Lattice Design Considerations

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Lectures

- Linac basics (two lectures)
- Basic theory to understand the lattice design (two lectures)
- Static imperfections and their beam-based correction
- Dynamic imperfections and their beam-based correction
- Multi-bunch effects
- Parameter optimisation and summary

This Lecture

- Introduction
  - motivation, basic recipe to design your own linac
- Single particle dynamics basics
  - matrix formalism, first basic matrices and FODO cell, Twiss parameters, acceleration
- Multi particle (single bunch) basics
  - emittance, impact of energy spread, single bunch beam break-up
- Imperfections
- Simulations

Why is the Main Linac Important?

- The two main parameters are important for the physics experiments
  - collision energy
  - luminosity, a measure for the rate of events at the interaction point
- The main linac is the main component to accelerate the beam

  ⇒ it is responsible for the beam energy
  - the main relevant parameter is the accelerating gradient
- The main linac is the main consumer of power

  ⇒ it is an important limitation for the beam current
  - the luminosity depends on the beam current
- The main linac is one of the main sources of emittance growth

  ⇒ the emittance is a parameter that affects the luminosity
- There is a third parameter which the main linac affect very much, the cost
  - is the society willing to pay for it?

Impact on Luminosity

- The luminosity can be written as

$$\mathcal{L} = H_D \frac{N^2 n_b f_r}{4\pi\sigma_x^*\sigma_y^*}$$

H_D a factor usually between 1 and 2, due to the beam-beam forces
N the number of particles per bunch
n_b the number of bunches per beam pulse (train)
f_r the frequency of trains
σ_x^* and σ_y^* the transverse dimensions at the interaction point

$$\mathcal{L} = H_D \frac{N}{4\pi\sigma_x^*} \frac{1}{\sqrt{\frac{\beta_y \epsilon_y}{\gamma}}} N n_b f_r = H_D \frac{N}{4\pi\sigma_x^*} \frac{1}{\sqrt{\frac{\beta_y \epsilon_y}{\gamma}}} \eta P_{wall}$$

- We will see that $\sigma_{x,y}$ can be written as the function of two parameters

$$\sigma_{x,y} = \sqrt{\frac{\beta_{x,y}\epsilon_{x,y}}{\gamma}}$$

Main Linac Lattice Design

- Which elements are needed?
  - accelerating structures

    it is obviously the purpose of the main linac to provide acceleration

    goal is usually to have the largest possible fraction of the linac filled with accelerating structures (fill factor)
  - guiding magnets

    otherwise the beam will not pass

    we will use quadrupoles
  - beam position monitors (BPMs)

    otherwise we do not see what the beam does

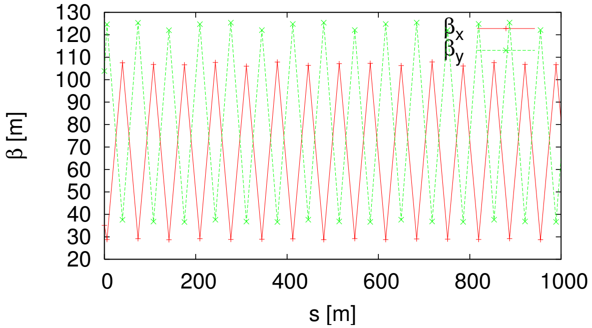
    needed to correct imperfections
  - some correctors

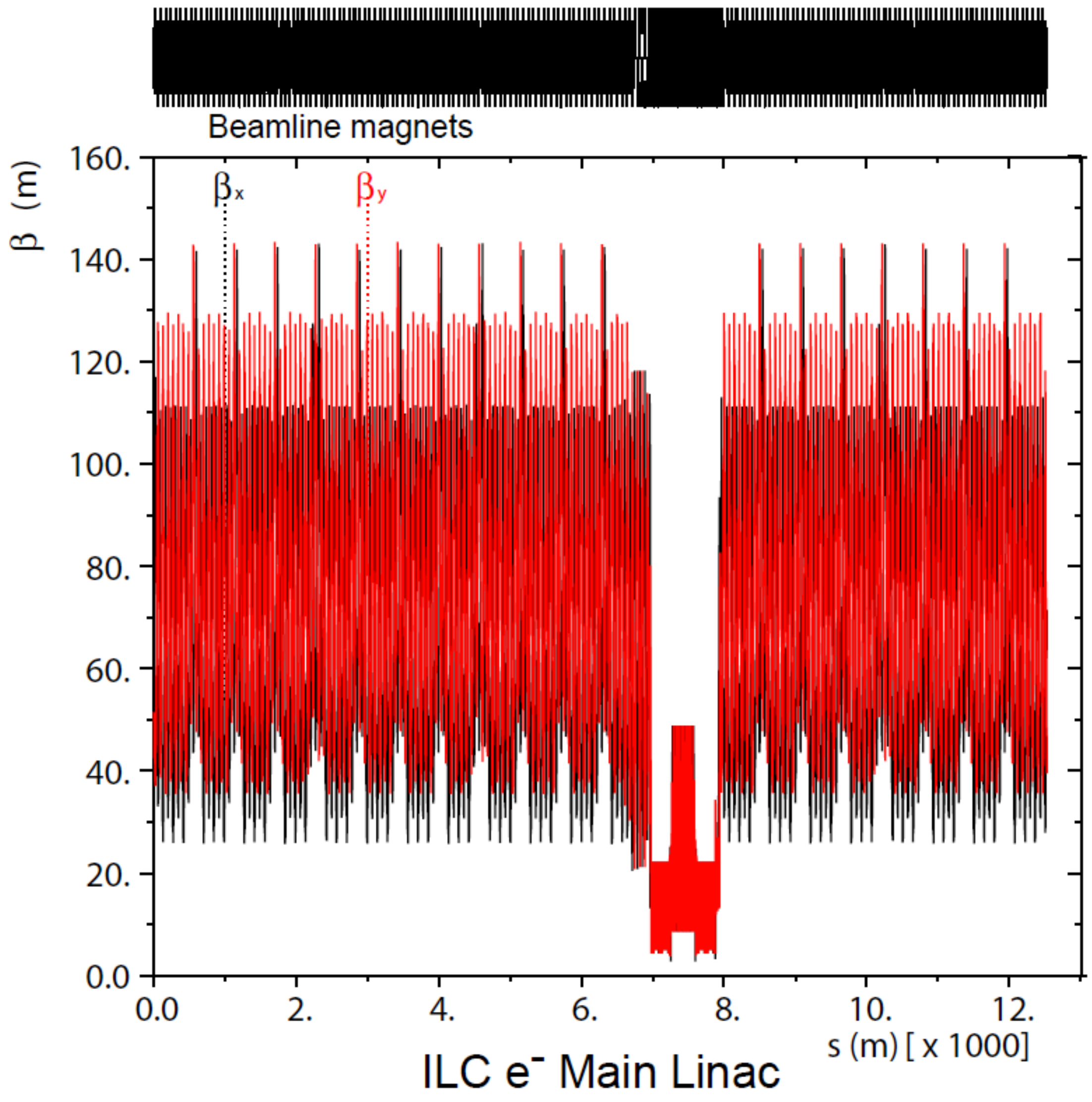
    because life is not perfect and needs to be corrected

parameter	symbol	ILC	CLIC
centre of mass energy	E_{cm}	$500\,\mathrm{GeV}$	$3000\,\mathrm{GeV}$
luminosity	$\mathcal{L}$	$2 \cdot 10^{34}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$	$6.5 \cdot 10^{34}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$
luminosity in peak	$\mathcal{L}_{0.01}$	$1.4 \cdot 10^{34}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$	$2 \cdot 10^{34}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$
initial energy	E_0	$15\,\mathrm{GeV}$	$9\,\mathrm{GeV}$
final energy	E_f	$250\,\mathrm{GeV}$	$1500\,\mathrm{GeV}$
charge per bunch	N	$2 \cdot 10^{10}$	$3.72 \cdot 10^{9}$
bunch length	σ_z	$300\,\mu\mathrm{m}$	$44\,\mu\mathrm{m}$
initial/final horizontal emittance	ϵ_x	$8400\,\mathrm{nm}/9400\,\mathrm{nm}$	$600\,\mathrm{nm}/660\,\mathrm{nm}$
initial/final vertical emittance	ϵ_y	$24\,\mathrm{nm}/34\,\mathrm{nm}$	$10\,\mathrm{nm}/20\,\mathrm{nm}$
bunches per pulse	n_b	2625	312
distance between bunches	n_b	$369\,\mathrm{ns}$	$0.5\,\mathrm{ns}$
repetition frequency	f_r	$5\,\mathrm{Hz}$	$50\,\mathrm{Hz}$

