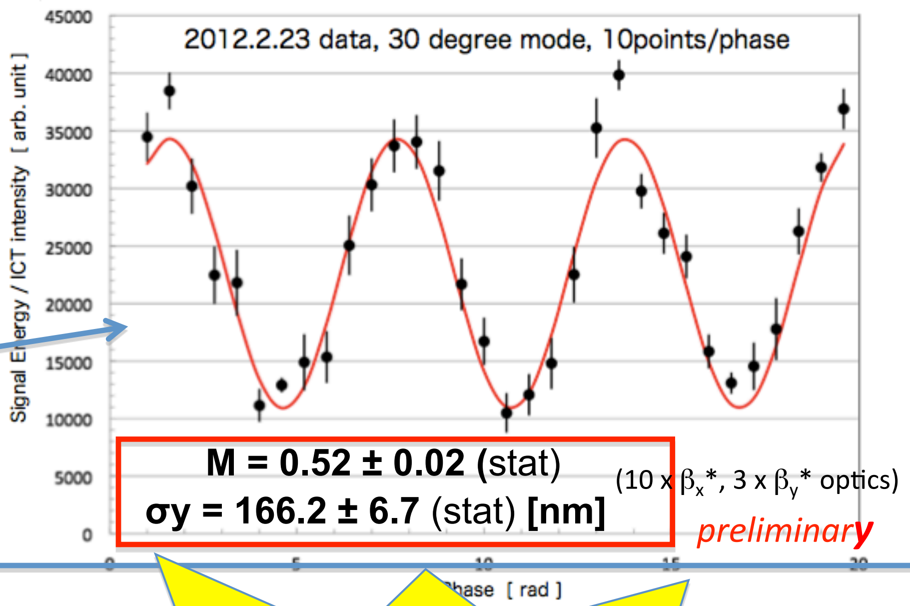
Beam time status in 2012

Spring run

Feb； 30 deg mode commissioned
(1st M detection on 2/17)

stable measurements of M ～ 0.55

- 2 - 8 ° mode: clear contrast（Mmeas ～0.9)
- Prepared 174 deg mode commissioning



M = 0.52 ± 0.02 (stat)
σy = 166.2 ± 6.7 (stat) **[nm]**

(10 x β_x*, 3 x β_y* optics)

preliminary

Major optics reform of 2012 summer

By IPBSM group@KEK

- **Suppress systematic errors**
- **Higher laser path stability / reliability**

12/20 ：
1st success in M detection
at 174 deg mode
10 x βx* ， **1 x βy***

preliminary

Winter run

- High M measured at 30 ° mode
- Contribute with **stable operation** to ATF2 beam focusing / tuning study

Last 2 days in Dec run
Measured many times   M = 0.15 – 0.25
（correspond to **σy ～ 70 – 82 nm**）

* IPBSM systematic errors uncorrected
** under low e beam intensity (～ 1E9 e / bunch)

Large step towards achieving ATF2 ‘s goal !!

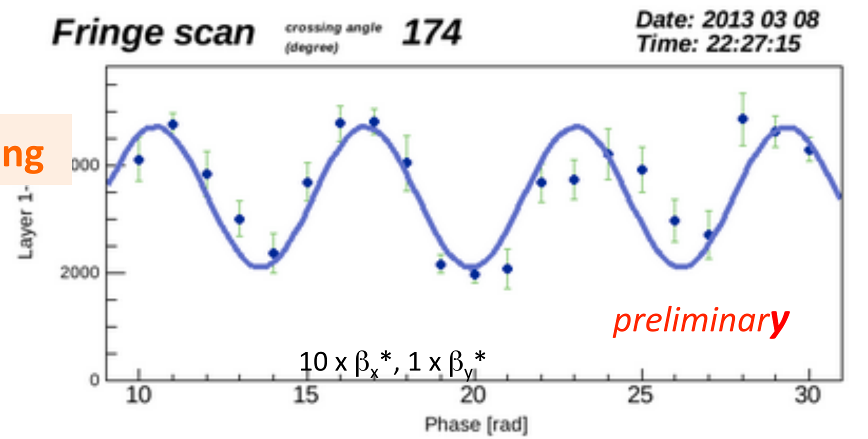
error studies ongoing aimed at deriving “true beamsize”

Beam time status in 2013 Spring

Stable IPBSM performance → major role in beam tuning

measured M over continuous **reiteration of linear /nonlinear@ tuning knobs** **@** 174 ° mode

dedicated data for error studies under analysis



Fit results: Av*(1.0+M*cos(x+Ph))

Modulation: 0.385 +/- 0.025

Beam Size: 58.4 + 2.0 -1.9 nm

174 ° mode ”consistency scan”

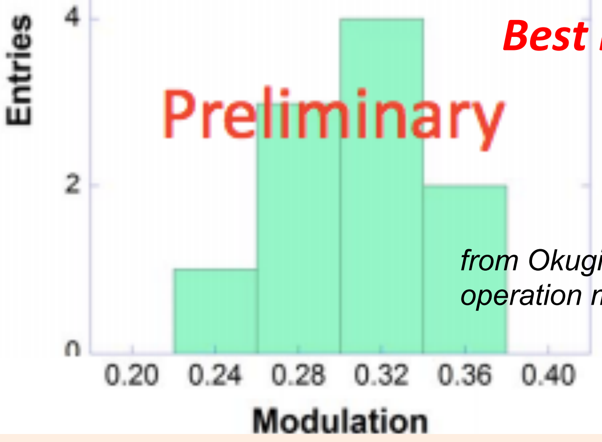
2013/03 /14

after IP-BSM roll alignment

after IP-BSM pitch alignment

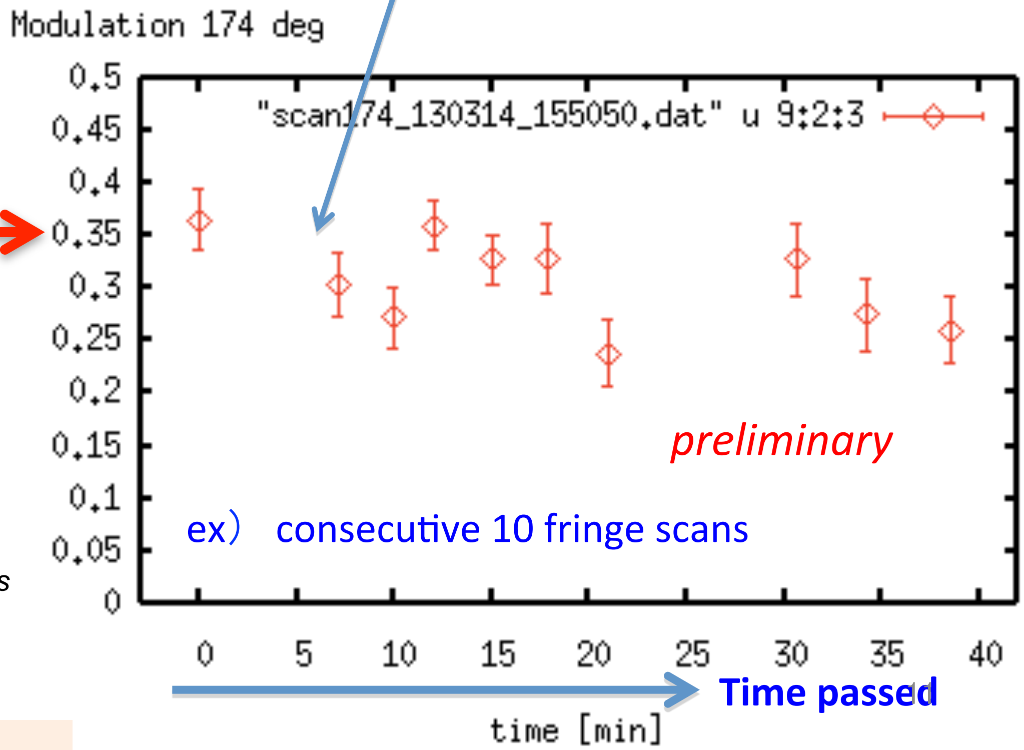
M ～ 0.306 ± 0.043 (RMS)
correspond to σy ～ 65 nm

Best record



from Okugi-san's Fri operation meeting slides

measure M vs time
after all conditions optimized



ex） consecutive 10 fringe scans

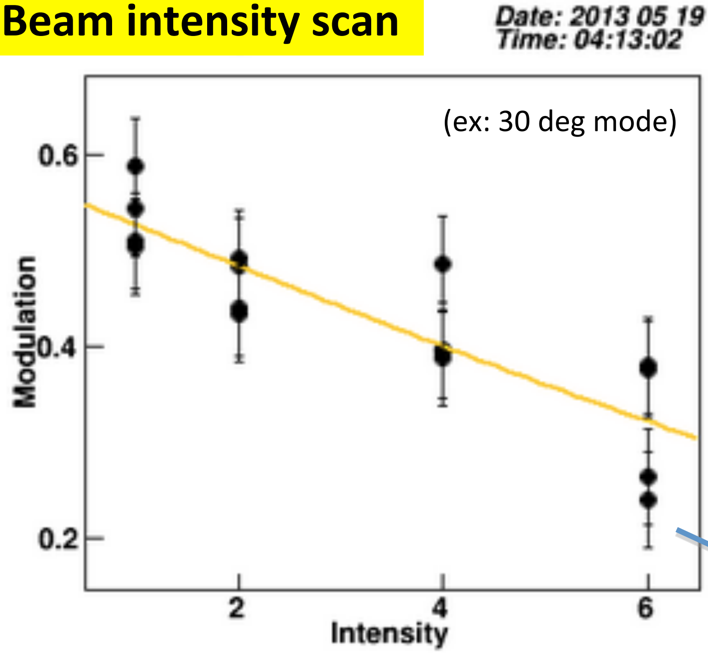
Time passed

moving towards goal of σy = 37 nm :

higher IPBSM precision and stability & *looser current limits of normal / skew sextupoles current*

Other studies using IPBSM

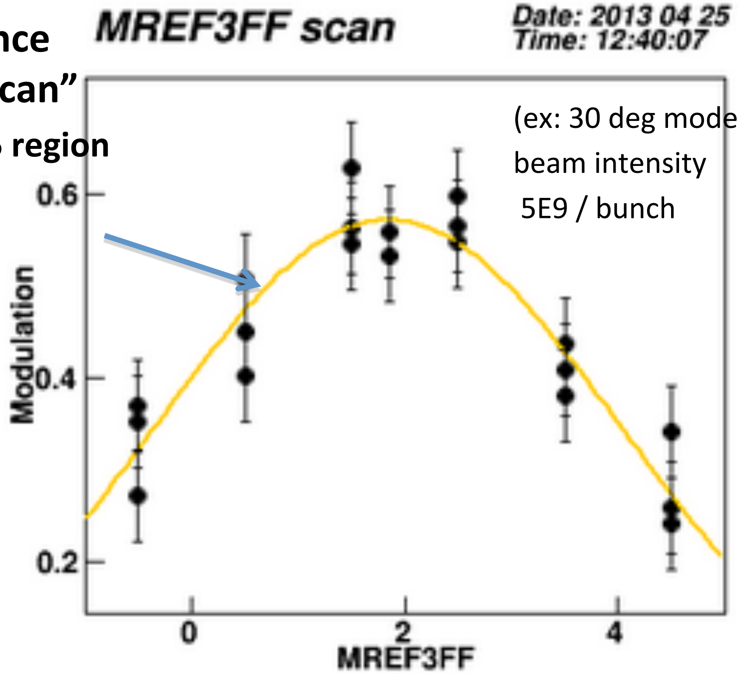
Beam intensity scan



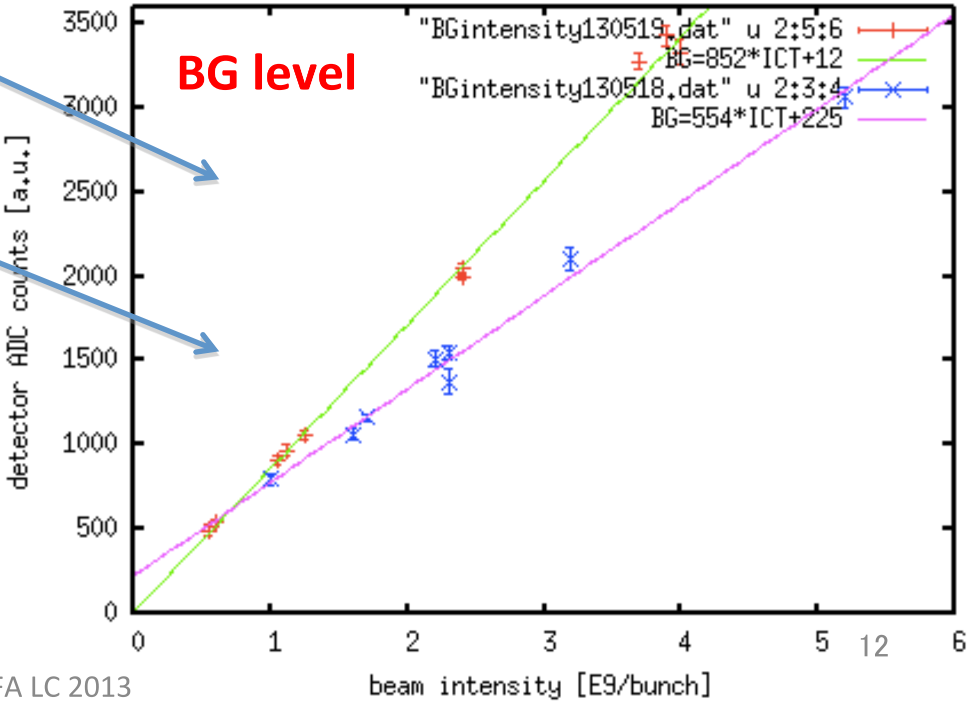
(ex: 30 deg mode)

wakefield studies

“Reference Cavity scan” in high β region



(ex: 30 deg mode)
beam intensity
5E9 / bunch



BG level

Check linearity of BG levels in IPBSM detector

→Observe **“steepness” of intensity dependence**

compare with other periods to test effects of orbit tuning and / or hardware improvement for wake suppression

others:

