Emittance Reconstruction from measured Beam Sizes

A. Faus-Golfe and Jesus Navarro Faus
IFIC Valencia

The **emittances** could be reconstructed from the **beam size measurement**s at different locations along the beamline. The **2D** (transverse) and **4D** (intrinsic) emittances could be obtained by **numerically solving three separated systems** of coupled equations. When the number of measurement stations is greater than four, these **systems are overdetermined**, and the numerical solutions can lead to unphysical results. The incidence of such meaningless results usually increases if the measurements are noisy. Some numerical rules could be used to study the conditioning of these systems.

The **main objective** of this work is to **study analytically** the conditions of **solvability of these systems of equations** and its implication in the emittance reconstruction algorithms used in the accelerators. The aim is to give some hints about the **optical constrains** and the **location** of the **measurement stations**.

Emittance Reconstruction   The formalism

The transverse beam envelope matrix:

$$\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ \sigma_2 & \sigma_5 & \sigma_6 & \sigma_7 \\ \sigma_3 & \sigma_6 & \sigma_8 & \sigma_9 \\ \sigma_4 & \sigma_7 & \sigma_9 & \sigma_{10} \end{pmatrix} \longrightarrow \begin{pmatrix} <x^2> & <xx'> & <xy> & <xy'> \\ <xx'> & <x'^2> & <x'y> & <x'y'> \\ <xy> & <x'y> & <y^2> & <yy'> \\ <xy'> & <x'y'> & <yy'> & <y'^2> \end{pmatrix}$$

The projected emittances (2D) ε_x and ε_y are:

X submatrix

$$\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ \sigma_2 & \sigma_5 & \sigma_6 & \sigma_7 \\ \sigma_3 & \sigma_6 & \sigma_8 & \sigma_9 \\ \sigma_4 & \sigma_7 & \sigma_9 & \sigma_{10} \end{pmatrix} \begin{array}{l} \longrightarrow \quad \epsilon_x = \sqrt{\sigma_1\sigma_5 - \sigma_2^2} \\ \longrightarrow \quad \epsilon_y = \sqrt{\sigma_8\sigma_{10} - \sigma_9^2} \end{array}$$

Y submatrix

Diagonalization of the beam matrix yields the intrinsic emittances ε_1 and ε_2 (4D) :

$$\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ \sigma_2 & \sigma_5 & \sigma_6 & \sigma_7 \\ \sigma_3 & \sigma_6 & \sigma_8 & \sigma_9 \\ \sigma_4 & \sigma_7 & \sigma_9 & \sigma_{10} \end{pmatrix} \longrightarrow \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \end{pmatrix}$$

Emittance Reconstruction The formalism

Experimentally only the horizontal σ_1 vertical $\boldsymbol{\sigma_8}$ are directly measured. The coupling term $\boldsymbol{\sigma_3}$ can be deduced by measuring the beam size along a tilted axis at an angle respect to the horizontal beam size.

At least **ten measurements** are required to **reconstruct the beam matrix**. The ten values could be obtained by changing the optics in a controlled manner at the location of the measurements or by **measuring the beam size** at **different locations**.

Assuming ***N*** measurement stations, for each measurement station labelled as ***i*** one obtain the following **systems of coupled equations**:

$$\hat{s}_1^{(i)} = R_{11}^{2(i)} s_1 + 2R_{11}^{(i)} R_{12}^{(i)} s_2 + R_{12}^{2(i)} s_5$$

$$\hat{s}_8^{(i)} = R_{33}^{2(i)} s_8 + 2R_{33}^{(i)} R_{34}^{(i)} s_9 + R_{34}^{2(i)} s_{10}$$

$$\hat{s}_3^{(i)} = R_{11}^{(i)} R_{33}^{(i)} s_3 + R_{11}^{(i)} R_{34}^{(i)} s_4 + R_{12}^{(i)} R_{33}^{(i)} s_6 + R_{12}^{(i)} R_{34}^{(i)} s_7$$