ILC Lattice

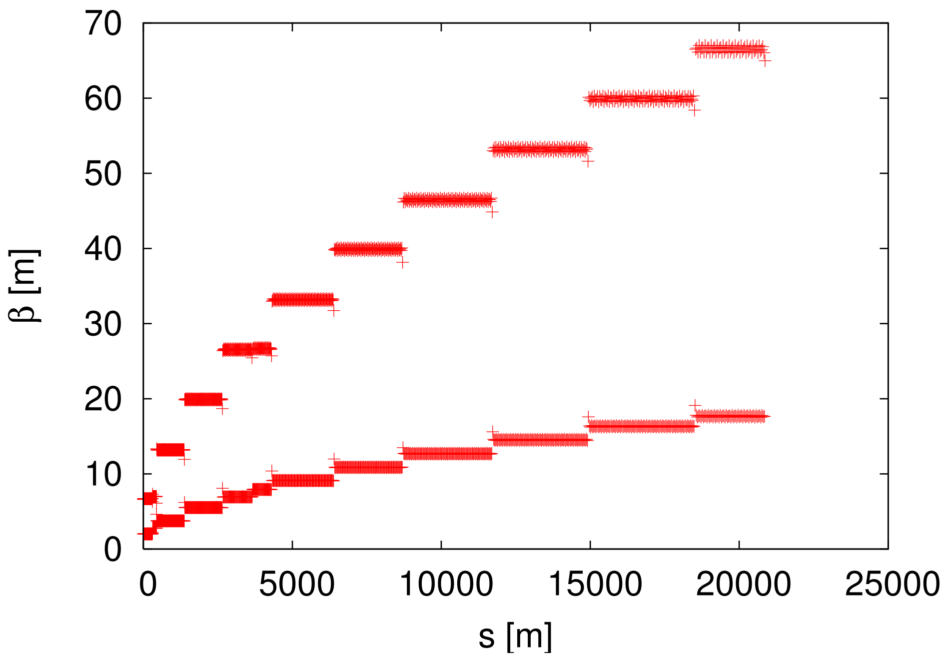
- In the ILC constant quadrupole spacing is chosen
- The phase advance per cell is constant
- The phase advance is different in the two planes
  - reduces some coupling effects between the two planes





CLIC Lattice Design

- Used $\beta \propto \sqrt{E}$, $\Delta\Phi = \mathrm{const}$
  - balances wakes and dispersion
  - roughly constant fill factor
  - phase advance is chosen to balance between wakefield and ground motion effects
- Preliminary lattice
  - made for $N = 3.7 \times 10^9$
  - quadrupole dimensions need to be confirmed
  - some optimisations remain to be done
- Total length 20867.6m
  - fill factor 78.6%



- 12 different sectors used
- Matching between sectors using 7 quadrupoles to allow for some energy bandwidth

CLIC Fill Factor

- Want to achieve a constant fill factor

  - to use all drive beams efficiently

- Scaling $f = f_0\sqrt{E/E_0}$ yields

$$L_q \propto \frac{E}{\sqrt{\frac{E}{E_0}}} \propto \sqrt{E}$$

using a quadrupole spacing of $L = L_0\sqrt{E/E_0}$ leads to

$$\frac{L_q}{L} \propto \frac{\sqrt{E}}{\sqrt{E}} \propto \mathrm{const}$$

⇒ The choice allows to maintain a roughly constant fill factor

⇒ It maximises the focal strength along the machine

Design Requirements

- How do I check that a lattice design is a good one?

  we will try to find an optimum solution later but first let us understand the criteria

- Test emittance growth of a perfect beam in the perfect machine

  ⇒ emittance growth must be small

  - if not improve lattice

- Test a beam with initial jitter in a perfect machine

  ⇒ beam must remain stable and relevant emittance must remain small

  - if beam is not stable redesign lattice (stronger focusing), reduce current or change structure

- Test beams in machines with realistic static imperfections

  ⇒ the emittance growth must be small

  - if not either lattice must be relaxed or alignment people must be pushed into R&D

- Test emittance growth in a machine with realistic dynamic imperfections

  ⇒ the emittance growth must remain small

  - if not either lattice must be relaxed or R&D on stabilisation is required

- Interaction with experts on RF, magnets, instrumentation, alignment and stability

  - put together what is considered reasonable by them

Main Linac Design Process

- Interactive process with interplay between
  - accelerating structure design
  - lattice design
  - beam parameters
  - hardware specifications which impact feasibility and cost
- Let us start with the lattice designers job
  - assume that we have a specific structure
  - beam parameters are given (except bunch length)
- Steps
  - choose lattice design type
  - adjust lattice parameters to have a stable beam
  - determine specifications for imperfections

Required Knowledge

- Single particle dynamics and the required formalism
- Multi-Particle Effects
  - particles at different energies
  - a bunch in the presence of wakefields
- Impact of static imperfections
  - origin of imperfections
  - methods to mitigate impact of imperfections
- Impact of dynamic imperfections
  - origin of imperfections
  - methods to mitigate impact of imperfections
- Multi-bunch effects

Coordinate Systems

- We use two frames, the laboratory frame and the beam frame
- The nominal direction of motion of the beam is called s in the laboratory frame, the beam moves toward increasing s
- The same direction is called z in the beam frame, with smaller z moving ahead of particles with larger z
- The transverse dimensions are x in the horizontal and y in the vertical plane, in both coordinate systems
- People use different systems so find out what they talk about

Single Particle Dynamics: Transfer Matrices

Particle Coordinates and Matrix Notation

- In one dimension one can describe a particle by

$$\frac{\partial x}{\partial s} = x' \qquad \frac{\partial x'}{\partial s} = f(s, x, x')$$

- Linear case can be described as

$$\frac{\partial x}{\partial s} = x' \qquad \frac{\partial x'}{\partial s} = f(s)x + g'(s)$$

- This leads to

$$x'' - f(s)x = g'(s)$$

- This can always be solved in the following form

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M(s) \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} + \begin{pmatrix} G(s) \\ g(s) \end{pmatrix}$$

In most cases $g' = 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M(s) \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Matrix Notation

- The transfer of a particle through the linac can be described by a matrix multiplication

$$\vec{x}_f = M\vec{x}_i$$

- For each element i a transfer matrix can be calculated M_i
- A sequence of the linac from element k to element m can be represented as

$$M_{k\rightarrow m} = M_{m-1}M_{m-2}\ldots M_{k+1}M_k$$

- This is close to the way the tracking of particle is implemented in simulation codes
- Note: the tranfermatrices are often also written as R

Simple Example

- Let us look at a simple example to determine the tranfer matrix
- A drift can be described by

$$\begin{aligned} x'(s) &= x'_0 \\ x(s) &= x_0 + s x'_0 \end{aligned}$$

this is equivalent to the following matrix

$$M_{drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

This transfer-matrix is also valid for BPMs

Field of a Quadrupole

- The field is designed to be

$$B_x(x,y) = B_0 y \qquad B_y(x,y) = B_0 x$$

- The Lorentz force is then

$$\vec{F} = q(\vec{v} \times B) = q \begin{pmatrix} v_y B_s - v_s B_y \\ v_s B_x - v_x B_s \\ v_x B_y - v_y B_x \end{pmatrix}$$

we approximate $v_x = v_y = 0$ and use $B_s = 0$

$$\vec{F} = q \begin{pmatrix} -v_s B_y \\ v_s B_x \\ 0 \end{pmatrix} = qcB_0 \begin{pmatrix} -x \\ y \\ 0 \end{pmatrix}$$

changing the field direction yields

$$\vec{F} = qcB_0 \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix}$$

$\Rightarrow$ A quadrupole focuses in one direction and defocuses in the other

Transfer Matrix of a Quadrupole

- A quadrupole (focusing plane)

$$x'' + kx = x'' + |k|x = 0$$

$$M_{QF} = \begin{pmatrix} \cos\left(\sqrt{k}L\right) & \frac{1}{\sqrt{k}}\sin\left(\sqrt{k}L\right) \\ -\sqrt{k}\sin\left(\sqrt{k}L\right) & \cos\left(\sqrt{k}L\right) \end{pmatrix}$$

- A quadrupole (defocusing plane)

$$x'' + kx = x'' - |k|x = 0$$

$$M_{QD} = \begin{pmatrix} \cosh\left(\sqrt{|k|}L\right) & \frac{1}{\sqrt{|k|}}\sinh\left(\sqrt{|k|}L\right) \\ -\sqrt{|k|}\sinh\left(\sqrt{|k|}L\right) & \cosh\left(\sqrt{|k|}L\right) \end{pmatrix}$$

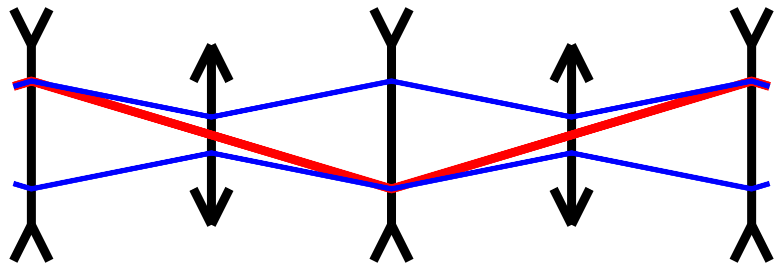
- Assuming a thin lens quadrupole one calculates $L \to 0$, $kL = K = 1/f$

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ -K & 1 \end{pmatrix} \qquad M_{QD} = \begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix}$$

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \qquad M_{QD} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

The FODO Lattice

- Each cell of a FODO lattice consists of a focusing and a defocusing quadrupole and two drifts



- For simplicity use the thin lens approximation for quadrupoles

$$M_{QD} = \begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix} \quad M_{QF} = \begin{pmatrix} 1 & 0 \\ -K & 1 \end{pmatrix} \quad M_D = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

- The transfer matrix from the centre of one focusing quadrupole to the centre of the next focusing quadrupole is then

$$M_{FODO} = \begin{pmatrix} 1 - K^2L^2/2 & L(2 + KL) \\ -K^2L/2(1 - KL/2) & 1 - K^2L^2/2 \end{pmatrix}$$

or

$$M_{FODO} = \begin{pmatrix} 1 - L^2/2f^2 & L(2 + L/f) \\ -L/(2f^2(1 - L/2f)) & 1 - L^2/(2f^2) \end{pmatrix}$$