- Test various linear / nonlinear tuning knobs
- IPBSM systematic error studies

Optics reform of 2012 summer

By IPBSM group@KEK

Proved greatly effective in 2012 winter run

Aim:

- **Suppress systematic error sources**
- **Higher alignment precision & reproducibility**

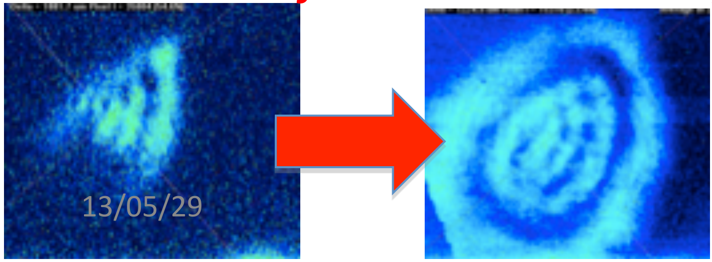
improvements	details
alignment precision ✓match focal point to IP ✓Injection position / angle into lens ✓Re-optimize expander / reducer	• focal point scan for all modes • CW laser + reference lines on new base plates • **new IP target** (screen monitor)
consistency , reproducibility before / after mode switching	• **θ mode switching technique** **{small linear stage + mirror actuators }** now: independent for each mode (before: shared rotating stages)
balanced profiles	suppress difference in path length & focal point

Tuning of main laser

by Spectra Physics

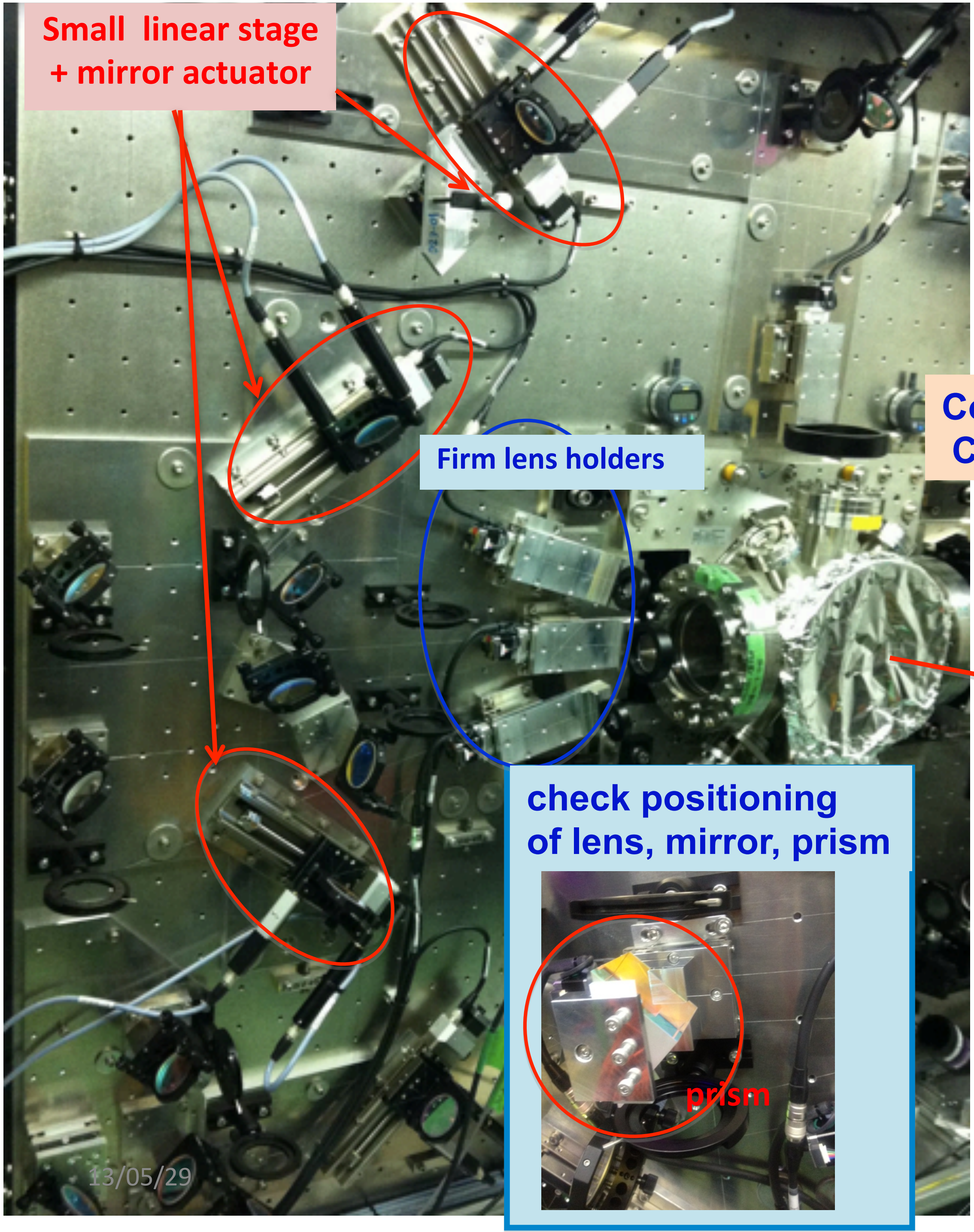
ex: spring 2012 :
Adjust curvature of laser cavity mirrors

Aim for a more Gaussian profile



- ❖ Reform laser profile and spatial coherence (adjust YAG rod & cavity mirrors)

- ❖ Exchange flash lamp
- ❖ seeding laser tuning (→ oscillation stability)



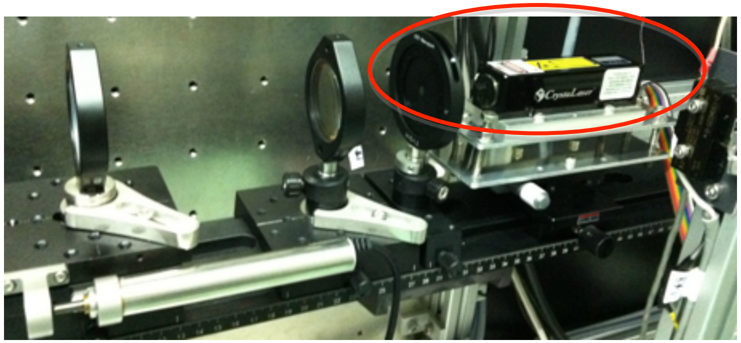
Small linear stage + mirror actuator

Firm lens holders

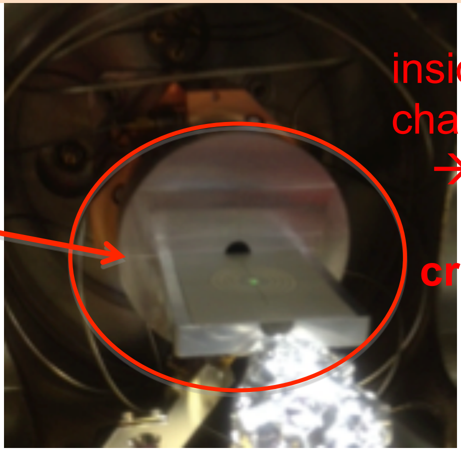
check positioning of lens, mirror, prism

prism

just after injection onto vertical table

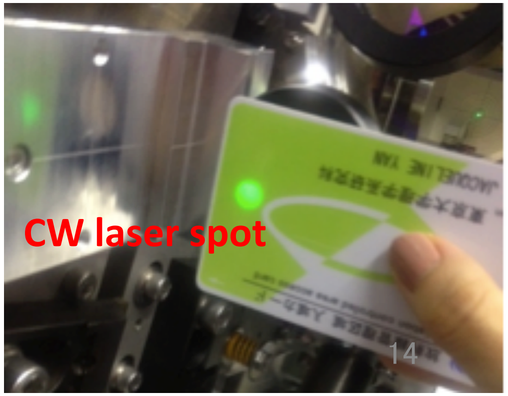


Confirm fine alignment using CW laser and transparent IP target



inside IP chamber

→**laser waist & crossing point**



CW laser spot

Performance Evaluation #1: Stability

Signal jitter sources
phase drift / jitter
Laser timing & power

Demonstration of stability in IPBSM operation : signal Jitter

long term stable performance is maintained under various scan conditions ➔ "standard"

Long range scans dedicated to error studies :

→ just as stable (jitter is not increased) compared to usual scans

(beam & IPBSM conditions, analysis method kept consistent)

Comp Sig. jitter is quite consistent at generally 20 – 25 % (@peak of fringe scans)

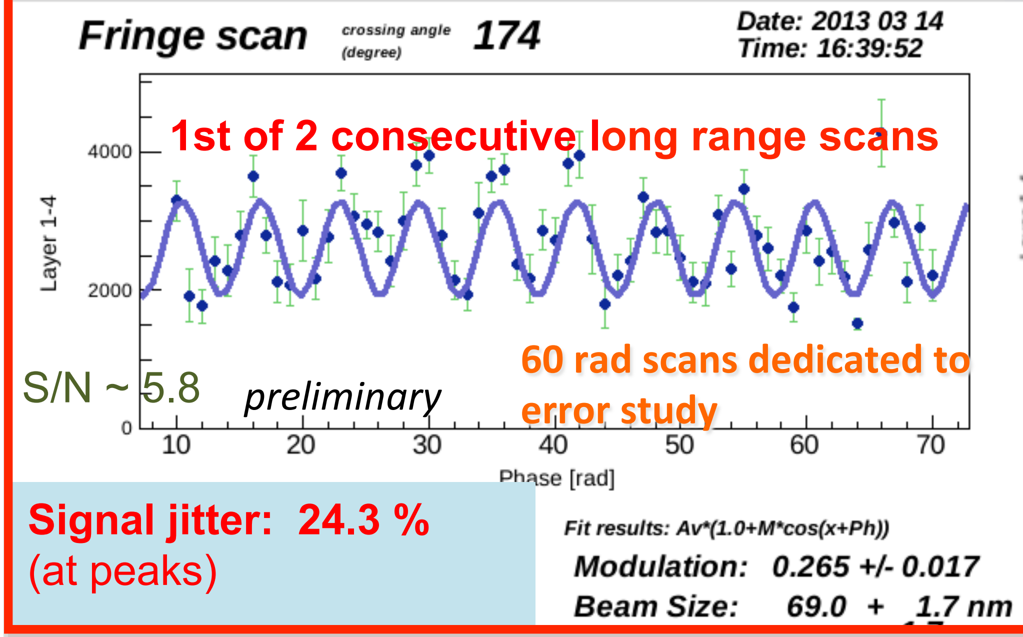
data	**range**	**Comp sig jitter (@peak of fringe scans)**
130314_155758	20 rad Nav = 10	21.1 %
130314_165737	20 rad Nav = 10	25.2 %
130314_163420	20 rad **Nav = 20**	24.3%
130314_163952	**60 rad** Nav = 10	25.4 %
130314_164840	**60 rad** Nav = 10	26.3 %

Usual scans
immediately before & after

Fine scan
Nav = 20 events
at each phase step

Long range scans
60 rad
(usually 20 rad)

Long scans from other periods show similar stability



1st of 2 consecutive long range scans

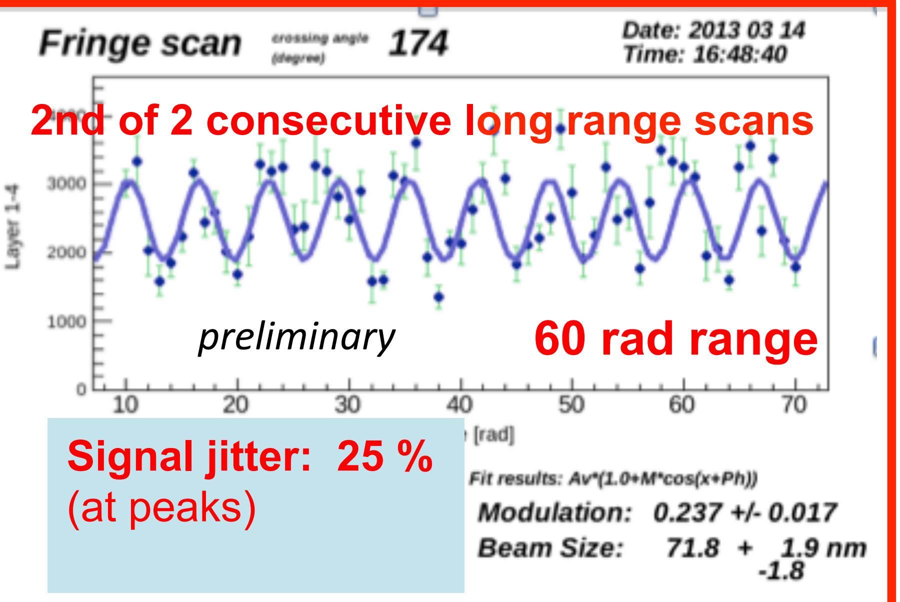
S/N ~ 5.8

preliminary

60 rad scans dedicated to error study

Signal jitter: 24.3 %
(at peaks)

