# Problems Lecture 1: Linac Basics

1) Calculate the relative longitudinal motion of two electrons with an energy of about $9\,\mathrm{GeV}$ and a difference of $3\%$ over a distance of $21\,\mathrm{km}$.

2) A superconducting linac consists of cavities with a length of $L = 1.1m$ and an external coupling of $Q_{ext} = 10^5$. It is operated in matched conditions (no reflected power) with a gradient $G_0 = 20MV/m$ and a beam current of $I_0 = 10mA$. and an external coupling of $Q_{ext} = 10^5$.
a) What is the input power $P$ required?
b) The management wants to double the beam current but keep the gradient the same. In order to stay matched, which input power does one need? Which other parameter needs to be changed and how?

3) A harmonic oscillator is a solution to Hill's equation for $K(s) = K_0 > 0$. Show that this fullfils the differential equation describing the development of $beta(s)$. What is $\beta(s)$?

4) How much energy is roughly stored in one ILC cavity at nominal gradient?

5) Show that the beta-function in a drift (i.e. $K(s) = 0$) around a waist (i.e. $\beta(s=0) = \beta_0$ and $\beta'(s=0) = 0$) is a parabola. (Optional)

# Solutions: Linac Basics

1) We calculate

$$\gamma = \frac{E_0}{mc^2} \approx \frac{9\,\mathrm{GeV}}{0.511\,\mathrm{MeV}} \approx 18000$$

then we use

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

to find an approximation for $\beta$

$$\beta \approx 1 - \frac{1}{2\gamma^2} \approx 1 - 1.5 \times 10^{-9}$$

over the length of a linac $21\,\mathrm{km}$ the longitudinal delay compared to light is $\approx\ 32\,\mu\mathrm{m}$ for two particles which have a energy difference of $\Delta\gamma$ the relative longitudinal motion would be

$$\beta_1 - \beta_2 \approx \frac{1}{2\gamma^2} - \frac{1}{2(\gamma+\Delta\gamma)^2} \approx \frac{\Delta\gamma}{\gamma^3}$$

for an example of 3% the motion is $\approx 1\,\mu\mathrm{m} \ll \sigma_z$

Note: Due to the acceleration the effect is even smaller

# Solutions: Linac Basics

2)

a) In matched conditions, the input power $P_0$ must be equal to the power $P_{beam}$ extracted by the beam.
We calculate

$$P_0 = P_{beam} = LG_0 I_0 = 1.1 \text{ m } 20 \text{ MV/m } 10 \text{ mA} = 220 \text{ kW}$$

b) We can calculate the required input power for matched conditions as above

$$P_{new} = P_{beam} = LG_0(2I_0) = 1.1 \text{ m} 20 \text{ MV/m} 20; \text{mA} = 440 \text{ kW} = 2P_0$$

So the input power has to double.
However we would not be matched any more if we did not change the coupling of the cavity to the RF. So we need to couple the cavity more strongly to the RF power source. The energy in the cavity $E$ remains the same since the gradient is not changed. But the input power doubles. We use

$$Q_{ext,new} = \frac{E}{P_{new}}\omega = \frac{E}{2P_0}\omega = \frac{1}{2}\frac{E}{P}\omega = \frac{1}{2}Q_{ext} = 0.5 \cdot 10^5$$

# Solutions: Linac Basics

3) We use $K(s) = \text{const} > 0$.

- We know the solution for a harmonic oszillation with a fixed amplitude

$$x = A\cos(\phi(s) + \phi_0)$$

for the beta-function this should correspond to a constant value of beta, which we call $\beta_0$

- We now need to check that this fulfills the differential equation for $\beta$

  - Ansatz: $\beta = \beta_0$, $\beta' = 0$ and $\beta'' = 0$:

$$\frac{\beta''\beta}{2} - \frac{\beta'^2}{4} + K\beta^2 = 1$$

$$\Rightarrow K\beta_0^2 = 1$$

Hence

$$\beta_0 = \frac{1}{\sqrt{K}}$$

Now one can plug this into the equation of motion to see that one recovers the known solution for a harmonic oszillator. $\epsilon$ is defined by the initial condition. See next page

# Solutions: Linac Basics

Replacing $K$ by $1/\beta_0^2$ in the original equation

$$x''(s) + \frac{1}{\beta_0^2} x = 0$$

we know the solution for a harmonic oscillator

$$x = A \cos\left(\frac{s}{\beta_0} + \phi_0\right)$$

On the other hand, assuming $\beta(s) = \beta_0$ in the Hill's equation we obtain

$$x = \sqrt{\epsilon\beta_0} \cos\left(\phi(s) + \phi_0\right) = \sqrt{\epsilon\beta_0} \cos\left(\frac{s}{\beta_0} + \phi_0\right)$$

So the harmonic oscillator is as expected a special case for the Hill's equation.
This is obviously no surprise, but the goal has been to make you use the equation a bit.

# Solutions: Linac Basics

4) Assuming $R/Q = 1\,\mathrm{k}\Omega$ and a length of about $L = 1\,\mathrm{m}$, we find approximately $120\,\mathrm{J}$

$$E = \frac{G^2 L^2}{2\pi f_{RF} R/Q}$$

# Solutions: Linac Basics

5) $K = 0$

Remark: the problem stated that the solution is a parabola, but we could find that ourselfs by considering that in a drift the position evolves as $x(s) = x_0 + x'(0)s$

Ansatz: $\beta$ is a polynom of second order

- We use $\beta'(s=0) = 0$ (this is the property of the waist as stated in the problem), this leads to the assumption $\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$

We use the equation for the propagation of $\beta$

$$\frac{\beta''\beta}{2} - \frac{\beta'^2}{4} + K\beta^2 = 1$$

And use $K = 0$ to find

$$\Rightarrow \frac{\beta''\beta}{2} - \frac{\beta'^2}{4} = 1$$

Now we determined $\beta'(s)$ and $\beta''(s)$ from the Ansatz and plug it into the equation

$$\Rightarrow \frac{1}{2}\left(\beta_0 + \frac{s^2}{\beta_0}\right)\left(\frac{2}{\beta_0}\right) - \frac{1}{4}\left(\frac{2s}{\beta_0}\right)^2 = 1$$

And we find that this is true

$$\Rightarrow 1 + \frac{s^2}{\beta_0^2} - \frac{s^2}{\beta_0^2} = 1$$

Hence the Ansatz has been correct.