Calculation of the FODO Cell

$$M_{FODO} = M_{QF/2} M_L M_{QD} M_L M_{QF/2}$$

$$M_{FODO} = \begin{pmatrix} 1 & 0 \\ -K/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -K/2 & 1 \end{pmatrix}$$

$$M_{FODO} = \begin{pmatrix} 1 & 0 \\ -K/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix} \begin{pmatrix} 1 - KL/2 & L \\ -K/2 & 1 \end{pmatrix}$$

$$M_{FODO} = \begin{pmatrix} 1 & 0 \\ -K/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - KL/2 & L \\ K/2(1 - KL) & 1 + KL \end{pmatrix}$$

$$M_{FODO} = \begin{pmatrix} 1 & 0 \\ -K/2 & 1 \end{pmatrix} \begin{pmatrix} 1 - K^2L^2/2 & L(2 + KL) \\ K/2(1 - KL) & 1 + KL \end{pmatrix}$$

$$M_{FODO} = \begin{pmatrix} 1 - K^2L^2/2 & L(2 + KL) \\ -K^2L/2(1 - KL/2) & 1 - K^2L^2/2 \end{pmatrix}$$

Single Particle Dyanmics: Twiss Parameters

Sorry, this is a bit tough, but very important

Reminder Hill's Equation

$$x''(s) + K(s)x(s) = 0$$

Defining

$$\phi(s) = \int_0^s \frac{1}{\beta(s')} ds'$$

We find the solution

$$x(s) = \sqrt{\epsilon\beta(s)} \cos\left(\phi(s) + \phi_0\right)$$

and

$$x'(s) = \sqrt{\frac{\epsilon}{\beta(s)}} \left[\frac{\beta'}{2} \cos\left(\phi(s) + \phi_0\right) - \sin\left(\phi(s) + \phi_0\right)\right]$$

β has to fulfill

$$\frac{\beta''\beta}{2} - \frac{\beta'^2}{4} + K\beta^2 = 1$$

Two new parameters are defined

$$\alpha = -\frac{\beta'}{2} \quad \gamma = \frac{1+\alpha^2}{\beta}$$

β, α and γ are called Twiss parameters

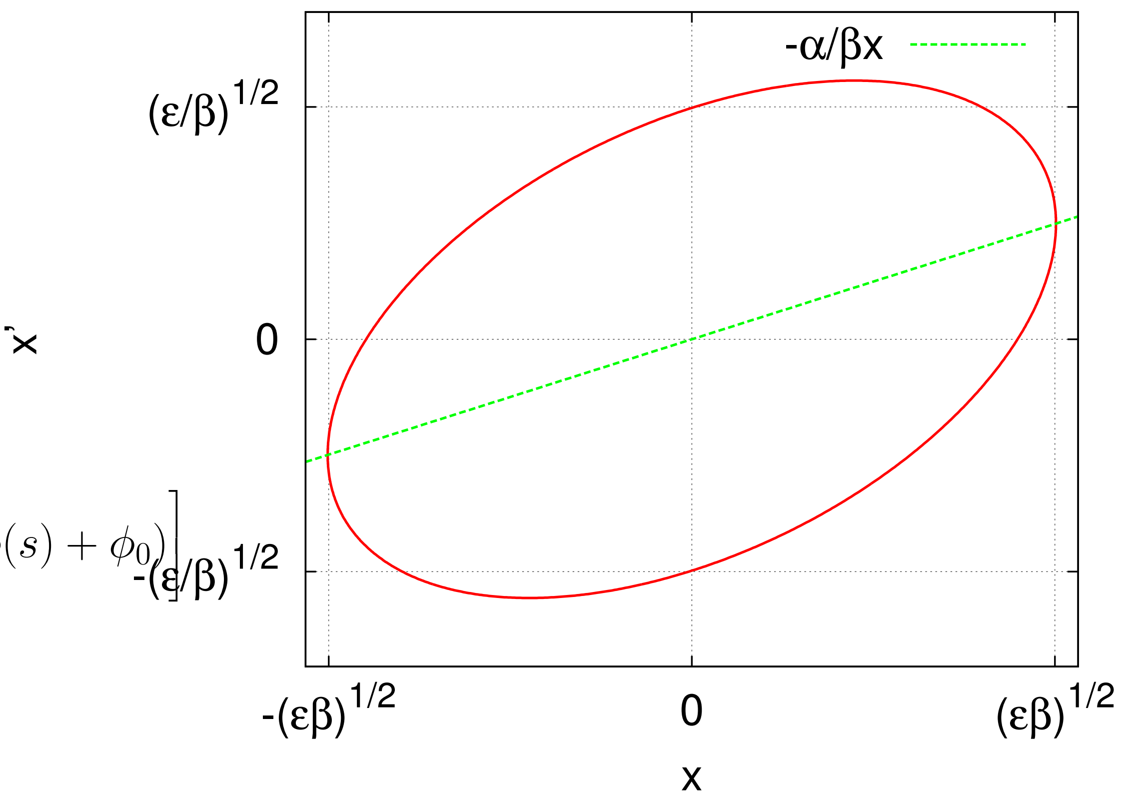
Phase Space Representation

- Particle position and angle at one point of the lattice is defined by an amplitude (action) that is preserved and its Twiss parameters

$$x(s) = \sqrt{\epsilon\beta(s)}\cos\left(\phi(s)+\phi_0\right)$$

$$x'(s) = \sqrt{\frac{\epsilon}{\beta(s)}}\left[\frac{\beta'}{2}\cos\left(\phi(s)+\phi_0\right) - \sin\left(\phi(s)+\phi_0\right)\right]$$

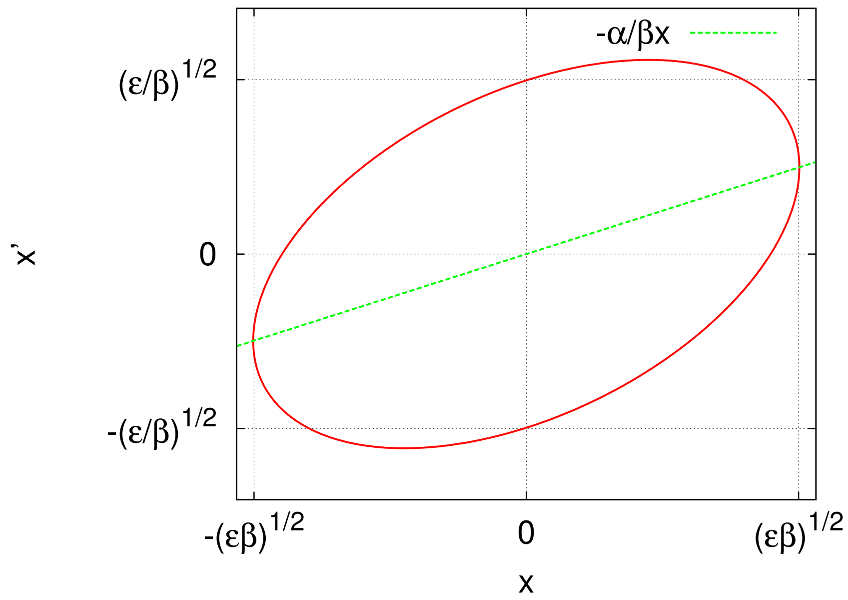
- All particles on the ellipse have the same action



Transformation into Normalised Phase Space

- We first need to remove the correlation between x and x'
  for this we use

$$\begin{pmatrix} 1 & 0 \\ \frac{\alpha}{\beta} & 1 \end{pmatrix}$$



- Then we normalise the amplitudes

$$\begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ 0 & \sqrt{\beta} \end{pmatrix}$$

- Both actions together

$$\begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ 0 & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\alpha}{\beta} & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

- Compare to Hill's equation

Testing Solutions of Hill's Equation

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta(s)}}\sqrt{\epsilon\beta(s)}\cos\left(\phi(s)+\phi_0\right) \\ \frac{\alpha}{\sqrt{\beta}}\sqrt{\epsilon\beta(s)}\cos\left(\phi(s)+\phi_0\right) - \sqrt{\beta}\sqrt{\frac{\epsilon}{\beta}}\sin(\phi+\phi_0) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_N \\ x'_N \end{pmatrix} = \begin{pmatrix} \sqrt{\epsilon}\cos(\phi+\phi_0) \\ -\sqrt{\epsilon}\sin(\phi+\phi_0) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_N \\ x'_N \end{pmatrix} = \sqrt{\epsilon}\begin{pmatrix} \cos(\phi+\phi_0) \\ -\sin(\phi+\phi_0) \end{pmatrix}$$

$\Rightarrow$ Not a surprise

- In normalised phase space the particle is characterised by a single-particle emittance ϵ and the phase ϕ_0
  - we could also replace ϵ by the action J with $\epsilon = 2J$

Trajectory Along the Machine

- In normalised phase space only the phase changes (no external force)

$$\begin{pmatrix} x_N(s_2) \\ x'_N(s_2) \end{pmatrix} = \begin{pmatrix} \cos(\phi(s_2)-\phi(s_1)) & \sin(\phi(s_2)-\phi(s_1)) \\ -\sin(\phi(s_2)-\phi(s_1)) & \cos(\phi(s_2)-\phi(s_1)) \end{pmatrix} \begin{pmatrix} x_N(s_1) \\ x'_N(s_1) \end{pmatrix}$$