Fit results: Av(1.0+M*cos(x+Ph))*
Modulation: 0.265 +/- 0.017
Beam Size: 69.0 + 1.7 nm



2nd of 2 consecutive long range scans

preliminary

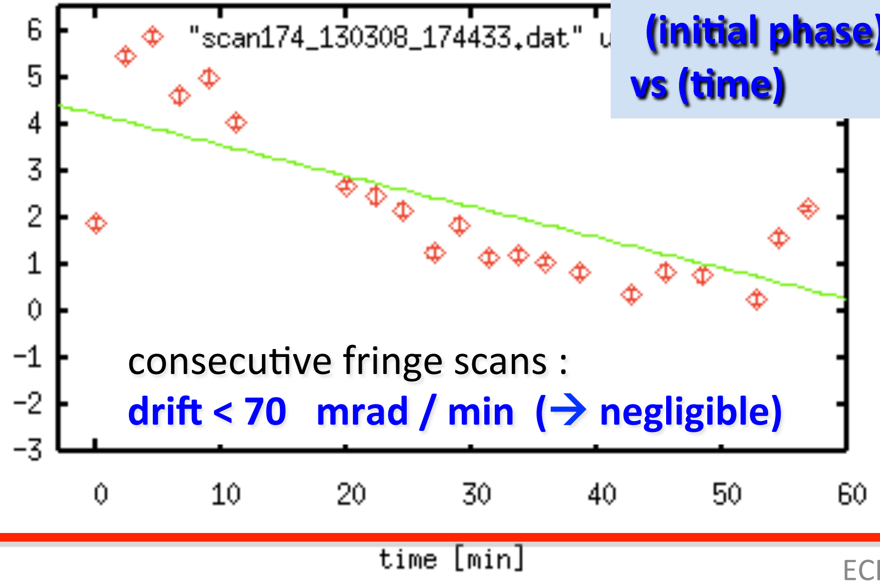
60 rad range

Signal jitter: 25 %
(at peaks)

Fit results: Av(1.0+M*cos(x+Ph))*
Modulation: 0.237 +/- 0.017
Beam Size: 71.8 + 1.9 nm -1.8

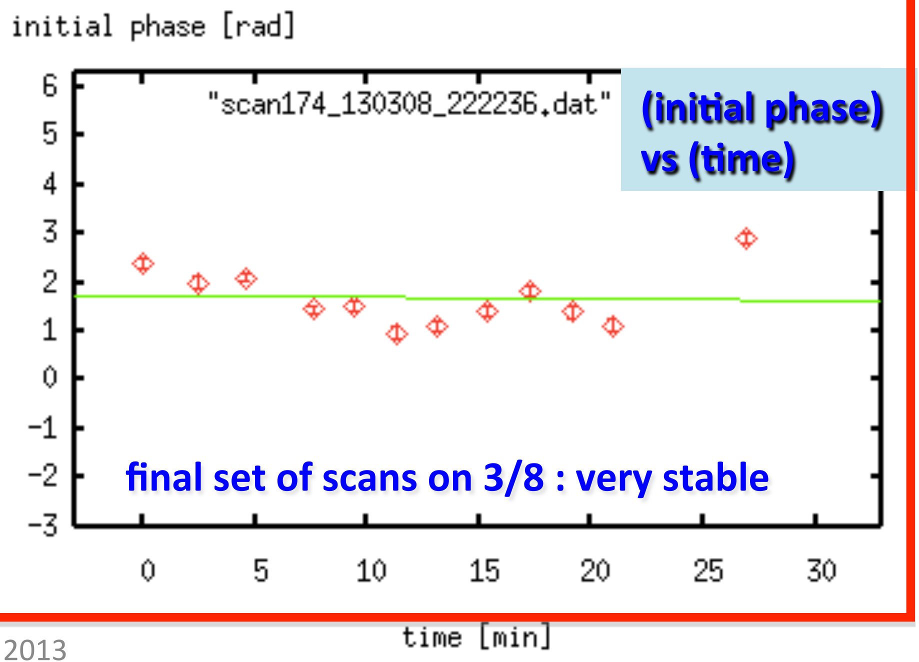
Stability is maintained for long range scans (fluctuation / drift e.g. BG, phase, timing, power, ect…)

Phase Drift



(initial phase) vs (time)

consecutive fringe scans :
drift < 70 mrad / min (→ negligible)



(initial phase) vs (time)

final set of scans on 3/8 : very stable

Study of Signal Fluctuation

Spring, 2013: 174 deg mode
contribution to Sig Jitter ΔE_{sig} / $E_{sig,\ avg}$

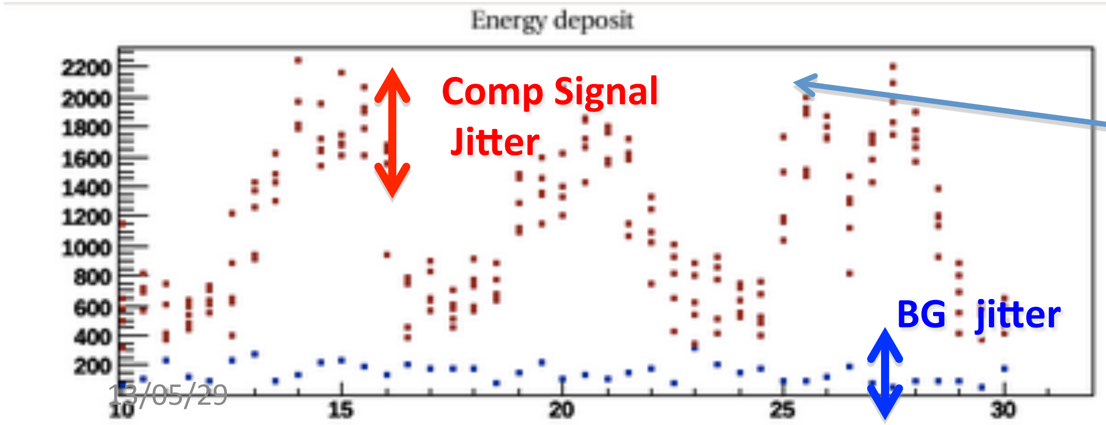
Prepared offline veto for large timing, power jittered events

Laser timing	6 - 7 % (monitored by PIN-PD signal)
Laser power	~ 1 % (from photo-diode)
Relative beam –laser position	***under investigation***
BG fluc	< 10%
detector energy resolution	< 1 %
Comp γ stat.	~ 3 %
ICT monitor accuracy	< 5 %

varies with beam condition

* Intrinsic CsI detector energy resolution (GEANT4 sim.)

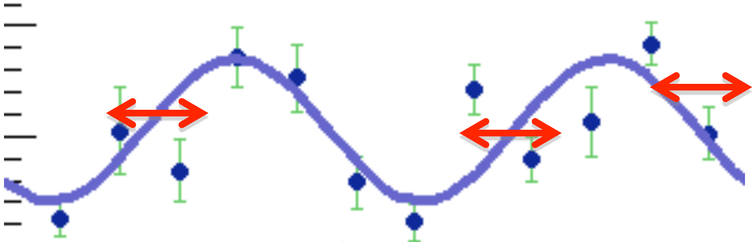
measured Comp sig energy normalized by beam intensity



signal jitter derived directly from actual fringe scans
(peaks) : **20 – 25%**

Phase Jitter / Relative Position Jitter

- hard to separate from other fluctuation sources (laser pointing jitters, drifts, ect….)
- jitters can vary greatly over time



Can't push all fluctuation to phase jitters

take high statistics scans (Nav ~ 100) under optimized conditions for dedicated analysis

Issue 1: Δy ←→ M reduction

Important to grasp residual M reduction factors in order to derive the true beamsize

$$y \rightarrow y + \Delta y$$

$$\sigma_y^2 \rightarrow \sigma_y^2 + (\Delta y)^2$$

$$\boxed{C_{\Delta y} = \exp\left(-2\left(k_y \Delta y\right)^2\right)}$$

if **Δy < 0.3 * σy**

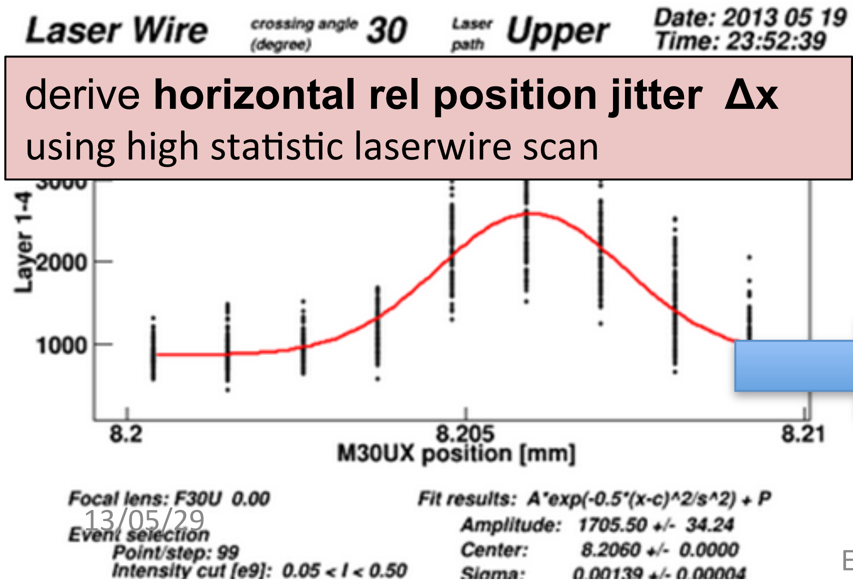
(ATF2 beamline design)

CΔy > 90 % for σy* = 65 nm

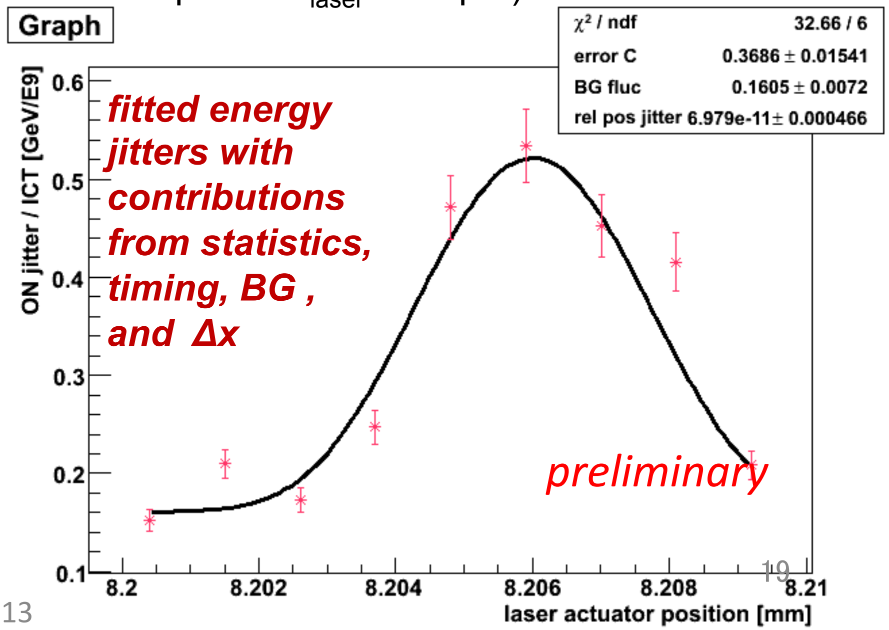
$$k_y = \frac{2\pi}{\lambda}\sin\left(\frac{\theta}{2}\right)$$