Introducing the matrices:

$$M_X = \begin{pmatrix} R_{11}^{2(1)} & 2R_{11}^{(1)}R_{12}^{(1)} & R_{12}^{2(1)} \\ R_{11}^{2(2)} & 2R_{11}^{(1)}R_{12}^{(2)} & R_{12}^{2(2)} \\ \dots & \dots & \dots \\ R_{11}^{2(N)} & 2R_{11}^{(N)}R_{12}^{(N)} & R_{12}^{2(N)} \end{pmatrix}$$

$$M_Y = \begin{pmatrix} R_{33}^{2(1)} & 2R_{33}^{(1)}R_{34}^{(1)} & R_{34}^{2(1)} \\ R_{33}^{2(2)} & 2R_{33}^{(1)}R_{34}^{(2)} & R_{34}^{2(2)} \\ \dots & \dots & \dots \\ R_{33}^{2(N)} & 2R_{33}^{(N)}R_{34}^{(N)} & R_{34}^{2(N)} \end{pmatrix}$$

$$M_{XY} = \begin{pmatrix} R_{11}^{(1)}R_{33}^{(1)} & R_{11}^{(1)}R_{34}^{(1)} & R_{12}^{(1)}R_{33}^{(1)} & R_{12}^{(1)}R_{34}^{(1)} \\ R_{11}^{(2)}R_{33}^{(2)} & R_{11}^{(2)}R_{34}^{(2)} & R_{12}^{(2)}R_{33}^{(2)} & R_{12}^{(2)}R_{34}^{(2)} \\ \dots & \dots & \dots & \dots \\ R_{11}^{(N)}R_{33}^{(N)} & R_{11}^{(N)}R_{34}^{(N)} & R_{12}^{(N)}R_{33}^{(N)} & R_{12}^{(N)}R_{34}^{(N)} \end{pmatrix}$$

Emittance Reconstruction   The formalism

The equations could be expressed as:

$$M_X \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_5 \end{pmatrix} = \begin{pmatrix} \hat{\sigma}_1^{(1)} \\ \hat{\sigma}_1^{(2)} \\ \\ \dots \\ \hat{\sigma}_1^{(N)} \end{pmatrix} \qquad M_Y \begin{pmatrix} \sigma_8 \\ \sigma_9 \\ \sigma_{10} \end{pmatrix} = \begin{pmatrix} \hat{\sigma}_8^{(1)} \\ \hat{\sigma}_8^{(2)} \\ \\ \dots \\ \hat{\sigma}_8^{(N)} \end{pmatrix} \qquad M_{XY} \begin{pmatrix} \sigma_3 \\ \sigma_4 \\ \\ \sigma_6 \\ \sigma_7 \end{pmatrix} = \begin{pmatrix} \hat{\sigma}_3^{(1)} \\ \hat{\sigma}_3^{(2)} \\ \\ \dots \\ \hat{\sigma}_3^{(N)} \end{pmatrix}$$

- with 3 stations only the projected emittance (2D) could we reconstructed
- with 4 stations coupled beam matrix could be reconstructed but the first two systems are overdetermined.
- with >4 stations coupled beam matrix could be reconstructed but the three systems are overdetermined.

Emittance Reconstruction

Analytical conditions: Projected Emittance (2D)