- Phase advance is given by

$$\phi(s) = \int_0^s \frac{1}{\beta(s')} ds'$$

- Very useful to study impact of perturbations
- Can consider a complex amplitude

$$x_N = \mathrm{re}(A\exp(i\phi_0)) \qquad x'_N = \mathrm{im}(A\exp(i\phi_0))$$

we will use that later

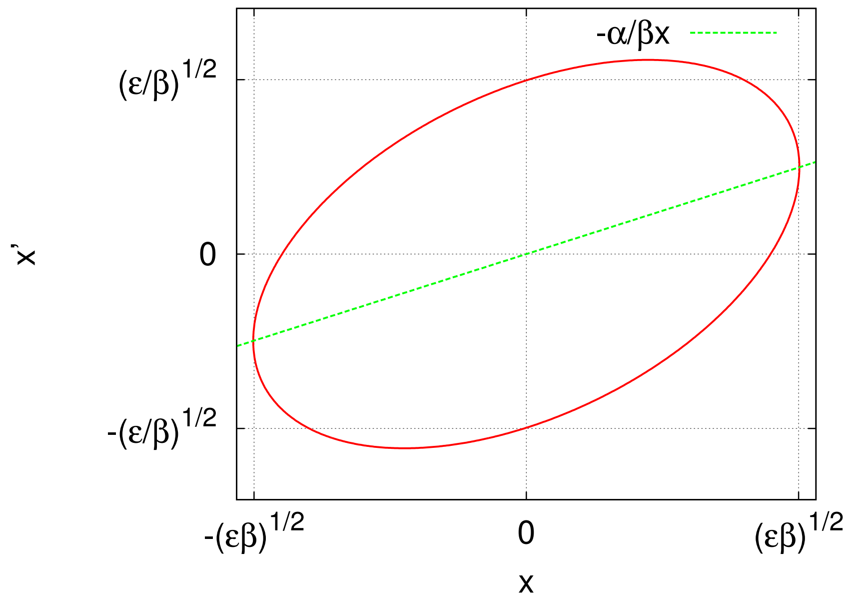
Transformation from Normalised Phase Space

- We first undo the amplitude normalisation

$$\begin{pmatrix} \sqrt{\beta} & 0 \\ 0 & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

- Then we add the correlation

$$\begin{pmatrix} 1 & 0 \\ -\frac{\alpha}{\beta} & 1 \end{pmatrix}$$



- Then we put both together we obtain the inverse of the other transfer matrix

$$\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

Periodic Solutions for FODO Lattice

- We aim to find a periodic solution for the beta-function of the FODO lattice
  - "matched solution"
- We use the transfer matrix into the normalised coordinates, some phase advance and a transformation back into real coordinates assuming the same Twiss parameters at both points

$$M_{period} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

  We chose a point where $\alpha = 0$ to make our life simple and find

$$M_{period} = \begin{pmatrix} \cos\mu & \beta\sin\mu \\ -\frac{1}{\beta}\sin\mu & \cos\mu \end{pmatrix}$$

- The periodic solutions for the beta-function can be found by solving

$$\beta^2 = -m_{1,2}/m_{2,1}$$

Periodic Solutions for FODO Lattice (cont)

- Using

$$M_{FODO} = \begin{pmatrix} 1 - K^2L^2/2 & L(2+KL) \\ -K^2L/2(1-KL/2) & 1-K^2L^2/2 \end{pmatrix} \qquad M_{period} = \begin{pmatrix} \cos\mu & \beta\sin\mu \\ -\frac{1}{\beta}\sin\mu & \cos\mu \end{pmatrix}$$

- Solving

$$\beta^2 = \frac{L(2+K/L)}{K^2L/2(1-KL/2)}$$

yields

$$\hat{\beta} = \frac{2}{K}\sqrt{\frac{1+KL/2}{1-KL/2}}$$

for the beta-function in the defocusing quadrupole one finds

$$\check{\beta} = \frac{2}{K}\sqrt{\frac{1-KL/2}{1+KL/2}}$$

Periodic Solutions for FODO Lattice (cont)

- Using

$$M_{FODO} = \begin{pmatrix} 1 - K^2L^2/2 & L(2+KL) \\ -K^2L/2(1-KL/2) & 1-K^2L^2/2 \end{pmatrix} \qquad M_{period} = \begin{pmatrix} \cos\mu & \beta\sin\mu \\ -\frac{1}{\beta}\sin\mu & \cos\mu \end{pmatrix}$$

- The phase advance $\Delta\phi$ obviously is given by

$$\cos\mu = 1 - \frac{K^2L^2}{2}$$

with the solution

$$\sin\frac{\mu}{2} = \frac{KL}{2}$$

FODO Cell with Different Quadrupole Strength

- The focusing and defocusing quadrupole do not need to have the same strength
- In this case find

$$\cos\mu_1 = 1 + K_2L - K_1L - \frac{K_1K_2L^2}{2}$$

and

$$\cos\mu_2 = 1 + K_1L - K_2L - \frac{K_1K_2L^2}{2}$$

- This is stable if $|1 + K_2L - K_1L - \frac{K_1K_2L^2}{2}| < 1$ and $|1 + K_1L - K_2L - \frac{K_1K_2L^2}{2}| < 1$
- Such a lattice is used in the ILC case ($\mu_x = 60^\circ$ and $\mu_y = 75^\circ$)
  - different phase advance in the two planes reduces coupling of resonant effects

Evolution of Twiss Parameters

The twiss parameters between the quadrupole centres can be calculated using

$$\begin{pmatrix} \beta_2 & -\alpha_2 \\ -\alpha_2 & \gamma_2 \end{pmatrix} = M_{1\rightarrow 2} \begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix} M_{1\rightarrow 2}^T$$

Here γ is the third Twiss parameter

$$\gamma = \frac{1+\alpha^2}{\beta}$$

In the following, I will not use it to avoid confusion

- Example: evolution in a drift

$$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ L & 1 \end{pmatrix} = \begin{pmatrix} \beta - 2\alpha L + \gamma L^2 & -\alpha + \gamma L \\ -\alpha + \gamma L & \gamma \end{pmatrix}$$

  if we start with $\alpha = 0$ we find $\beta = \beta_0 + \frac{L^2}{\beta_0}$

- Note: from symmetry $\alpha = 0$ in the quadrupole centres

Evidence

- We calculate the change of Twiss parameters for a small distance with constant K:

$$\begin{pmatrix} \beta_2 & -\alpha_2 \\ -\alpha_2 & \gamma_2 \end{pmatrix} = \begin{pmatrix} 1 & \delta \\ -K\delta & 1 \end{pmatrix} \begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} 1 & -K\delta \\ \delta & 1 \end{pmatrix} \quad (1)$$

$$= \begin{pmatrix} 1 & \delta \\ -K\delta & 1 \end{pmatrix} \begin{pmatrix} \beta_1 - \alpha_1\delta & -\alpha_1 - K\delta\beta_1 \\ -\alpha_1 + \gamma_1\delta & \gamma_1 + K\delta\alpha_1 \end{pmatrix} \quad (2)$$

$$= \begin{pmatrix} \beta_1 - 2\delta\alpha_1 + \delta^2\gamma_1 & -\alpha_1 + \delta\gamma_1 - K\delta\beta_1 + K\delta^2\alpha_1 \\ -\alpha_1 + \delta\gamma_1 - K\delta\beta_1 + K\delta^2\alpha_1 & \gamma_1 + K\delta\alpha_1 + K\delta\alpha_1 + K^2\delta^2\beta_1 \end{pmatrix} \quad (3)$$

- Now we calculate the derivative using

$$\frac{\begin{pmatrix} \beta_2 & -\alpha_2 \\ -\alpha_2 & \gamma_2 \end{pmatrix} - \begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix}}{\delta}$$

This yields for $\lim_{\delta\to 0}$

$$\beta' = \frac{\beta_2 - \beta_1}{\delta} = -2\alpha_1$$

and

$$\alpha' = \frac{\alpha_2 - \alpha_1}{\delta} = -\gamma_1 + K\beta_1$$

Evidence (cont.)

- We can compare the matrix results

$$\beta' = -2\alpha_1$$

to the definition of α:

$$\beta' = -2\alpha$$

and we can compare the result for α'

$$\alpha' = -\gamma + K\beta$$

to Hills equation:

$$\frac{\beta''\beta}{2} - \frac{\beta'^2}{4} + K\beta^2 = 1$$

this can be written as

$$-\alpha'\beta - \alpha^2 + K\beta^2 - 1 = 0$$

and

$$\alpha' = -\frac{1+\alpha^2}{\beta} + K\beta$$

which results in

$$\alpha' = -\gamma + K\beta$$

Single Particle Dyanmics: Acceleration

Transfer Matrix with Acceleration

- The inner part of an accelerating structure (assume constant and static electric field G that points parallel to s)

The transverse angle can be calculated using the conservation of the transverse momentum

$$x'(s) \;=\; x'(0)\frac{E_0}{E_0 + eGs} \tag{4}$$

$$\tag{5}$$

Simply integration yields the equation for the position

$$x(s) \;=\; x(0) + \frac{\ln\left(1 + \frac{eG}{E_0}s\right)}{\frac{eG}{E_0}}x'(0) \tag{6}$$

This yields the matrix

$$M_{acc,in} = \begin{pmatrix} 1 & L\frac{\ln\left(1+\frac{eG}{E_0}L\right)}{\frac{eGL}{E_0}} \\ 0 & \frac{E_0}{E_0+\frac{eGL}{E_0}} \end{pmatrix}$$

and replacing $eGL/E_0 = \delta$ we find

$$M_{acc,in} = \begin{pmatrix} 1 & L\frac{\ln(1+\delta s)}{\delta} \\ 0 & \frac{1}{1+\delta} \end{pmatrix}$$

Accelerating Structure End Fields

- Accelerating structure end fields are important

  - often wrong in textbooks

- As exercise: calculate the thin lens end-field kick of an accelerating structure for a particle moving with the speed of light

  - assume a homogeneous longitudinal electric field in the structure

  - use Gauss law and assume no charge inside the cylinder

  Note: it is not strictly correct to consider static electric field since we are dealing with RF fields, but it gives you some insight nevertheless. You are welcome to include the magnetic fields.