Issue 2 : fluctuation source during fringe scan

If **Δx ∼2.5 μm** cause ∼ 4 % signal jitters (assume Gaussian profile σ_{laser} = 10 μm)



derive **horizontal rel position jitter Δx**
using high statistic laserwire scan



fitted energy jitters with contributions from statistics, timing, BG , and Δx

preliminary

Investigate Signal Fluctuation

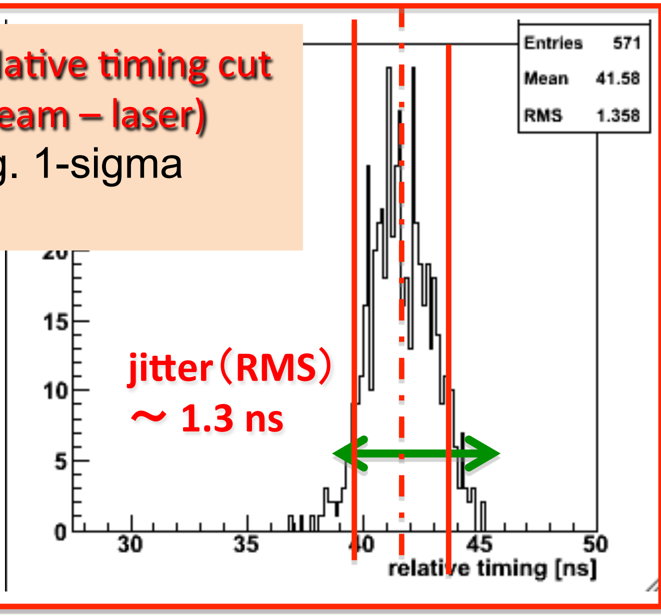
[1] improve hardware
[2] data selection

Observe ΔEsig dependence on Esig :

$$\Delta E_{sig} = \sqrt{C_{const}^2 + \left(C_{sqrt}\sqrt{Esig}\right)^2 + \left(C_{linear}Esig\right)^2}$$

possibly veto jittered points under clearly identified causes

Goal: **achieve precise M_{meas} ($\sigma y_{,meas}$)**



Relative timing cut (beam – laser)
e.g. 1-sigma

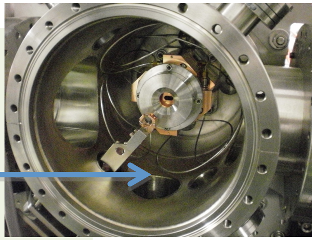
jitter (RMS) ~ 1.3 ns

e beam orbit

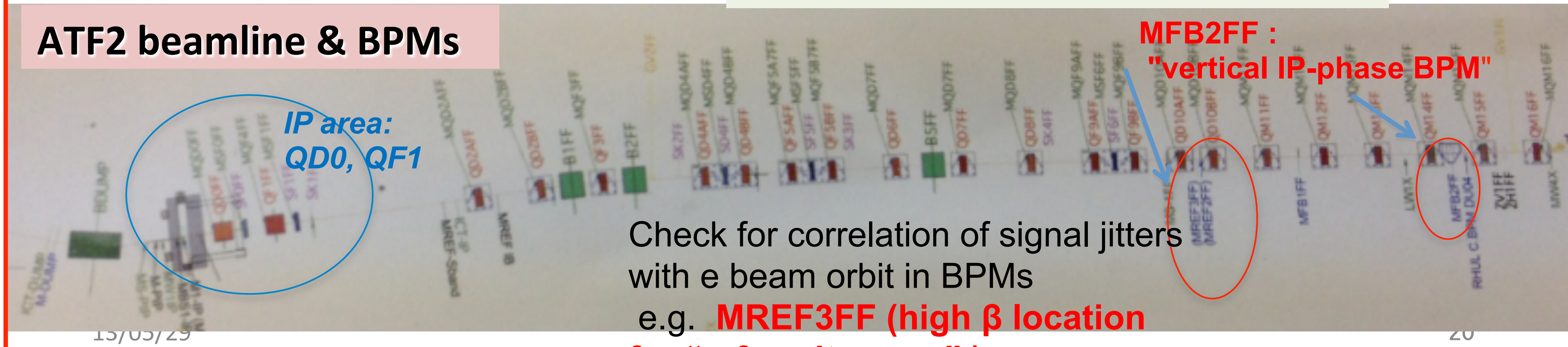
synchronize fringe scan data with all ATF2 monitors e.g. BPMs, ICT monitors

ex): check y position jitter@IP using
MFB2FF : "vertical IP-phase BPM"

Anticipate O(nm) res. measurement of beam position jitter at IP by **IPBPMs**
(under commissioning)



ATF2 beamline & BPMs



Check for correlation of signal jitters with e beam orbit in BPMs
e.g. **MREF3FF (high β location for "ref cavity scan")**

Performance Evaluation #2:

Modulation Reduction Factors

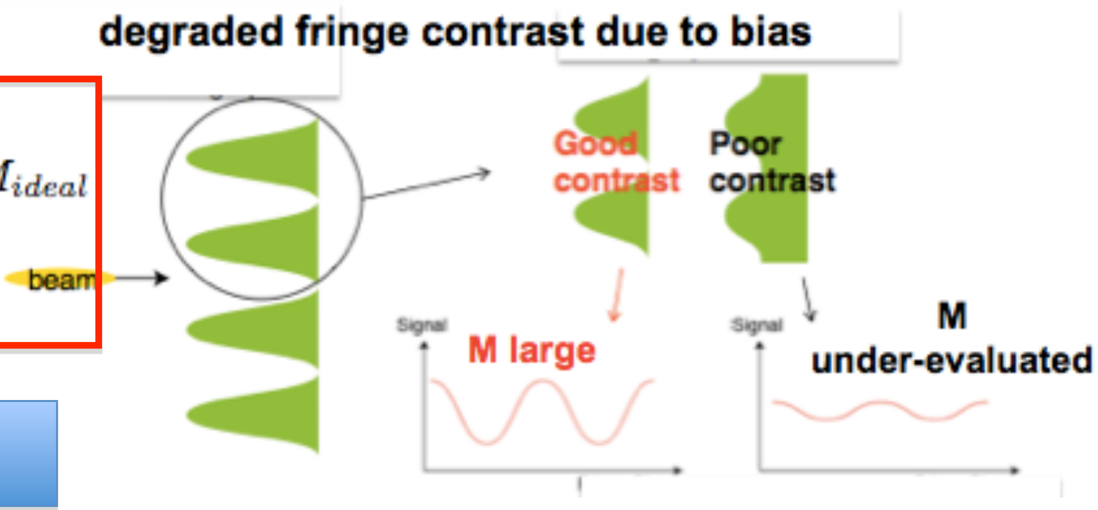
Study of M reduction

Under-evaluate M, over-evaluate σ_y

Modulation Reduction Factor

$$M_{meas} = C_1 C_2 .... M_{ideal} = \left(\prod_i C_i\right) M_{ideal}$$

$$\sigma^2_{y,ideal} + \frac{1}{2k_y^2}\left|\sum lnC_i\right|$$



How to evaluate M reduction?

（1） "Direct Method"

consecutive mode switching , under same beam condition (e.g. : 2 ° → 7 ° → 30 °)

use a σy that yields very high M at low θ mode → observe upper limit on M_{meas}

Note) **apply to a particular dedicated data sample**

（2） "Indirect Method"

Evaluate each individual factor offline and "sum up"

Note) **represents the typical conditions of a particular period**

however …… **hard to derive overall M reduction**

(e.g. some factors lack quantitative evaluation, vary over time, only can get **"worst limit")**

Plan for assessment of M reduction factors

priorities

1st : suppress M reduction → aim for Ctotal ~ 1

2nd: precisely evaluate any residual errors → derive the "true beam size"

how to find out **bias due to "uncertain" individual factors:** (e.g. relative position jitter, spatial coherence)

test using **"direct method"**

At a low θ mode : measure a large M (near resolution limit) using a sufficiently small σy
compare results with higher θ modes

example:

if we measure M corresponding to σy = 350 nm at 7 deg mode

expect **M = 0.98 at 2.75 deg mode** *(try to keep within 2-8 deg)*

what if we get only 0.95 ??? → **Ctotal ~ 0.97** → ***no individual bias factor worse than 0.97***

Note:

- conditions may vary over time → confirm with repeated measurements
- need prove that these factors are really independent of θ

Individual M Reduction Factors

Represent typical condition of a particular period

Error source	**M reduction factor** Spring 2013, 174 deg
Fringe tilt (z, t)	Beamtime→ final optimization by **"tilt scan"** **Limited by alignment precision** Major bias if unattended to
profile imbalance	Cpro > 98.5% *assume Gaussian laser profile (spot size)*
power imbalance	Cpow > 99 % **power measured directly for each path**
Laser polarization	Optimized to **"S state"** using λ / 2 plate **Measured polarization and half mirror reflective properties**
Phase drift	not major issue **drift : < 70 mrad / min** during consecutive fringe scans
Laser path alignment	Ct,pos : ~ 99 %, Cz,pos : > 98 % Resolution of mirror actuators aligning laser to beam

Still quantitatively uncertain under evaluation:

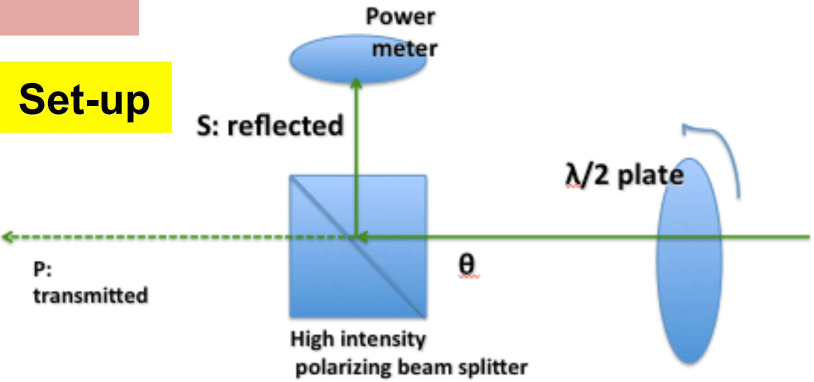
- **relative position jitter (phase jitter)**
- **Spatial coherence**

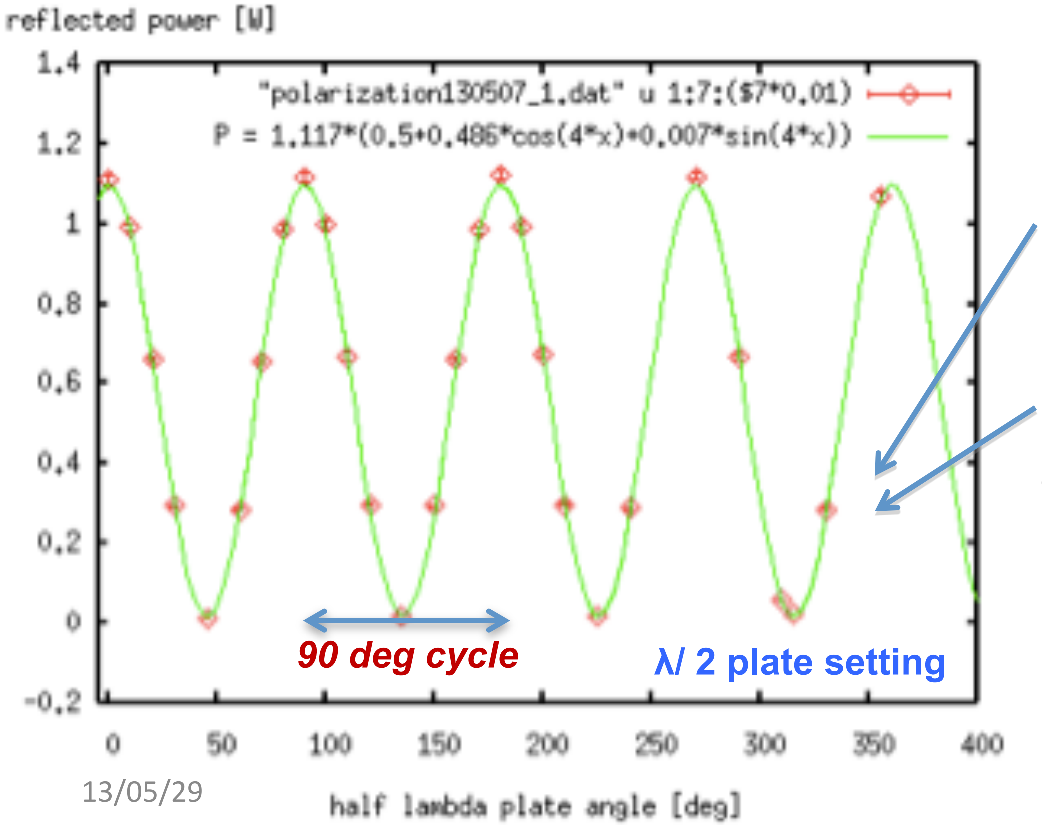
Could be major bias

laser polarization related measurements

IPBSM laser optics is designed for pure linear S polarization

Set-up





results

polarization measured just after injection onto vertical table

- **very close to linearly S polarization**
- should be very little polarization related M reduction

"P contamination": **P_p/P_s = (1.46± 0.06) %**

power ratio

to precisely confirm there is no residual M reduction …….

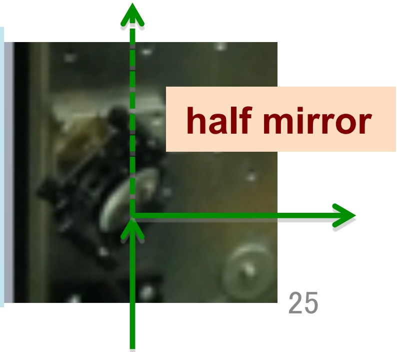
next plan individual measurements for upper and lower paths near IP

Hardware prepared → carry out in June

also measured **reflective properties of "half mirror"**

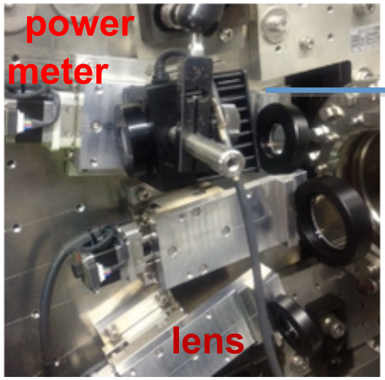
Rs = 50.3 %, Rp = 20.1 %

Match catalog specifications !!