In the general case of ***N*** measurement stations we have M_X and M_X^*

$$M_X = \begin{pmatrix} R_{11}^{2(1)} & 2R_{11}^{(1)}R_{12}^{(1)} & R_{12}^{2(1)} \\ R_{11}^{2(2)} & 2R_{11}^{(1)}R_{12}^{(2)} & R_{12}^{2(2)} \\ \cdots & \cdots & \cdots \\ R_{11}^{2(N)} & 2R_{11}^{(N)}R_{12}^{(N)} & R_{12}^{2(N)} \end{pmatrix} \qquad M_X^* = \begin{pmatrix} R_{11}^{2(1)} & 2R_{11}^{(1)}R_{12}^{(1)} & R_{12}^{2(1)} & \hat{\sigma}_1^{(1)} \\ R_{11}^{2(2)} & 2R_{11}^{(2)}R_{12}^{(2)} & R_{12}^{2(2)} & \hat{\sigma}_1^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ R_{11}^{2(N)} & 2R_{11}^{(N)}R_{12}^{(N)} & R_{12}^{2(N)} & \hat{\sigma}_1^{(N)} \end{pmatrix}$$

The system has **unique solution ($\sigma_1\ \sigma_2\ \sigma_5$)** if and only if the rank of both M_X and M_X^* is equal to three. That means that the determinants of all 3x3 minors of M_X should not vanish and the determinants of all 4x4 minors of M_X^* should be zero.

In terms of Twiss parameters the determinants of such minors are:

$$\Delta_{3x}(ijk) = 2\,\beta_x^{(i)}\beta_x^{(j)}\beta_x^{(k)} \sin\phi_x^{(ji)} \sin\phi_x^{(ki)} \sin\phi_x^{(kj)}$$

$$A_{4x}(ijkl) = -\hat{\sigma}_1^{(i)}\Delta_{3x}(jkl) + \hat{\sigma}_1^{(j)}\Delta_{3x}(ikl) - \hat{\sigma}_1^{(k)}\Delta_{3x}(ijl) + \hat{\sigma}_1^{(l)}\Delta_{3x}(ijk)$$

Emittance Reconstruction Analytical conditions: Projected Emittance (2D)

This give us two conditions:

Condition1:

$$\phi_x^{(ji)} \neq n\pi, \forall (i,j)$$

This is the only required condition in the case of **3** measurement stations.

For **4 or more stations** a second condition is required to get a unique solution.

Condition 2:

$$-\hat{\sigma}_1^{(i)}\Delta_{3x}(jkl)+\hat{\sigma}_1^{(j)}\Delta_{3x}(ikl)-\hat{\sigma}_1^{(k)}\Delta_{3x}(ijl)+\hat{\sigma}_1^{(l)}\Delta_{3x}(ijk)=0, \forall (ijkl)$$

Condition 1 tell us that the measurement stations should be located at places where the phase advances correspond to different snapshots of the beam. Condition 2 involve the β and the measurements, one can see that in general the equality cannot be exactly satisfied. One could replace the zero by some previously fixed error value, related with the error of the measurements.

Idem for vertical plane.

Emittance Reconstruction

Analytical conditions: Coupling Terms

In the general case of ***N*** measurement stations we have M_{XY} and M_{XY}^*

$$M_{XY} = \begin{pmatrix} R_{11}^{(1)}R_{33}^{(1)} & R_{11}^{(1)}R_{34}^{(1)} & R_{12}^{(1)}R_{33}^{(1)} & R_{12}^{(1)}R_{34}^{(1)} \\ R_{11}^{(2)}R_{33}^{(2)} & R_{11}^{(2)}R_{34}^{(2)} & R_{12}^{(2)}R_{33}^{(2)} & R_{12}^{(2)}R_{34}^{(2)} \\ \dots & \dots & \dots & \dots \\ R_{11}^{(N)}R_{33}^{(N)} & R_{11}^{(N)}R_{34}^{(N)} & R_{12}^{(N)}R_{33}^{(N)} & R_{12}^{(N)}R_{34}^{(N)} \end{pmatrix} \quad M_{XY}^* = \begin{pmatrix} R_{11}^{(1)}R_{33}^{(1)} & R_{11}^{(1)}R_{34}^{(1)} & R_{12}^{(1)}R_{33}^{(1)} & R_{12}^{(1)}R_{34}^{(1)} & \hat{s}_3^{(1)} \\ R_{11}^{(2)}R_{33}^{(2)} & R_{11}^{(2)}R_{34}^{(2)} & R_{12}^{(2)}R_{33}^{(2)} & R_{12}^{(2)}R_{34}^{(2)} & \hat{s}_3^{(2)} \\ \dots & \dots & \dots & \dots & \dots \\ R_{11}^{(N)}R_{33}^{(N)} & R_{11}^{(N)}R_{34}^{(N)} & R_{12}^{(N)}R_{33}^{(N)} & R_{12}^{(N)}R_{34}^{(N)} & \hat{s}_3^{(N)} \end{pmatrix}$$