Solution

- the flux through a circle with radius r is

$$\Phi_l = G\pi r^2$$

- The flux through the mantle of the cylinder must be the same size but opposite sign

$$\Phi_\perp = \int_{s_1}^{s_2} G_\perp 2\pi r ds = -\Phi_l$$

- The transverse deflection is given by

$$\Delta x' = \int_{s_1}^{s_2} eG_\perp ds \frac{1}{E}$$

- Hence, we find

$$\Delta x' = -\frac{eG\pi r^2}{2\pi r}\frac{1}{E} = -\frac{eG}{2E}x$$

Full Transfer Matrix

- Now we add the transverse deflection to the structure

$$M_{acc} = \begin{pmatrix} 1 & 0 \\ -\frac{\delta}{2L(1+\delta)} & 1 \end{pmatrix} \begin{pmatrix} 1 & L\frac{\ln(1+\delta)}{\delta} \\ 0 & \frac{1}{1+\delta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\delta}{2L} & 1 \end{pmatrix}$$

$$M_{acc} = \begin{pmatrix} 1 - \frac{1}{2}\ln\left(1+\delta\right) & L\frac{\ln(1+\delta)}{\delta} \\ -\frac{\delta\ln(1+\delta)}{4L(1+\delta)} & \frac{1+\frac{1}{2}\ln(1+\delta)}{1+\delta} \end{pmatrix}$$

For $\delta \ll 1$

$$M_{acc} \approx \begin{pmatrix} 1 - \frac{1}{2}\delta & L\left(1 - \frac{1}{2}\delta\right) \\ 0 & 1 - \frac{1}{2}\delta \end{pmatrix}$$

$\Rightarrow$ Taking into account end fields makes the transfer matrix of the accelerating structure look more like a drift that shrinks the transverse beam size and divergence

Normalised Phase Space Revisited

- We used

$$\begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

to go into a normalised phase space. With acceleration the ellipse size is changing (remember we did not use the canonical variables)

- So we need instead to use

$$\sqrt{\gamma} \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

and for the transformation back

$$\frac{1}{\sqrt{\gamma}} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

Multi-Particle Dynamics

Beam Size

- A beam consists of many particles with coordinates $\vec{x}_i$
- We need to describe the statistical properties of these particles
- A convenient method is to use the sigma-matrix (which should have been called sigma-square-matrix)

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x x' \rangle & \langle x'^2 \rangle \end{pmatrix}$$

which can be calculated from a matrix X representing the beam

$$X = \frac{1}{\sqrt{n}} \begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \ldots & \vec{x}_n \end{pmatrix}$$

this allows to calculate

$$\Sigma = X X^T$$

- The transfer of this ensemble through the machine can be easily calculated

$$X_2 = M X_1$$

$$\Rightarrow \Sigma_2 = X_2 X_2^T = M X_1 (M X_1)^T = M X_1 X_1^T M^T = M \Sigma_1 M$$

Emittance

- We define the projected geometric emittance with the help of the sigma-matrix

$$\epsilon^2 = \det(\Sigma)$$

- If we assume a Gaussian beam, the area of the ellipse described by one sigma is $\pi\epsilon$
- In a linac it is easier to use the normalised emittance

$$\epsilon_N = \gamma\epsilon$$

  this value does not change with acceleration

- In this lecture we will always use the normalised emittance, without the index
- It should be noted that different definitions for the emittance exist
  - we use the projected emittance
  - but one could remove correlations before
- We usually define the emittance of a single bunch but in some cases we can also use the multi-pulse emittance, the overly of consecutive pulses

Beam Representation with Emittance

- We can represent a beam as

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{pmatrix}$$

- For Gaussian beams we can also use

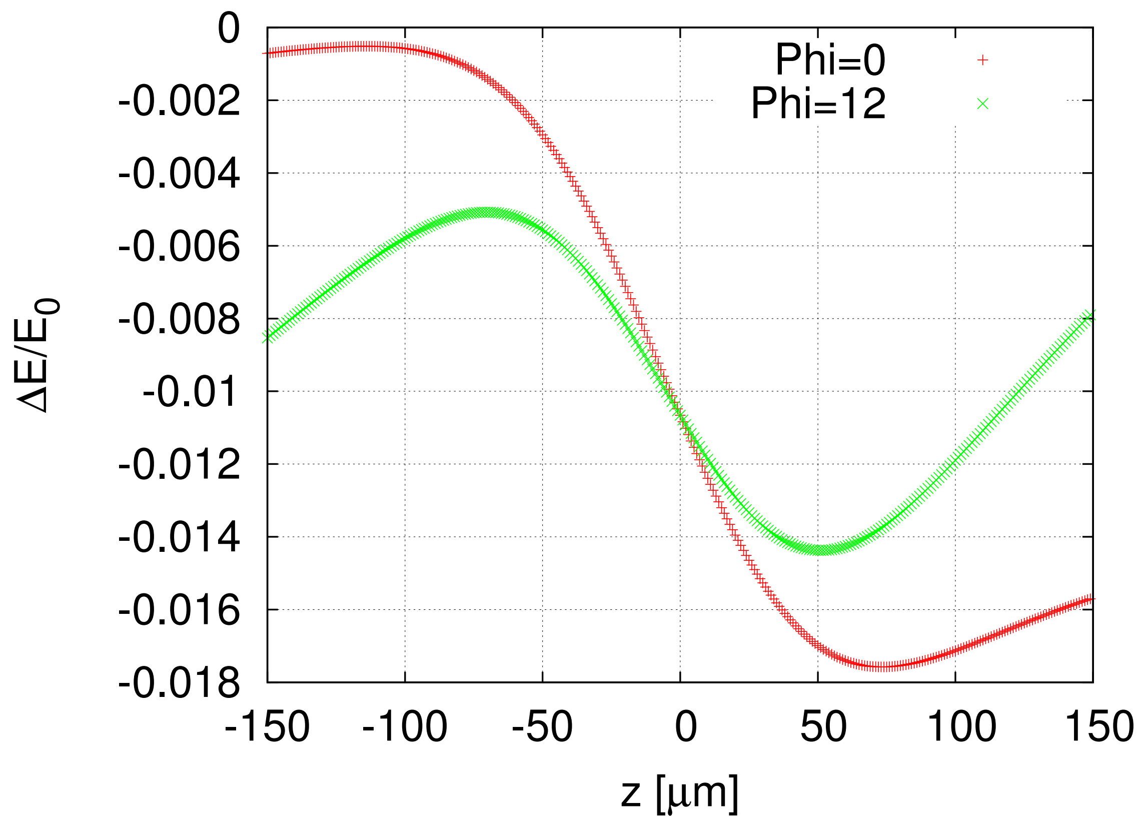
$$\Sigma = \begin{pmatrix} \beta\epsilon/\gamma & -\alpha\epsilon/\gamma \\ -\alpha\epsilon/\gamma & \gamma_T\epsilon/\gamma \end{pmatrix}$$

  Here, the Twiss parameter is named γ_T to distinguish it from the Lorentz-factor γ

Energy Spread

- If we do not run at the crest of the RF we can compensate the longitudinal single bunch wakefields
- But we are still left with some energy spread

  ⇒ need to understand the impact of the lattice design



Filamentation

- Using

$$\sin\frac{\mu}{2} = \frac{KL}{2}$$

and

$$K = \frac{E_0}{E}K_0 = \frac{1}{1+\delta}K_0$$

we can calculate the phase advance difference as

$$\sin\left(\frac{\mu_0 + \Delta\mu}{2}\right) = \frac{KL}{2(1+\delta)}$$

we develop the left hand side

$$\Rightarrow \sin\left(\frac{\mu_0}{2}\right)\cos\left(\frac{\Delta\mu}{2}\right) + \cos\left(\frac{\mu_0}{2}\right)\sin\left(\frac{\Delta\mu}{2}\right) = \frac{KL}{2(1+\delta)}$$

we approximate both sides

$$\Rightarrow \sin\left(\frac{\mu_0}{2}\right) + \cos\left(\frac{\mu_0}{2}\right)\frac{\Delta\mu}{2} \approx \sin\frac{\mu_0}{2}(1-\delta)$$

this yields

$$\Rightarrow \Delta\mu \approx -2\tan\left(\frac{\mu_0}{2}\right)\delta$$

- In CLIC we have roughly 200 betatron oscillations and $\mu = 1.26$ and $2\tan\mu/2 \approx 1.45$

$\Rightarrow$ A gradient difference of initial energy and gradient of one percent leads to a phase difference of 170°

Beta-Functions

In a similar fashion we can calculate the difference in beta-function

$$\frac{\hat{\beta}}{\hat{\beta}_0} = \frac{\frac{2}{K}\sqrt{\frac{1+KL/2}{1-KL/2}}}{\frac{2}{K_0}\sqrt{\frac{1+K_0L/2}{1-K_0L/2}}}$$

$$\Rightarrow \frac{\hat{\beta}}{\hat{\beta}_0} = \frac{1}{1+\delta}\sqrt{\frac{1+KL(1+\delta)/2}{1+K_0L/2}}\sqrt{\frac{1-K_0L/2}{1-KL(1+\delta)/2}}$$

$$\Rightarrow \frac{\hat{\beta}}{\hat{\beta}_0} = \frac{1}{1+\delta}\sqrt{\frac{1+\delta(K_0L/2)-(1+\delta)(K_0L/2)^2}{1-\delta(K_0L/2)-(1+\delta)(K_0L/2)^2}}$$

$$\Rightarrow \frac{\hat{\beta}}{\hat{\beta}_0} \approx \frac{1}{1+\delta}\left(1+\frac{K_0L/2}{1-(K_0L/2)^2}\delta\right)$$

and similarly for $\check{\beta}$

$$\Rightarrow \frac{\check{\beta}}{\check{\beta}_0} \approx \frac{1}{1+\delta}\left(1-\frac{K_0L/2}{1-(K_0L/2)^2}\delta\right)$$