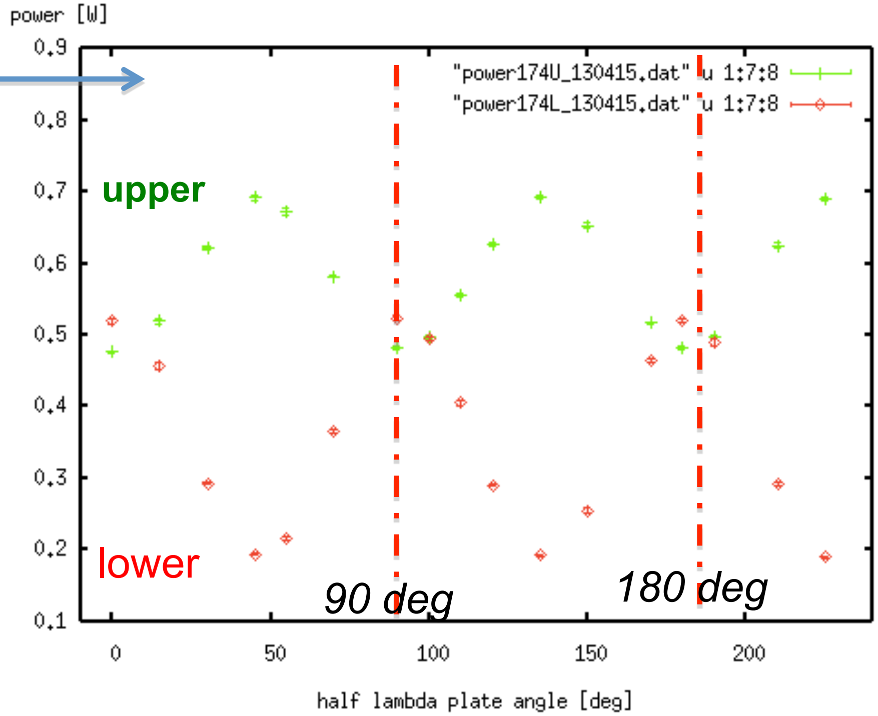


laser polarization and power balance

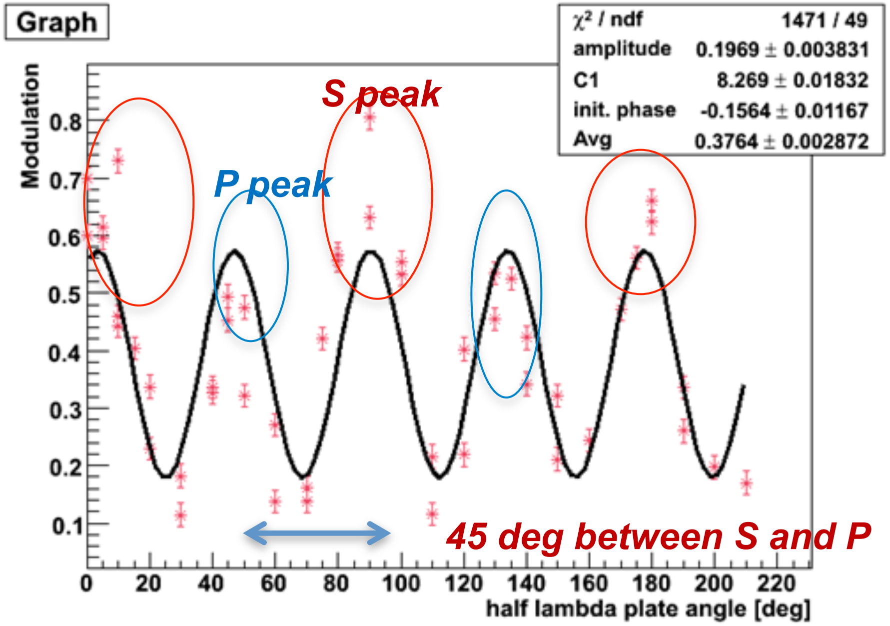
Rotate λ/2 plate and measure high power Immediately in front of final focus lenses



investigate power balance: U vs L path



During Beamtime
"λ/2 plate scan " to maximize M



Rotate λ/2 plate angle

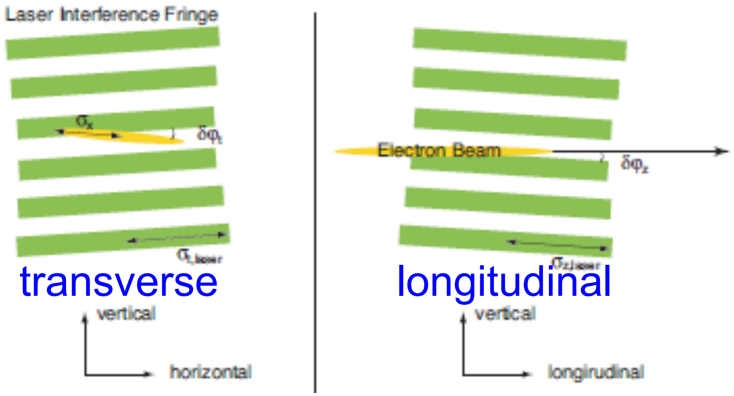
"S peaks" (maximum M)
also yield best power balance
→ Minimize M reduction

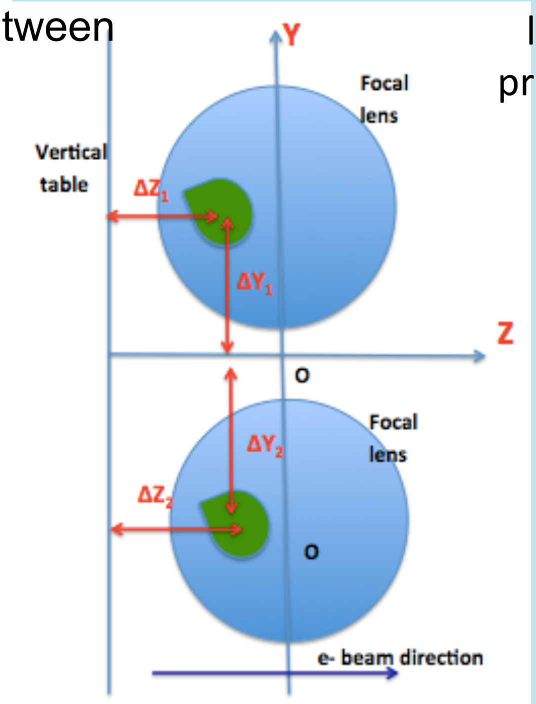
$$C_{pow} = \frac{2\sqrt{P_U / P_L}}{1 + P_U / P_L}$$

M reduction factor due to power imbalance

Fringe Tilt

Mismatch in axis between fringe and beam





laser path observed on lens: precision ~ 0.5 mm (few mrad)

issues:

- Position drifted by the time we scan
- e beam may also be rotated in transverse

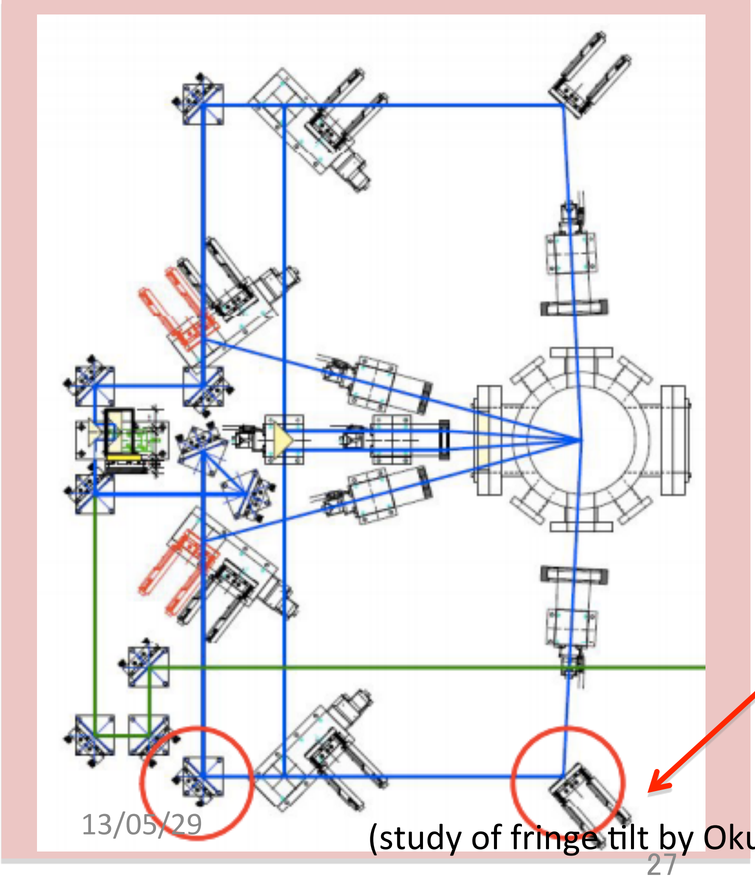
Current method : "tilt scan"

fringe pitch / roll adjustment:

observe M reduction " C_{tilt} "

(70 - 80% if uncorrected)

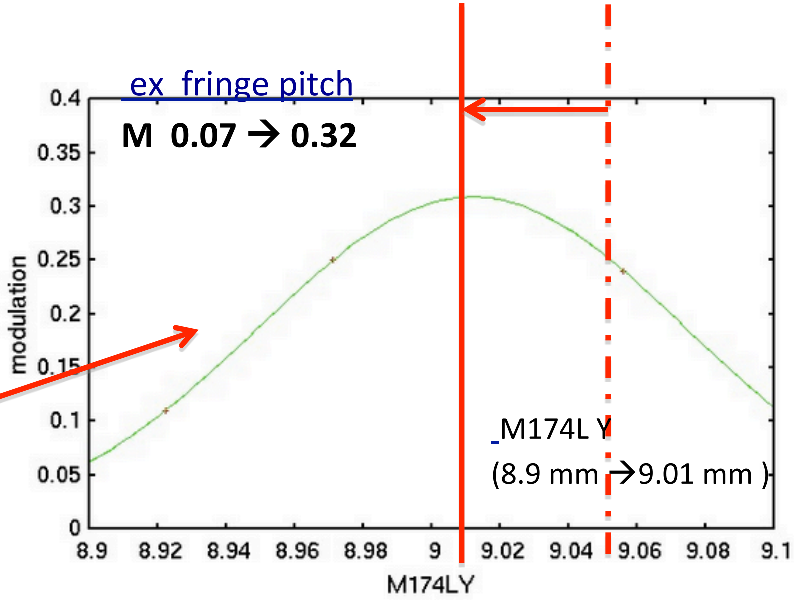
directly use e beam as reference for tilt adjustment



important adjustment to eliminate M reduction

Mirrors for adjusting tilt

(study of fringe tilt by Okugi-san)



ex fringe pitch

M 0.07 → 0.32

M174L Y (8.9 mm →9.01 mm)

Summary

Shintake Monitor (IPBSM)

beamsize monitor using laser interference

- ❖ Only existing device capable of measuring $\sigma_y < 100$ nm
- ❖ **Indispensible for achieving ATF2 goals and realizing ILC**

< Status >

- ❖ **contribute with stable operation to continuous beam size tuning**
- ❖ **Consistent measurement of M ∼ 0.3** (174 ° mode) *at low beam intensity*
  correspond to σ_y ~ 65 nm (assuming no M reduction)
- ❖ Application of various linear / non-linear multi- knobs
- ❖ dedicated studies of e beam and IPBSM errors

<towards performance improvement>

Performance significantly improved by laser optics reforms

suppressed error sources, improved laser path reliability & reproducibility

Goals

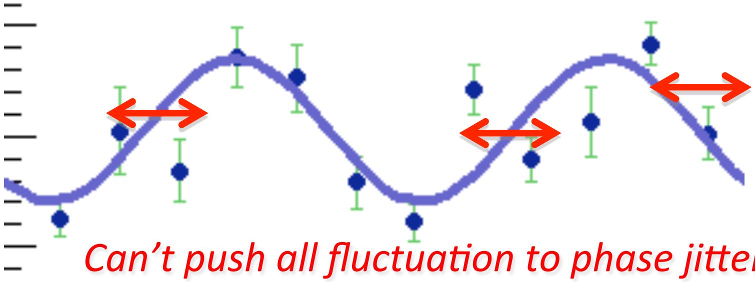
Towards confirming $\sigma_y = 37$ nm

- ◆Maintain / improve **beamtime performance :** e.g. stability, precision
- ◆Assess **residual systematic errors** → **derive the "true beam size"**
- ◆ **stable measurements of σy < 50 nm** within this run

Backup

Phase Jitter / Relative Position Jitter

- hard to separate from other fluctuation sources (laser pointing jitters, drifts, ect….)
- jitters can vary greatly over time



Can't push all fluctuation to phase jitters

take high statistics scans (Nav ~ 100) under optimized conditions for dedicated analysis