The system has **unique solution ($\sigma_3\ \sigma_4\ \sigma_6\ \sigma_7$)** if and only if the rank of both M_{XY} and M_{XY}^* is equal to four. That means that the determinants of all 4x4 minors of M_{XY} should not vanish and the determinants of all 5x5 minors of M_{Xy}^* should be zero.

Emittance Reconstruction Analytical conditions: Coupling Terms

In terms of Twiss parameters the determinants of such minors are:

$$
\begin{aligned}
&-8\left(\beta_x^{(i)}\beta_y^{(i)}\beta_x^{(j)}\beta_y^{(j)}\beta_x^{(k)}\beta_y^{(k)}\beta_x^{(l)}\beta_y^{(l)}\right)^{-1/2}\Delta_4(ijkl) = \\
&\cos(\phi_x^{(ji)}+\phi_x^{(lk)})\left[\cos(\phi_y^{(ki)}+\phi_y^{(lj)})-\cos(\phi_y^{(ki)}-\phi_y^{(lj)})\right] \\
&+\cos(\phi_x^{(ki)}+\phi_x^{(lj)})\left[\cos(\phi_y^{(ji)}-\phi_y^{(lk)})-\cos(\phi_y^{(ji)}+\phi_y^{(lk)})\right] \\
&+\cos(\phi_x^{(ji)}-\phi_x^{(lk)})\left[\cos(\phi_y^{(ji)}+\phi_y^{(lk)})-\cos(\phi_y^{(ki)}+\phi_y^{(lj)})\right]
\end{aligned}
$$

$$
\begin{aligned}
&A_5(ijklm) = +\hat{\sigma}_3^{(i)}\Delta_4(jklm) - \hat{\sigma}_3^{(j)}\Delta_4(iklm) + \hat{\sigma}_3^{(k)}\Delta_4(ijlm) \\
&- \hat{\sigma}_3^{(l)}\Delta_4(ijkm) + \hat{\sigma}_3^{(m)}\Delta_4(ijkl)
\end{aligned}
$$

Emittance Reconstruction Analytical conditions: Coupling Terms

This give us two more conditions:

Condition 3:

$$\cos(\phi_x^{(ji)}+\phi_x^{(lk)})\left[\cos(\phi_y^{(ki)}+\phi_y^{(lj)})-\cos(\phi_y^{(ki)}-\phi_y^{(lj)})\right]$$
$$+\cos(\phi_x^{(ki)}+\phi_x^{(lj)})\left[\cos(\phi_y^{(ji)}-\phi_y^{(lk)})-\cos(\phi_y^{(ji)}+\phi_y^{(lk)})\right]$$
$$+\cos(\phi_x^{(ji)}-\phi_x^{(lk)})\left[\cos(\phi_y^{(ji)}+\phi_y^{(lk)})-\cos(\phi_y^{(ki)}+\phi_y^{(lj)})\right]\neq 0,\ \forall\ (ijkl)$$

This is the only required condition in the case of **4** measurement stations.

In the particular case where $\phi_x^{(ji)}=\phi_y^{(ji)}$ the system has no solution.

For **5 or more stations** an additional condition is required to get a unique solution.