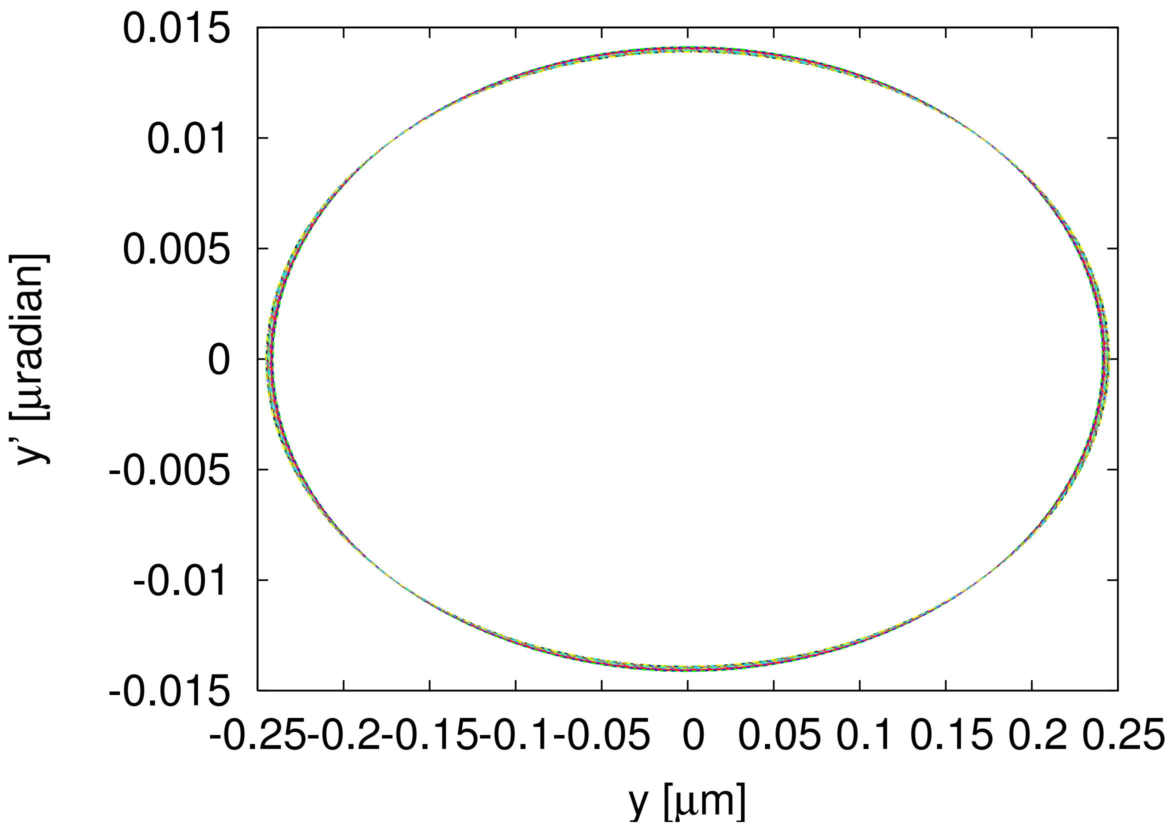
$\Rightarrow$ Beta-function do not vary strongly

Final Beam with Energy Spread

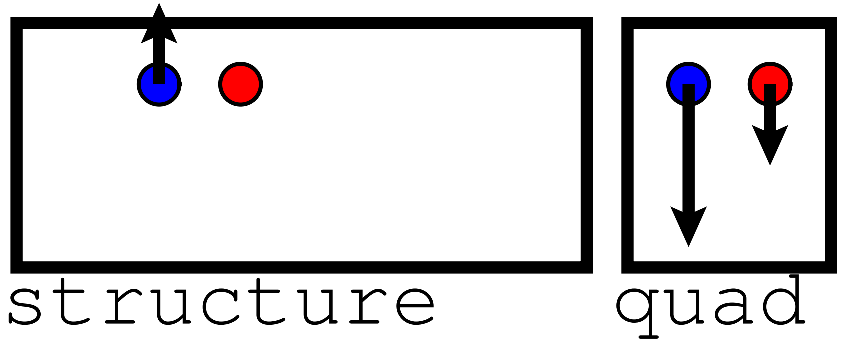
- The final beam ellipses at different energies look quite similar
  - plot shows all beam ellipses in $\pm 3\sigma_z$

  $\Rightarrow$ the resulting emittance growth is negligible

  - one of the reasons the FODO lattice is so nice



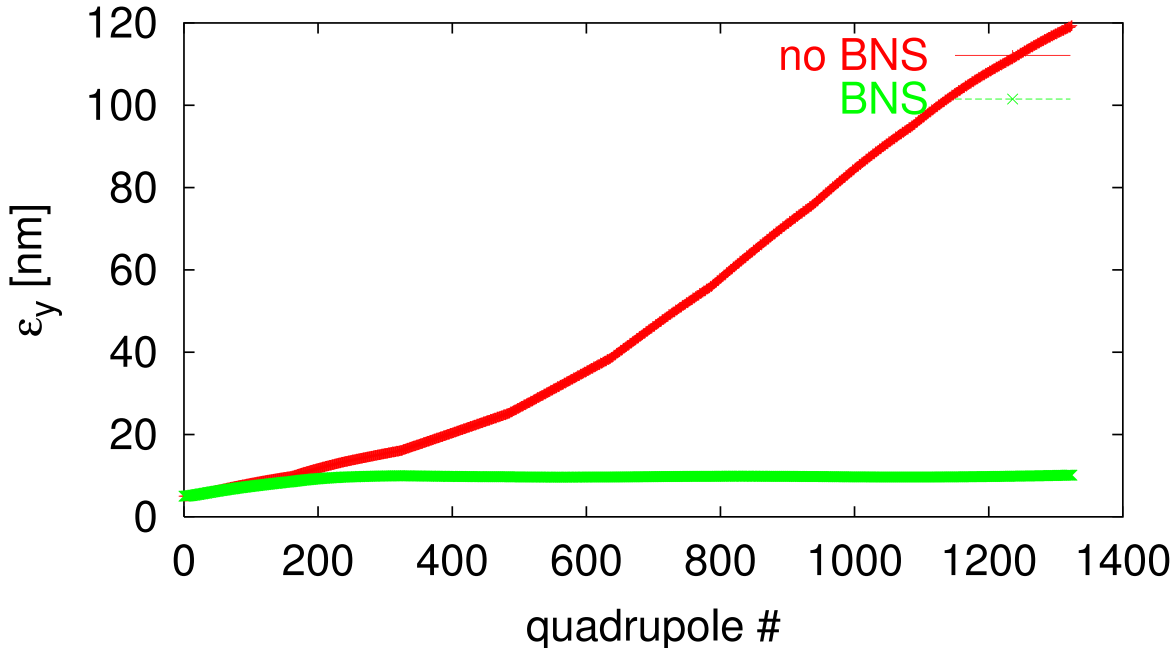
Beam Stability and BNS Damping



- Transverse wakes act as defocusing force on tail

  ⇒ beam jitter is exponentially amplified

- BNS damping prevents this growth
  - manipulate RF phases to have energy spread
  - take spread out at end



Two-Particle Wakefield Model

- Assume bunch can be represented by two particles and constant $K(s) = 1/\beta^2$
  - second particle is kicked by transverse wakefield
  - initial oscillation

$$x_1'' + \frac{1}{\beta^2} x_1 = 0 \qquad x_2'' + \frac{1}{\beta^2} x_2 = \frac{Ne^2 W_\perp}{P_L c} x_1$$

$$x_1 = x_0 \cos\left(\frac{s}{\beta}\right)$$

$$x_2'' + \frac{1}{\beta^2} x_2 = x_0 \frac{Ne^2 W_\perp}{P_L c} \cos\left(\frac{s}{\beta}\right)$$

- Solution is simple with an ansatz

$$x_2 = x_0 \cos\left(\frac{s}{\beta}\right) + \left(\frac{x_0 N e^2 W_\perp \beta}{2E} s\right) \sin\left(\frac{s}{\beta}\right)$$

⇒ Amplitude of second particle oscillation is growing

⇒ The bunch charge and length matter as well as the lattice

⇒ Have a closer look into wakefields

BNS Damping solution

- First particle performs a harmonic oscillation

$$x_1(s) = x_0 \cos\left(\frac{s}{\beta_1}\right)$$

- We want the second particle to perform the **same** oscillation
- Modify unperturbed oscillation frequency of second particle

$$x_2 = x_0 \cos\left(\frac{s}{\beta_2}\right)$$

- Leads to

$$x_2'' + \frac{1}{\beta_2^2} x_2 = x_0 \frac{Ne^2 W_\perp}{P_L c} \cos\left(\frac{s}{\beta_1}\right) = x_1 \frac{Ne^2 W_\perp}{P_L c}$$