Issue 1: Δy ←→ M reduction

Important to grasp residual M reduction factors in order to derive the true beamsize

$$y \rightarrow y + \Delta y$$

$$\sigma_y^2 \rightarrow \sigma_y^2 + (\Delta y)^2$$

$$C_{\Delta y} = \exp\left(-2\left(k_y \Delta y\right)^2\right)$$

if **Δy < 0.3 * σy**

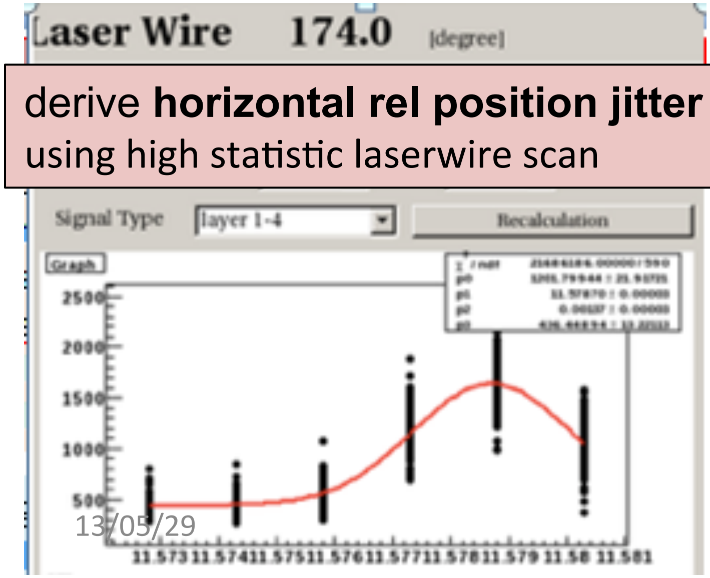
(ATF2 beamline design)

CΔy > 90 % for σy* = 65 nm

$$k_y = \frac{2\pi}{\lambda}\sin\left(\frac{\theta}{2}\right)$$

Issue 2 : fluctuation source during fringe scan

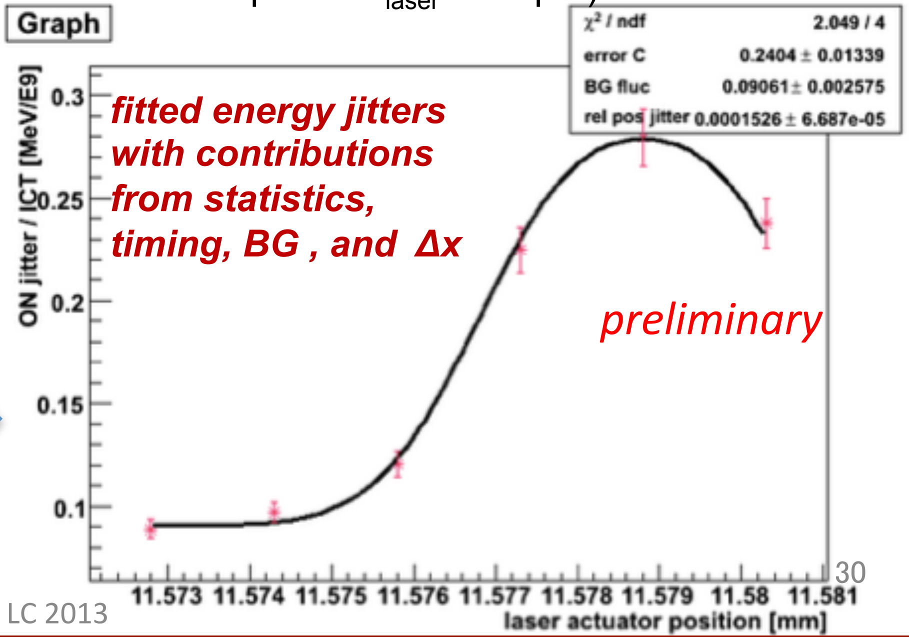
If **Δx ∼2.5 μm** cause ∼ 4 % signal jitters (assume Gaussian profile σ_{laser} = 10 μm)



Laser Wire 174.0 [degree]

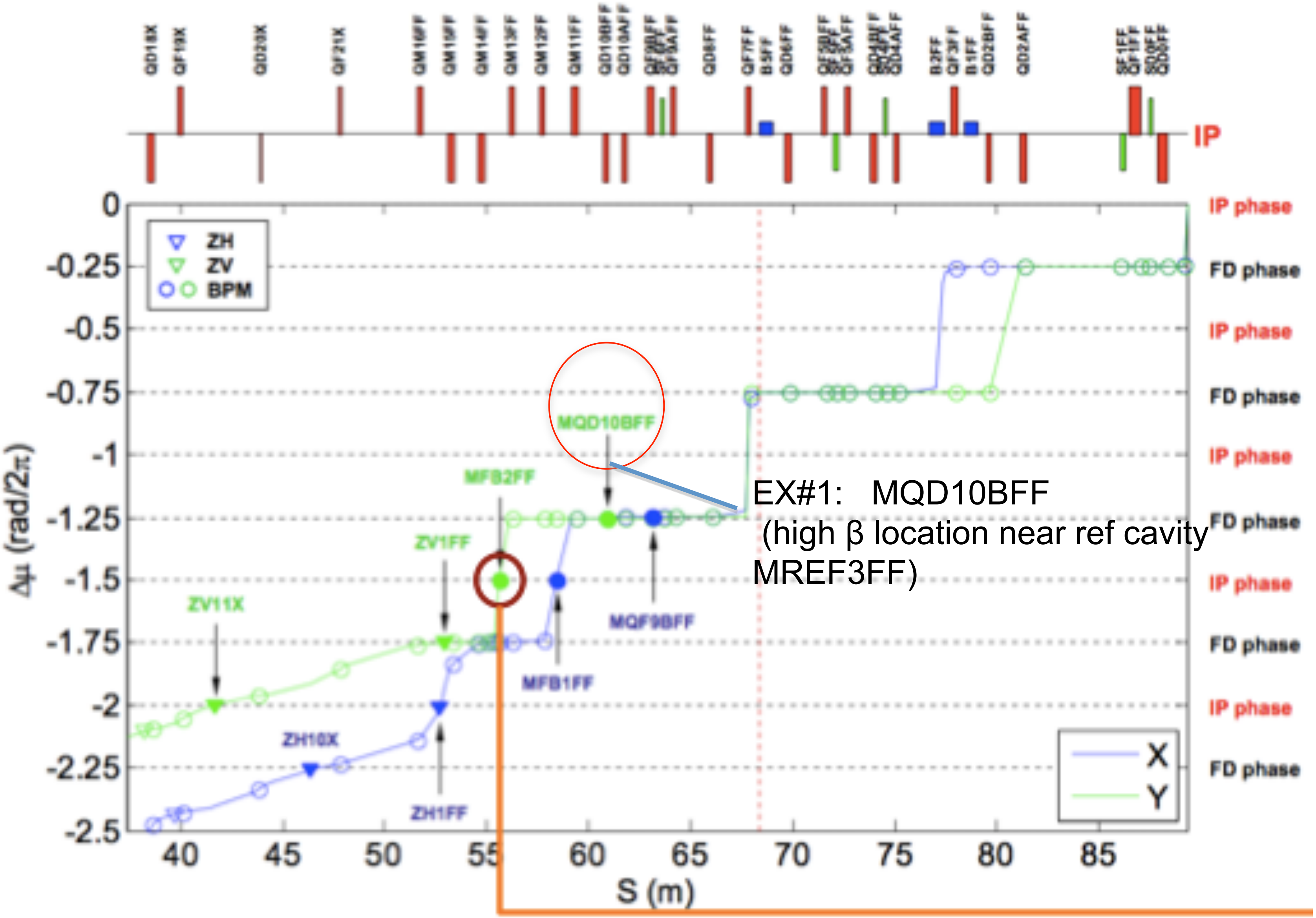
derive **horizontal rel position jitter Δx**

using high statistic laserwire scan



fitted energy jitters with contributions from statistics, timing, BG , and Δx

preliminary



EX#1: MQD10BFF
(high β location near ref cavity
MREF3FF)

Ex #2:
check y position jitter@IP
using **MFB2FF : "vertical IP-phase BPM"**

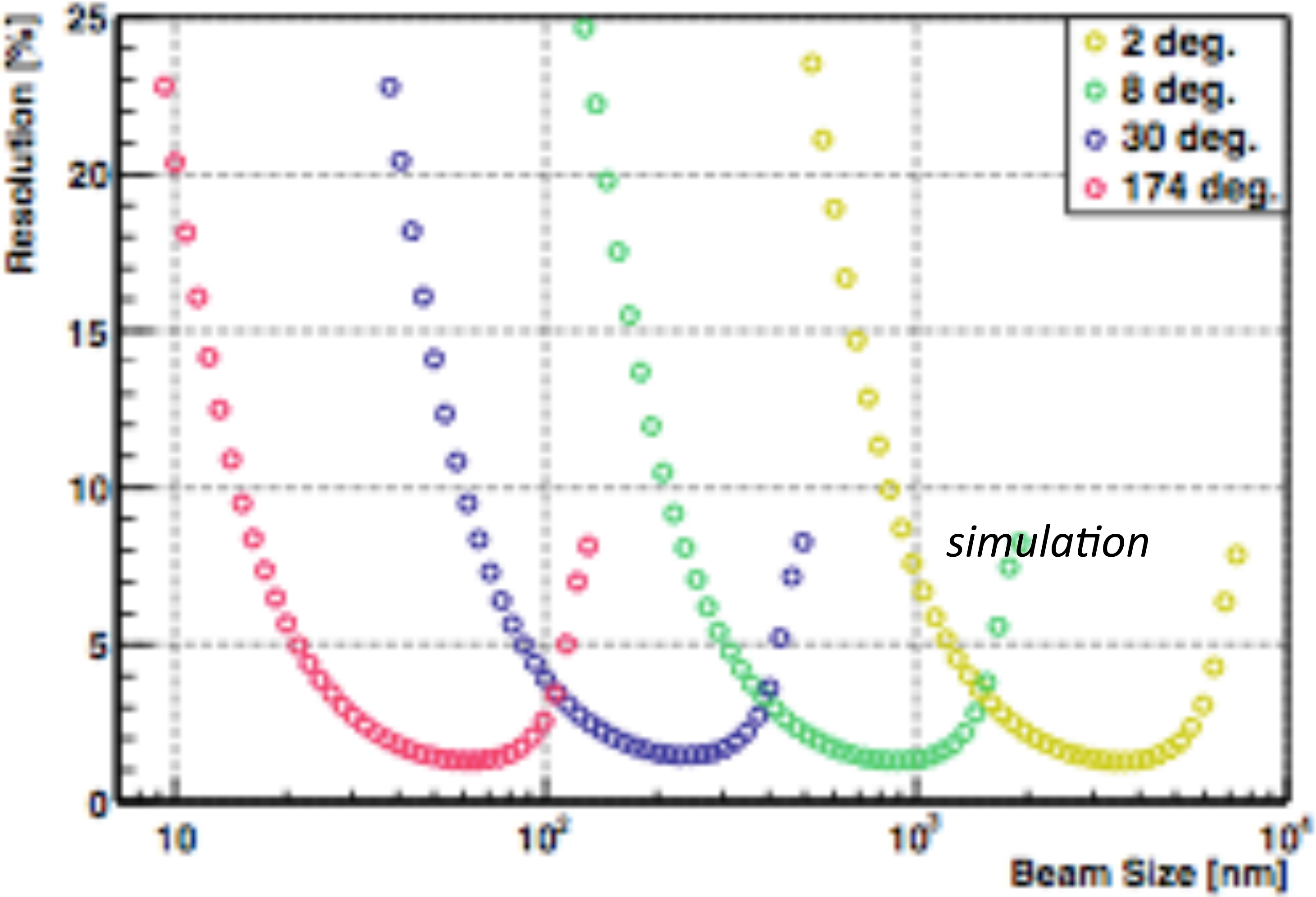
Expected Performance

$$37 \pm 2\,(\text{stat.})\,{}^{-0}_{+4}\,(\text{syst.})\ \ \text{nm}$$

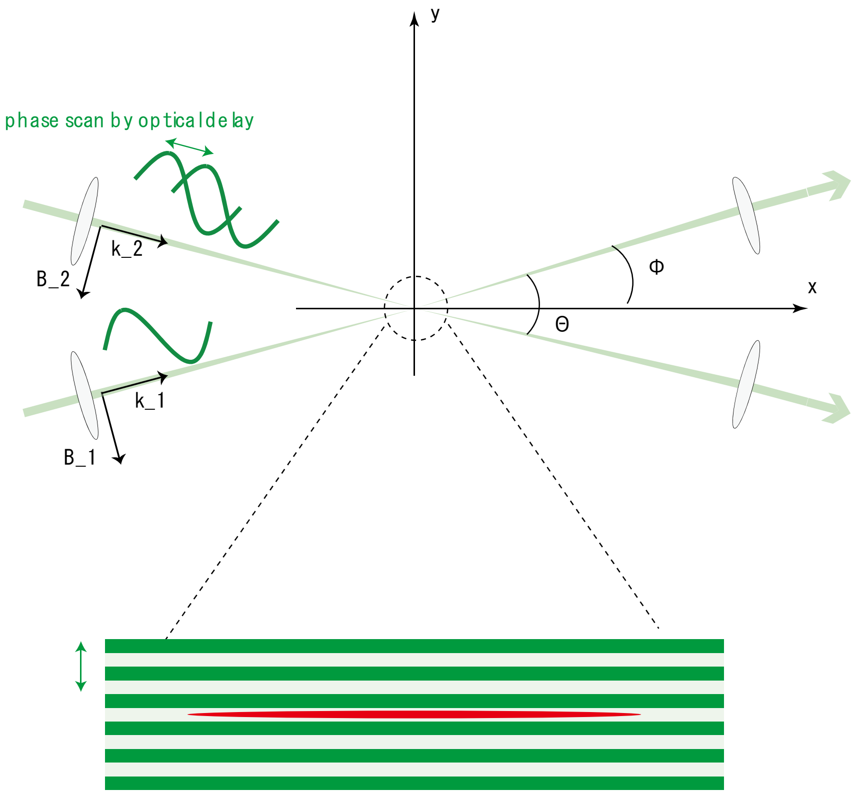
must select appropriate mode according to beam focusing