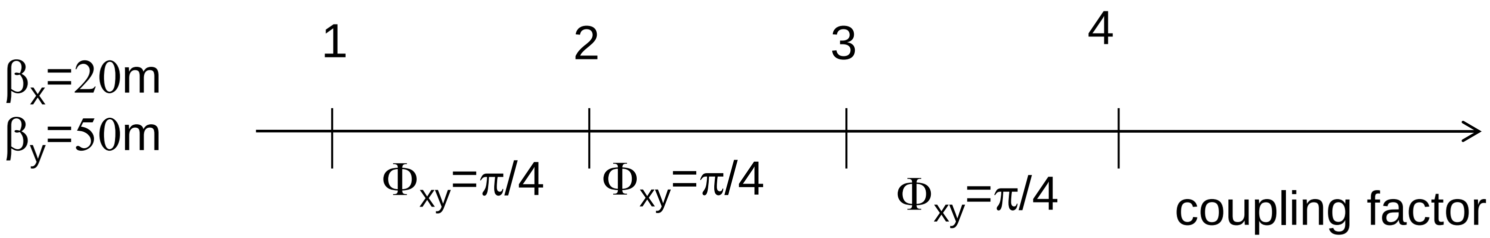
Condition 4:

$$\hat{\sigma}_3^{(i)}\Delta_4(jklm)-\hat{\sigma}_3^{(j)}\Delta_4(iklm)+\hat{\sigma}_3^{(k)}\Delta_4(ijlm)-\hat{\sigma}_3^{(l)}\Delta_4(ijkm)+\hat{\sigma}_3^{(m)}\Delta_4(ijkl)=0,\ \forall\ (ijklm)$$

One could replace the zero by some previously fixed error value, related with the error of the measurements.

Emittance Reconstruction

Some examples: NLC 2D diagnostic section



coupling factor r=0.1

1% measurement errors

Condition 1: not an integer

1	2	0.2500	0.2500
1	3	0.5000	0.5000
2	3	0.2500	0.2500
1	4	0.7500	0.7500
2	4	0.5000	0.5000
3	4	0.2500	0.2500