- Assuming

$$\frac{1}{\beta_2^2} = \frac{1}{\beta_1^2} + \frac{Ne^2 W_\perp}{P_L c}$$

- Yields simple solution

$$x_2 = x_0 \cos\left(\frac{s}{\beta_1}\right) = x_1$$

$\Rightarrow$ No more wakefield effect

Introduction of Energy Spread

- For BNS damping we want to achieve

$$\frac{1}{\beta_2^2} = \frac{1}{\beta_1^2} + \frac{Ne^2 W_\perp}{P_L c}$$

this can be achieved by reducing the energy of the second particle

- We express β_2 as a function of β_1 and the relative energy difference δ

$$\frac{1}{\beta_1^2(1-\delta)} = \frac{1}{\beta_1^2} + \frac{Ne^2 W_\perp}{P_L c}$$

this yields

$$\delta \approx \beta_1^2 \frac{Ne^2 W_\perp}{P_L c}$$

⇒ Want to keep β small

⇒ If we scale $\beta = \beta_0 \sqrt{E/E_0}$ we find

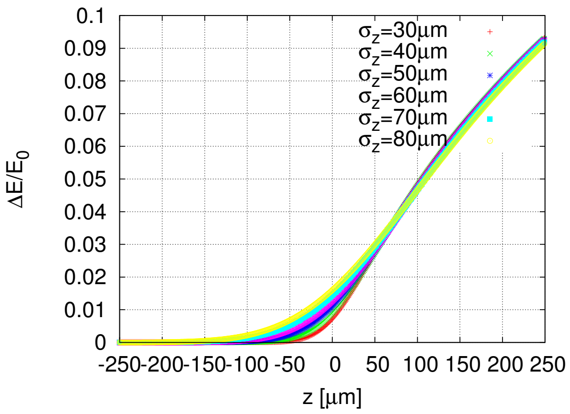
$$\delta \approx \beta_0^2 \frac{Ne^2 W_\perp}{E_0} = \mathrm{const}$$

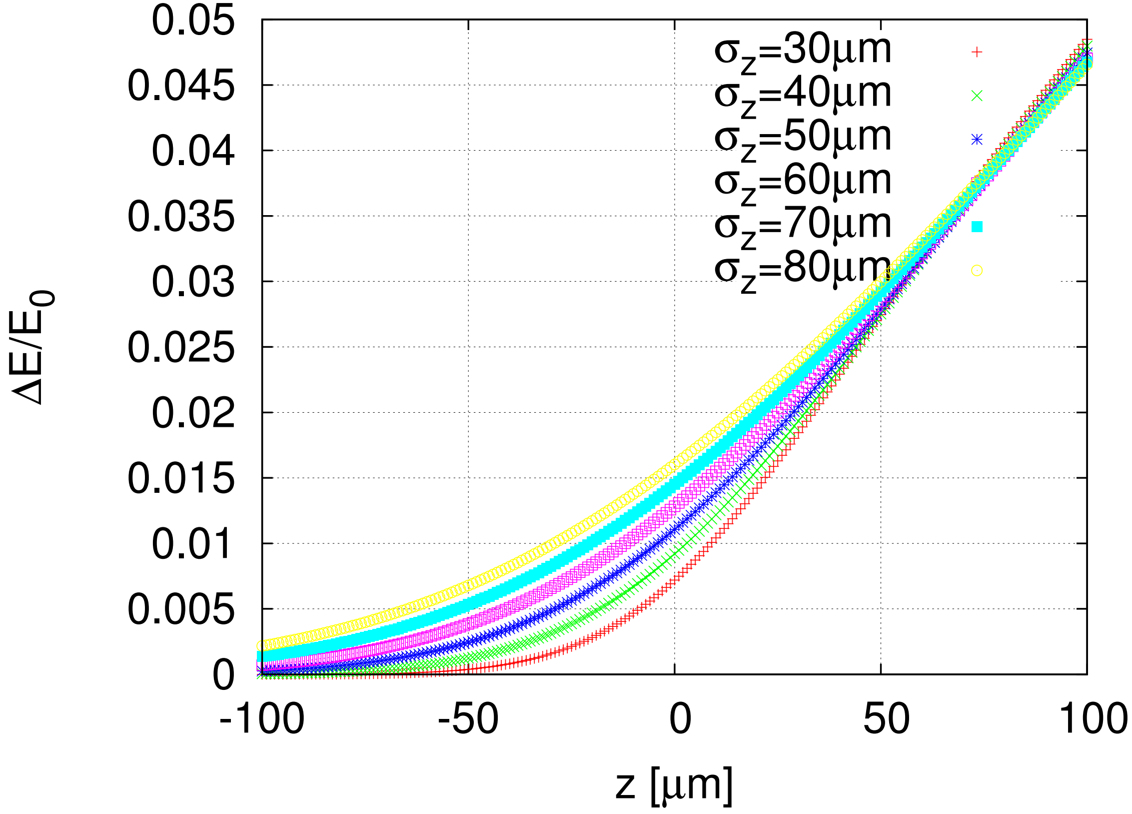
BNS Damping for a Bunch

- If each particle of the bunch should be damped we must require that the transverse sum-wake is matched by the energy spread

$$\int_{-\infty}^{z} W_{\perp}(z-z')N\rho(z')dz'$$

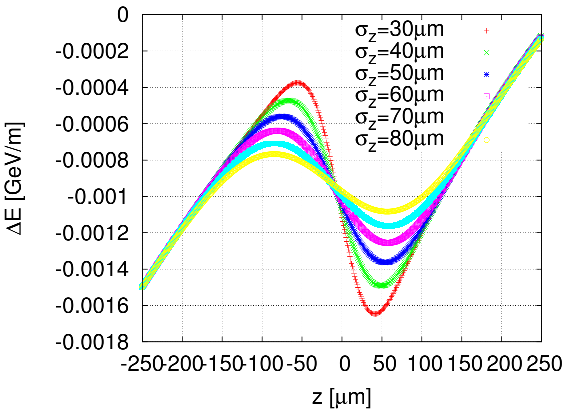
- Some examples assuming a rigid bunch

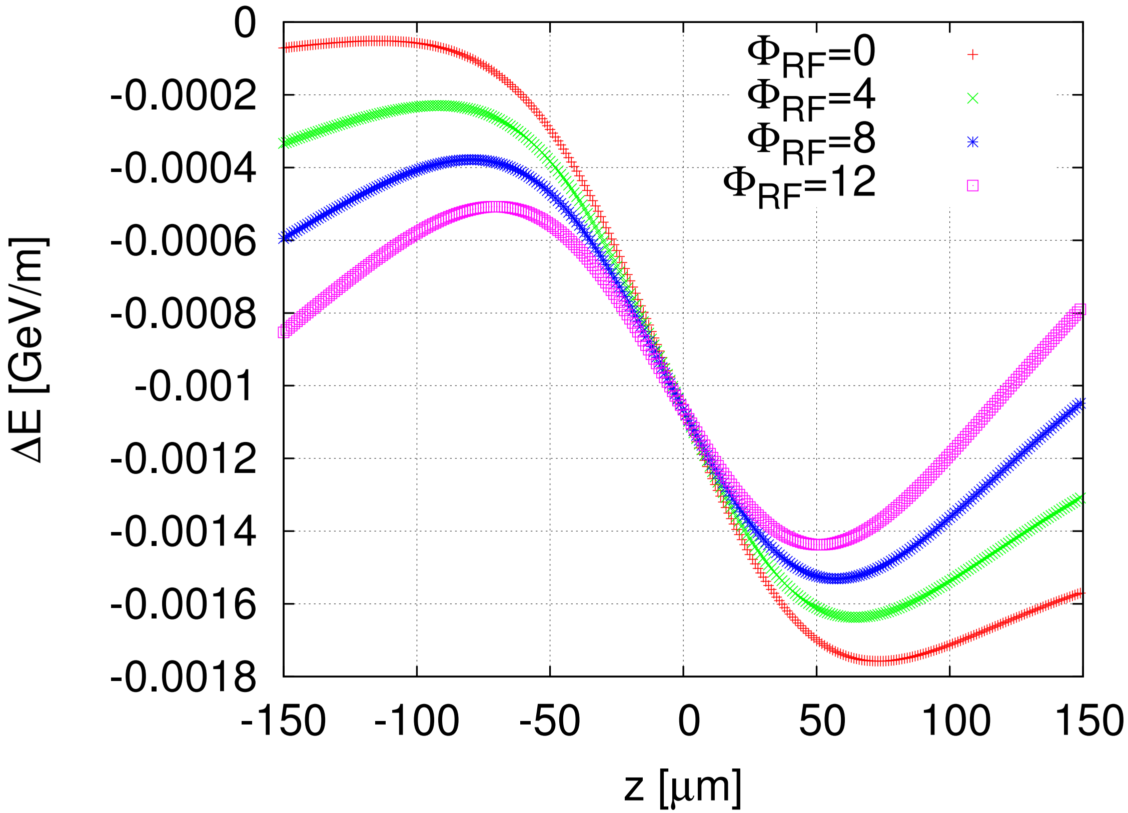




Energy Spread in the Linac

- In CLIC one uses one RF phase from the beginning of the linac
- At the end one runs at 30° to reduce the energy spread
  - yields an average phase of 12°