Resolution for each θ mode

Measures
σy* = 25 nm ∼few μm
with < **10% resolution**



Laser interference scheme



Wave number vector of two laser paths

$$\vec{k_1} = (k\cos\phi, k\sin\phi, 0) \equiv (k_x, k_y, 0)$$
$$\vec{k_2} = (k\cos\phi, -k\sin\phi, 0)$$

S-polarized laser

$$\vec{B_1} = B(\sin\phi, -\cos\phi, 0)\cos(\omega t - \vec{k_1}\cdot\vec{x} - \frac{\alpha}{2})$$
$$\vec{B_2} = B(-\sin\phi, -\cos\phi, 0)\cos(\omega t - \vec{k_2}\cdot\vec{x} + \frac{\alpha}{2})$$

$$\vec{B} = \vec{B_1} + \vec{B_2}$$
$$= 2B\begin{pmatrix} \sin\phi\sin(\omega t - k_x x)\sin(k_y y + \frac{\alpha}{2}) \\ -\cos\phi\cos(\omega t - k_x x)\cos(k_y y + \frac{\alpha}{2}) \\ 0 \end{pmatrix}$$

Time averages magnetic field causes inverse Compton scattering

$$< B_x^2 + B_y^2 >= B^2(1 + \cos\theta\cos(2k_y y + \alpha))$$

- phase shift at IP ←→ α
- wave number component along y-axis $2k_y = 2k\sin\phi$
- modulation depends on cosθ

Fringe pitch

$$d = \frac{\pi}{k_y} = \frac{\lambda}{2\sin\phi}$$

Calculation of beam size

Total signal energy measured by γ-detector

Convolution of

- Laser magnetic field : Sine curve
- Electron beam profile : Gaussian

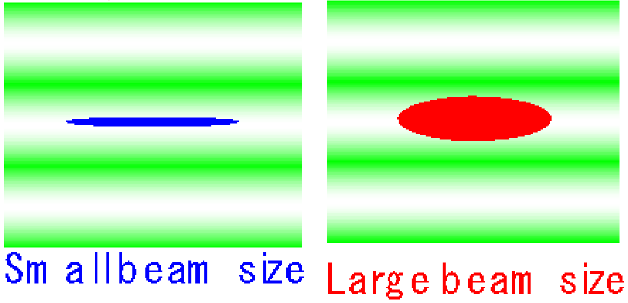
Laser magnetic field

Electron Beam profile with beam size σ_y along y-direction

$$S \propto \int dy' \boxed{< B_x^2 + B_y^2 >} \boxed{\frac{1}{\sqrt{2\pi\sigma_y^2}} \exp(\frac{-y'^2}{2\sigma_y^2})}$$

$$= \int dy' B^2 (1 + \cos\theta \cos(2k_y y + \alpha)) \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp(\frac{-y'^2}{2\sigma_y^2})$$

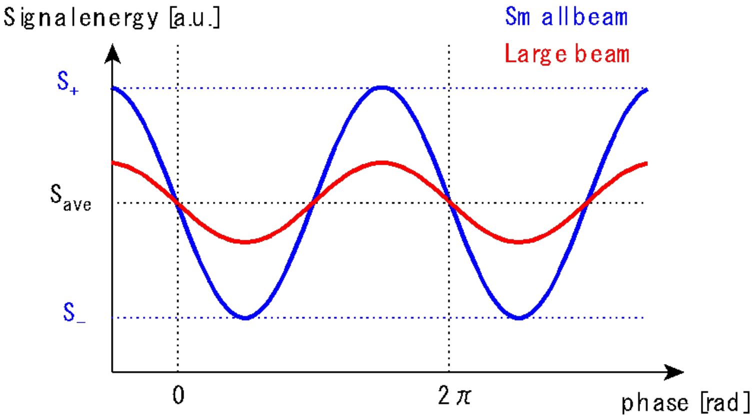
$$S = S_{ave}(1 + \cos((2k_y y) + \alpha) \cos\theta \exp(-2(k_y\sigma_y)^2))$$



$S_\pm$: Max / Min of Signal energy

$$S_+ = S_{ave}(1 + |\cos\theta| \exp(-2(k_y\sigma_y)^2))$$

$$S_- = S_{ave}(1 - |\cos\theta| \exp(-2(k_y\sigma_y)^2))$$

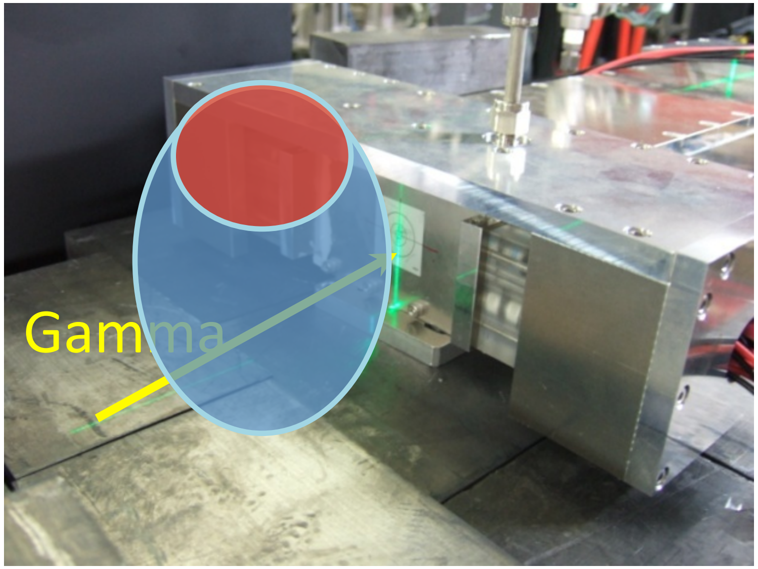


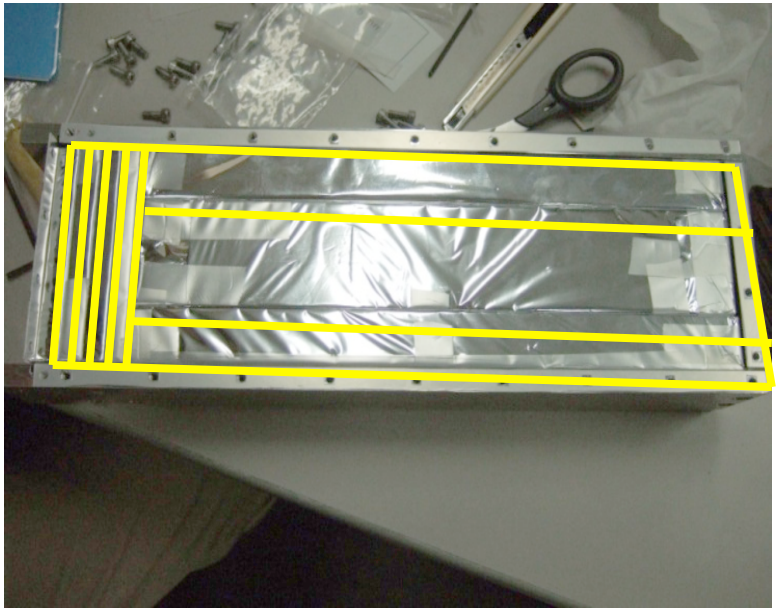
M : Modulation depth

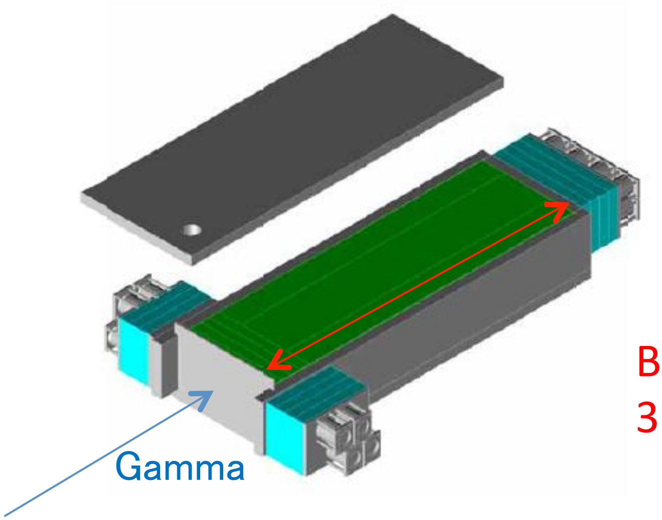
$$M = \frac{S_+ - S_-}{S_+ + S_-} = |\cos\theta| exp(-2(k_y\sigma_y)^2)$$

$$\sigma_y = \frac{1}{2k_y}\sqrt{2\ln(\frac{|\cos\theta|}{M})}$$

Gamma detector







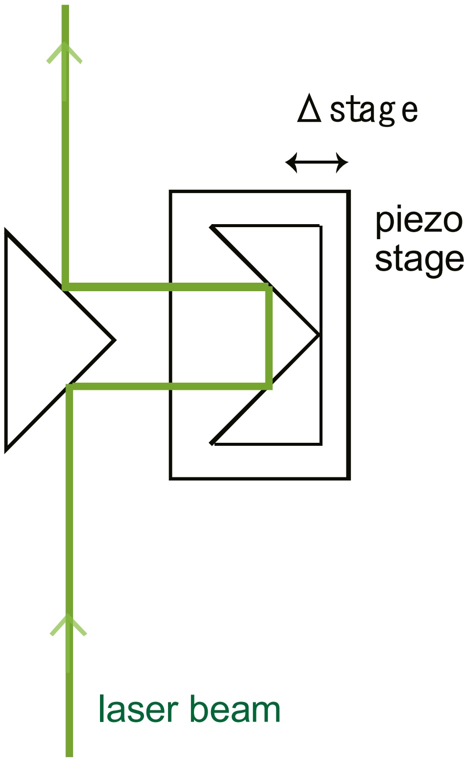
Calorimeter like gamma detector
- Multi layered CsI(TI) scintillator
- PMT R7400U
 (Hamamatsu Photonics)

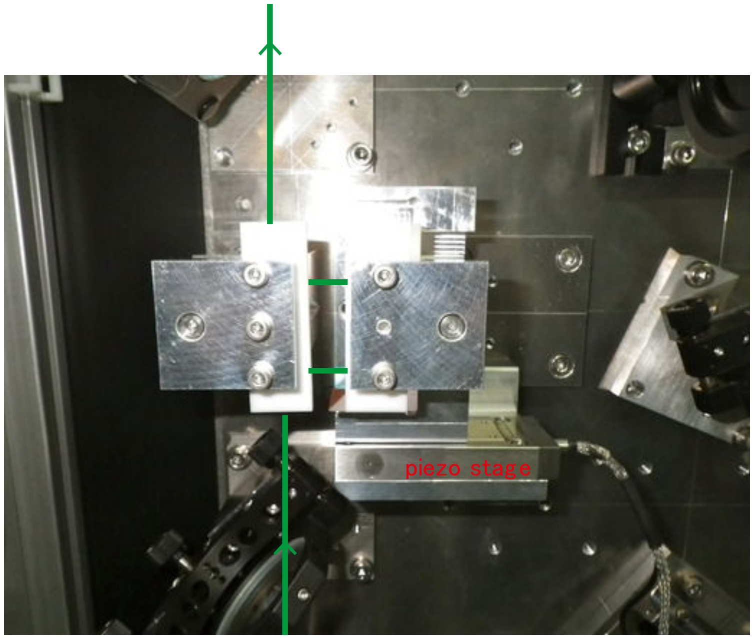
Beam longitudinal direction:
33cm (17.7radiation length)