Condition 2: ax4 (CL s1,fx3) = 0

1 2 3 4 0.000E+00 0.253E-07

No problems for solving the 2D systems

Condition 2: ay4 (CL s8,fy3) = 0

1 2 3 4 0.000E+00 0.200E-07

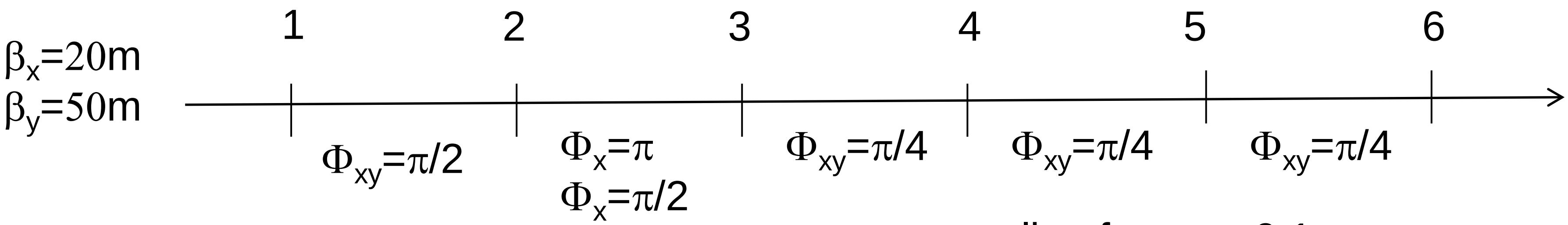
Condition 3: f4 different from 0

1 2 3 4 0.0000

Equivalent planes, not valid for 4D

Emittance Reconstruction

Some examples: NLC 4D diagnostic section



coupling factor r=0.1

1% measurement errors

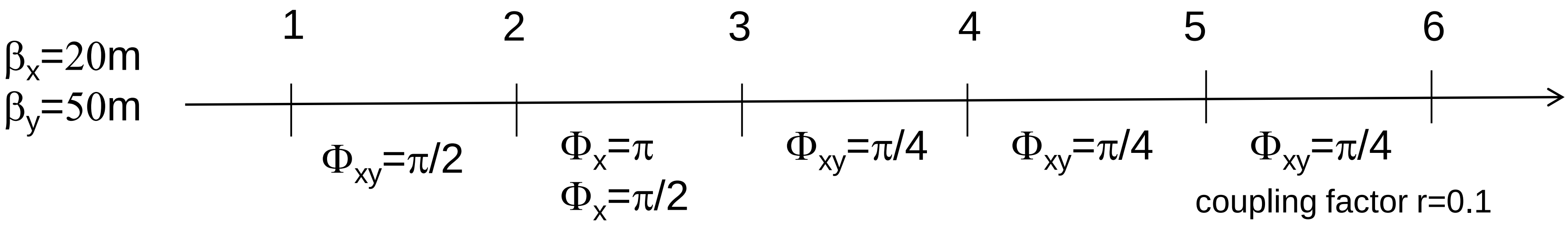
Condition 1: not an integer

1	2	0.5000	0.5000
1	3	1.5000	1.0000
2	3	1.0000	0.5000
1	4	1.7500	1.2500
2	4	1.2500	0.7500
3	4	0.2500	0.2500
1	5	2.0000	1.5000
2	5	1.5000	1.0000
3	5	0.5000	0.5000
4	5	0.2500	0.2500
1	6	2.2500	1.7500
2	6	1.7500	1.2500
3	6	0.7500	0.7500
4	6	0.5000	0.5000
5	6	0.2500	0.2500

Some stations are redundant

Emittance Reconstruction

Some examples: NLC 4D diagnostic section



coupling factor r=0.1

1% measurement errors

Condition 2: ax4 (CL s1,fx3) = 0

1	2	3	4	0.493E-22	0.179E-07
1	2	3	5	0.470E-37	0.981E-23
1	2	4	5	-0.106E-21	0.179E-07
1	3	4	5	-0.106E-21	0.179E-07
2	3	4	5	-0.776E-22	0.179E-07
1	2	3	6	-0.566E-22	0.179E-07
1	2	4	6	0.000E+00	0.253E-07
1	3	4	6	0.106E-21	0.253E-07
2	3	4	6	-0.106E-21	0.179E-07
1	2	5	6	0.722E-23	0.179E-07
1	3	5	6	0.722E-23	0.179E-07
2	3	5	6	-0.283E-22	0.179E-07
1	4	5	6	-0.261E-21	0.179E-07
2	4	5	6	0.000E+00	0.253E-07
3	4	5	6	0.000E+00	0.253E-07

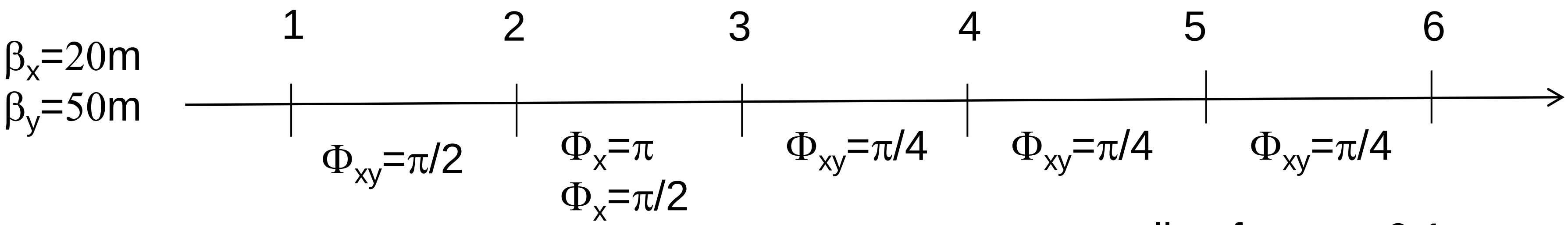
Condition 2: ay4 (CL s8,fy3) = 0

1	2	3	4	-0.541E-22	0.141E-07
1	2	3	5	0.000E+00	0.489E-23
1	2	4	5	0.000E+00	0.141E-07
1	3	4	5	-0.535E-22	0.141E-07
2	3	4	5	0.000E+00	0.141E-07
1	2	3	6	0.518E-22	0.141E-07
1	2	4	6	0.000E+00	0.200E-07
1	3	4	6	-0.529E-22	0.141E-07
2	3	4	6	0.000E+00	0.200E-07
1	2	5	6	0.541E-22	0.141E-07
1	3	5	6	-0.523E-22	0.141E-07
2	3	5	6	-0.541E-22	0.141E-07
1	4	5	6	-0.529E-22	0.200E-07
2	4	5	6	-0.535E-22	0.141E-07
3	4	5	6	0.000E+00	0.200E-07

Valid for 2D

Emittance Reconstruction

Some examples: NLC 4D diagnostic section



coupling factor r=0.1

1% measurement errors

Condition 3: f4 different from 0

1	2	3	4	-2.0000
1	2	3	5	-4.0000
1	2	4	5	-2.0000
1	3	4	5	2.0000
2	3	4	5	2.0000
1	2	3	6	-2.0000
1	2	4	6	-0.0000
1	3	4	6	2.0000
2	3	4	6	2.0000
1	2	5	6	2.0000
1	3	5	6	2.0000
2	3	5	6	2.0000
1	4	5	6	2.0000
2	4	5	6	2.0000
3	4	5	6	0.0000

two planes equivalent

Condition 4: a5 (CL s3, f4) equal to 0

1	2	3	4	5	-0.12E-26	0.19E-12
1	2	3	4	6	-0.16E-26	0.13E-12
1	2	3	5	6	0.89E-26	0.19E-12
1	2	4	5	6	-0.44E-26	0.13E-12
1	3	4	5	6	-0.69E-26	0.13E-12
2	3	4	5	6	0.65E-26	0.13E-12

Valid for 4D

Emittance Reconstruction

Some examples: ATF2 mOTR system

1	2	3	4
β_x=36.0m β_y=12.8m	β_x=58.7m β_y=8.6m	β_x=10.7m β_y=15.0m	β_x=2.4m β_y=2.7m

2π units

1 → 2	2 → 3	3 → 4
Φ_y=0.004 Φ_y=0.026	Φ_y=0.026 Φ_y=0.045	Φ_y=0.1 Φ_y=0.164

coupling factor r=0.1

1% measurement errors

Condition 1: not an integer

1	2	0.0080	0.0520
1	3	0.0600	0.1420
2	3	0.0520	0.0900
1	4	0.2600	0.4700
2	4	0.2520	0.4180
3	4	0.2000	0.3280