Width : 10 cm
Height : 5 cm

Phase control by optical delay line

Optical delay line (~10 cm)
Controlled by piezo stage



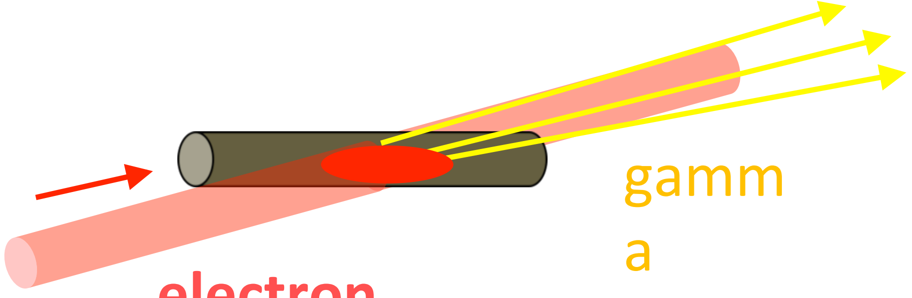


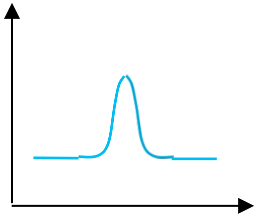
Movement by piezo stage : Δ_{stage} → Phase shift

$$\Delta\alpha = 2\pi\frac{\Delta_{stage}}{\lambda}$$

measurement scheme

wire scanner, laser wire

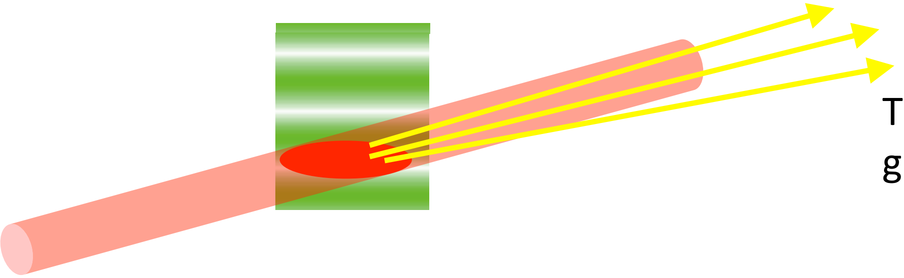


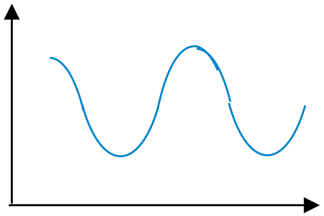


Calculate beam size from Gaussian sigma

measurable beamsize ~ 1μm

Shintake monitor



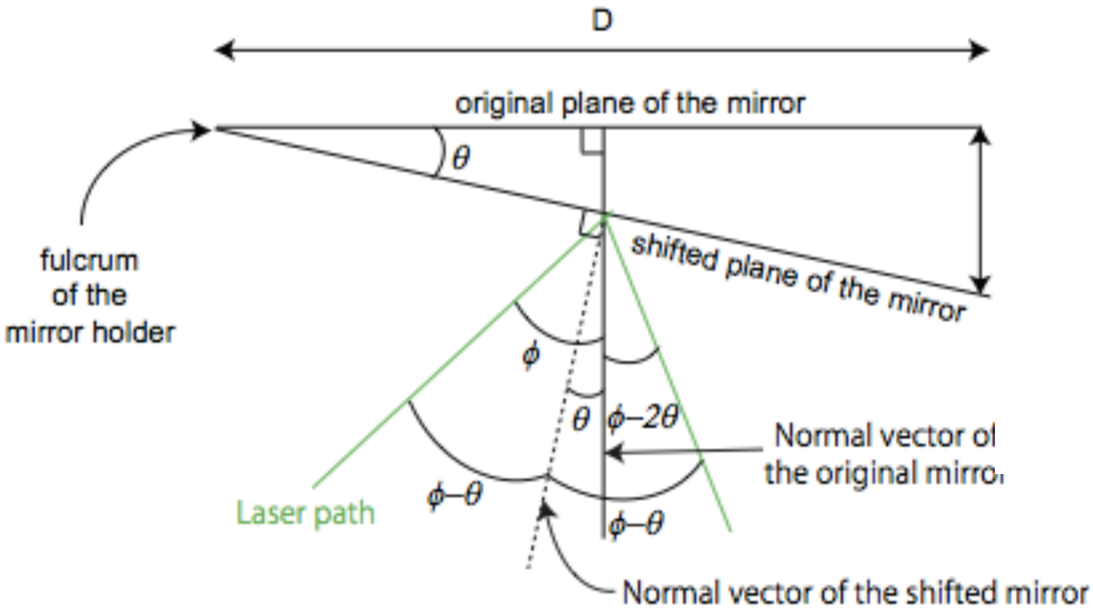


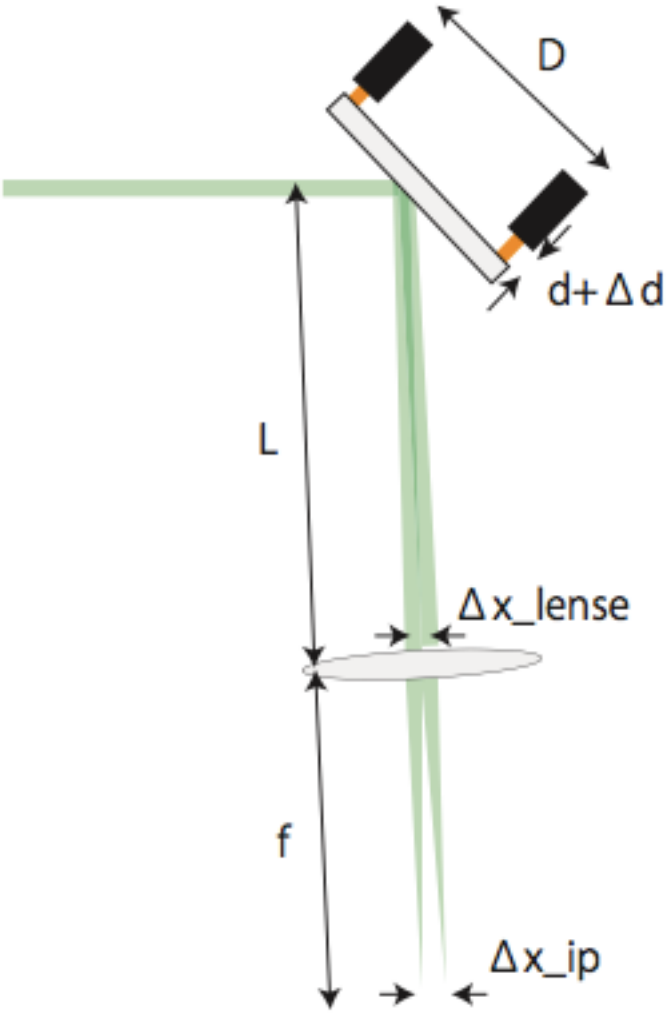
Calculate beam size from contras

of sine curve

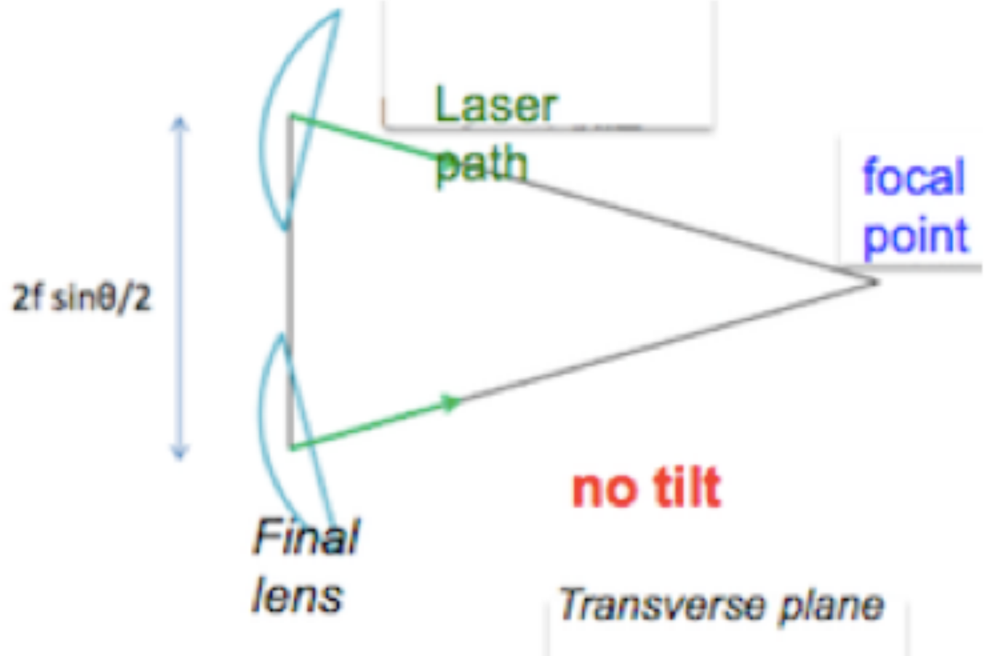
measurable beamsize < 100nm

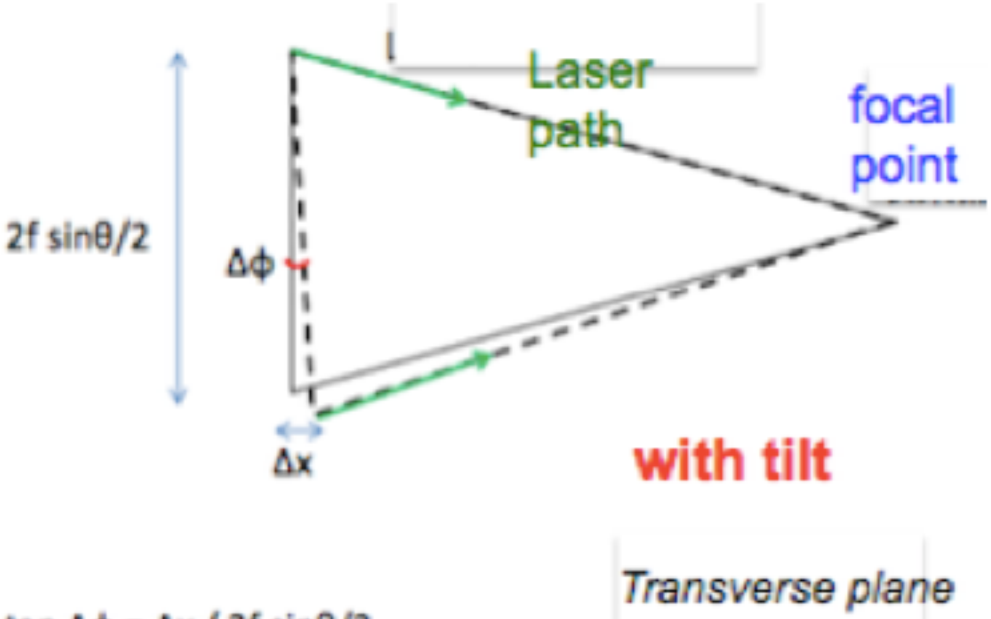
mode	D [mm]	L [mm]	f [mm]	$2\frac{L}{D}$
2 to 8 deg	61.9	900	250	29.0
30 deg	61.9	400	300	12.9
174 deg	61.9	350	250	11.3





$$\Delta x_{IP} = 2f\tan\frac{\Delta d}{D} \simeq 2\frac{f}{D}\Delta d$$





$\tan \Delta\phi = \Delta x / 2f \sin\theta/2$

laser path misalignment

precision of alignmnet by mirror actuator

- Δz, about 15-20% of σz,laser (from zscan)
- Δt about 5-10% of σt, laser * (from laserwire scan)

σz,laser about half of σt,laser

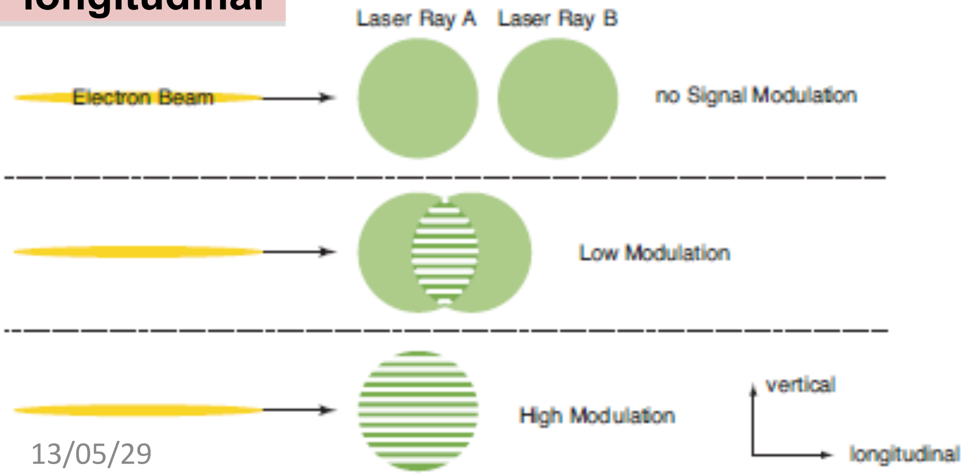
longitudinal **Cz- pos > 98.9 %**

transverse **Ct-pos ~ 99.9 %**

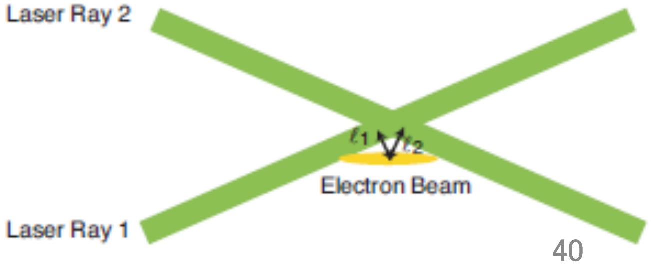
$$\text{longitudinal :} \quad C_{z,pos} = exp\left(-\frac{z_0^2}{2\sigma_{z,laser}^2}\right)$$

$$\text{transverse :} \quad C_{t,pos} = \frac{1}{cosh\left(-\frac{l_1^2}{4\sigma_{t,laser}^2}\right)}$$

longitudinal



transverse



Phase (relative position) jitter

If Δy ~ 0.3 σy
C ∼ 88.4% for 70 nm @ 174 deg
C∼ 96.2% for 150 nm@30 deg mode
C∼97.7% for 500 nm@7 deg mode

phase jitter observed from fringe scan: about 200 mrad ??
→ C ∼ 98 % (????)

$$C_{phase} = exp\left(-\frac{(\Delta\alpha)^2}{2}\right) \Longleftrightarrow C_{\Delta y} = exp\left(-2\left(k_y \Delta y\right)^2\right) \qquad \left(k_y = \frac{2\pi}{\lambda} sin\left(\frac{\theta}{2}\right)\right)$$

$$\Longrightarrow \quad \Delta y = \frac{\Delta\alpha}{2k_y} = \frac{\lambda\Delta\alpha}{4\pi sin\left(\theta/2\right)}$$