No problems for solving the 2D systems

Condition 2: ax4 (CL s1,fx3) = 0

1 2 3 4 0.117E-07 0.458E-06

Condition 2: ay4 (CL s8,fy3) = 0

1 2 3 4 -0.647E-11 0.947E-09

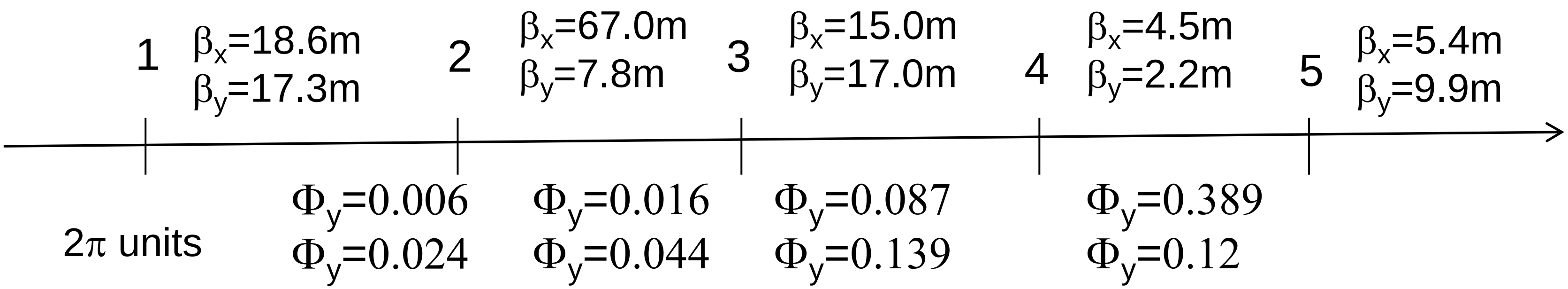
Condition 3: f4 different from 0

1 2 3 4 -0.0497

Valid for 4D

Emittance Reconstruction

Some examples: ATF2 WS system



coupling factor r=0.1

1% measurement errors

Condition 1: not an integer

1	2	0.0120	0.0480
1	3	0.0440	0.1360
2	3	0.0320	0.0880
1	4	0.2180	0.4140
2	4	0.2060	0.3660
3	4	0.1740	0.2780
1	5	0.9960	0.6540
2	5	0.9840	0.6060
3	5	0.9520	0.5180
4	5	0.7780	0.2400

No problems for solving the 2D systems

Emittance Reconstruction

Some examples: ATF2 WS system

1 β_x=18.6m β_y=17.3m

2 β_x=67.0m β_y=7.8m

3 β_x=15.0m β_y=17.0m

4 β_x=4.5m β_y=2.2m

5 β_x=5.4m β_y=9.9m

2π units

Φ_y=0.006 Φ_y=0.024

Φ_y=0.016 Φ_y=0.044

Φ_y=0.087 Φ_y=0.139

Φ_y=0.389 Φ_y=0.12

coupling factor r=0.1

1% measurement errors

Condition 2: ax4 (CL s1,fx3) = 0

1 2 3 4	-0.845E-08	0.389E-06
1 2 3 5	-0.207E-09	0.779E-08
1 2 4 5	-0.160E-08	0.615E-07
1 3 4 5	-0.889E-09	0.379E-07
2 3 4 5	0.509E-09	0.125E-06

Condition 2: ay4 (CL s8,fy3) = 0

1 2 3 4	0.234E-10	0.901E-09
1 2 3 5	0.121E-09	0.491E-08
1 2 4 5	0.208E-10	0.116E-08
1 3 4 5	-0.260E-10	0.275E-08
2 3 4 5	-0.303E-10	0.115E-08

No problems for solving the 2D systems

Emittance Reconstruction

Some examples: ATF2 WS system

1	2	3	4	5
β_x=18.6m	β_x=67.0m	β_x=15.0m	β_x=4.5m	β_x=5.4m
β_y=17.3m	β_y=7.8m	β_y=17.0m	β_y=2.2m	β_y=9.9m

2π units

Φ_y=0.006	Φ_y=0.016	Φ_y=0.087	Φ_y=0.389
Φ_y=0.024	Φ_y=0.044	Φ_y=0.139	Φ_y=0.12

coupling factor r=0.1

1% measurement errors

Condition 3: f4 different from 0

1 2 3 4	-0.0086
1 2 3 5	0.0047
1 2 4 5	0.0751
1 3 4 5	0.2328
2 3 4 5	0.1672

No problems for solving the 4D systems

Condition 4: a5 (CL s3, f4) equal to 0

1 2 3 4 5 -0.18E-10 0.19E-11

We have studied analytically the conditions of solvability of the systems of equations involved in the process of emittance reconstruction and we have obtained some rules about the locations of the measurements to avoid unphysical results.

Thanks for your attention