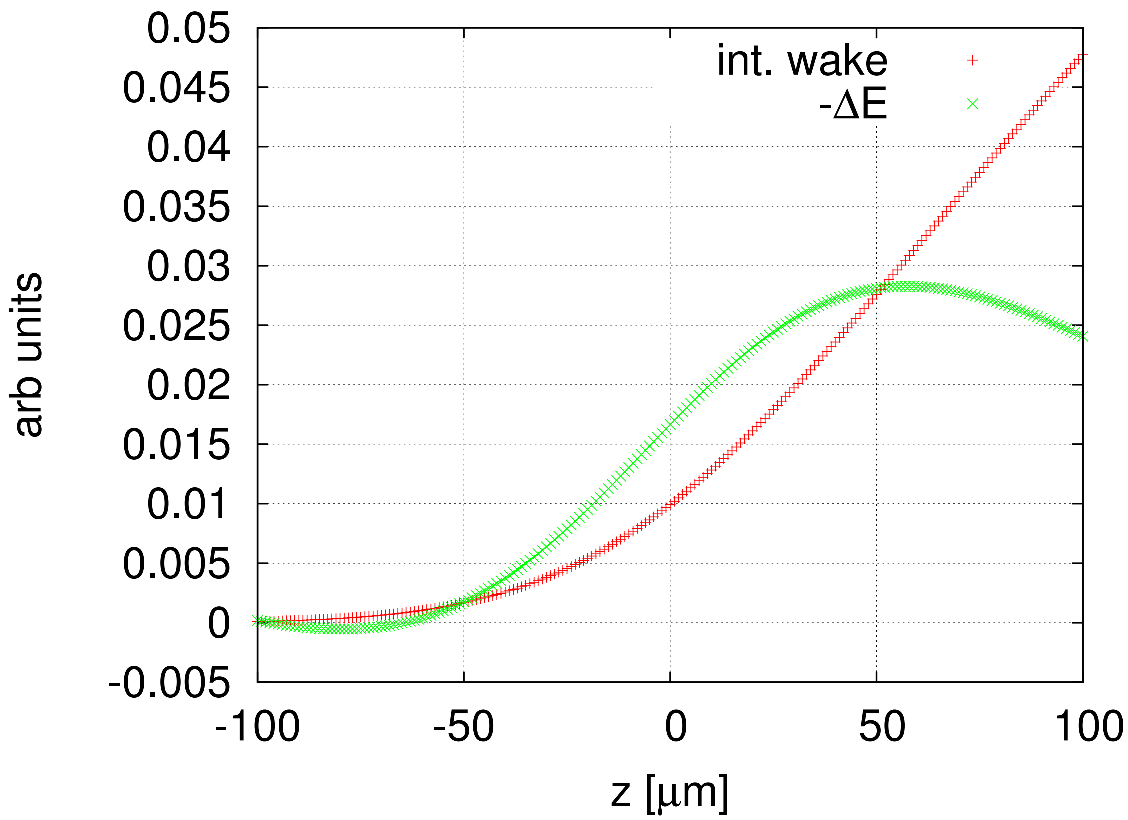


Beam Energy Spread and Wakefield

- We have to work with the energy spread in the beam
- The shape of the energy spread and the integrated wake are different

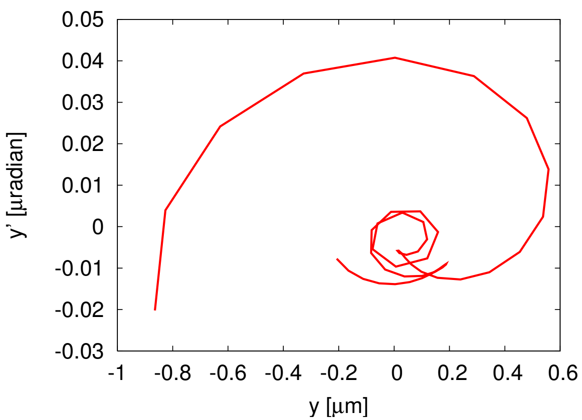
  ⇒ can only obtain some correction

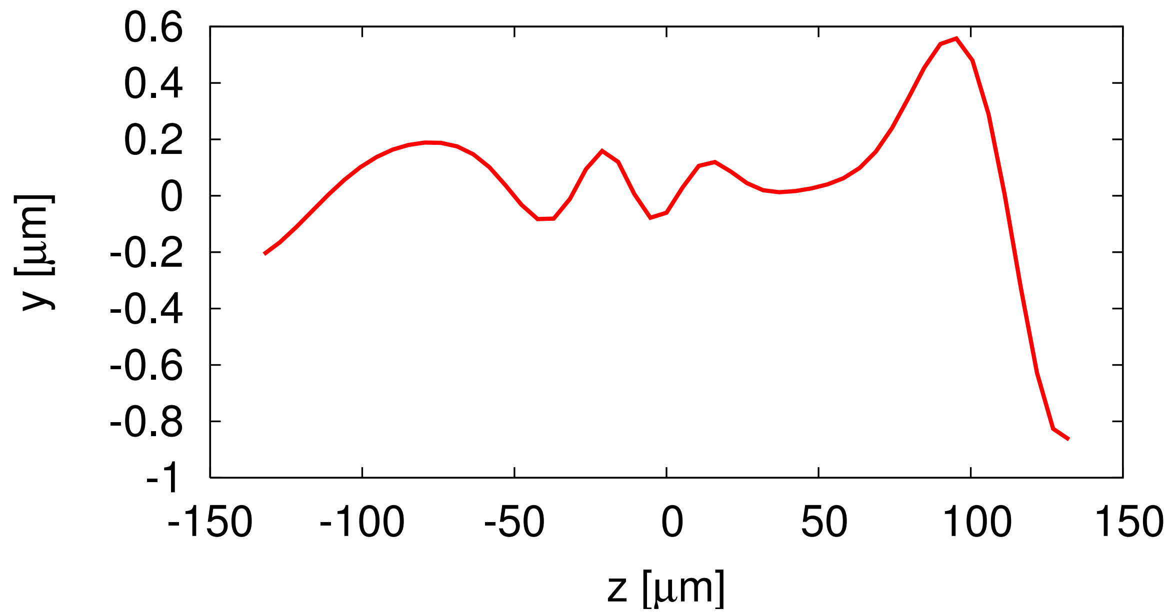
  ⇒ need to resort to simulations

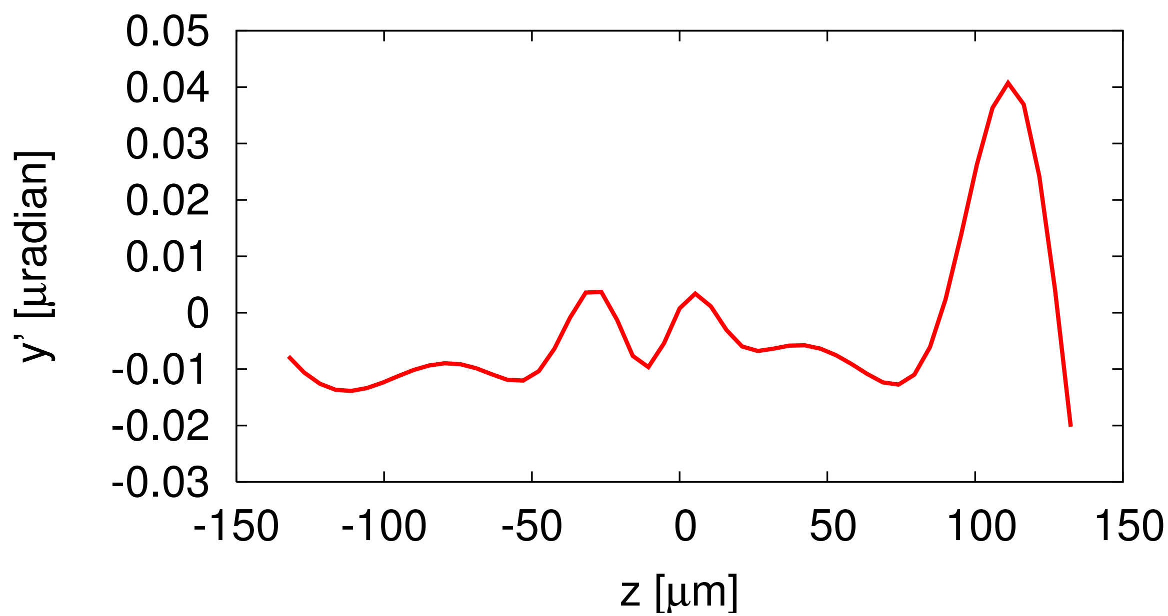


Final Bunch

- To illustrate the final bunch in CLIC with an initial offset of $1\sigma_y$



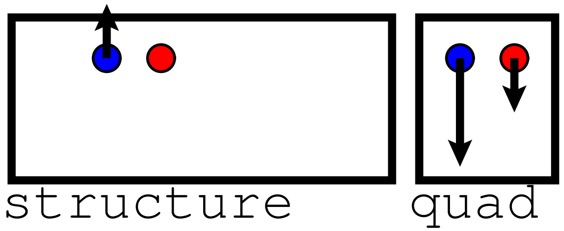




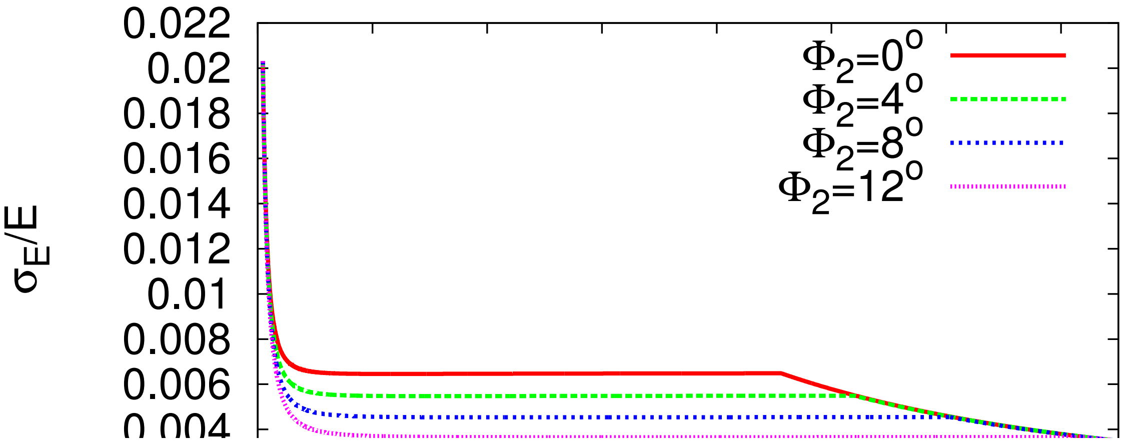
Energy Spread and Beam Stability

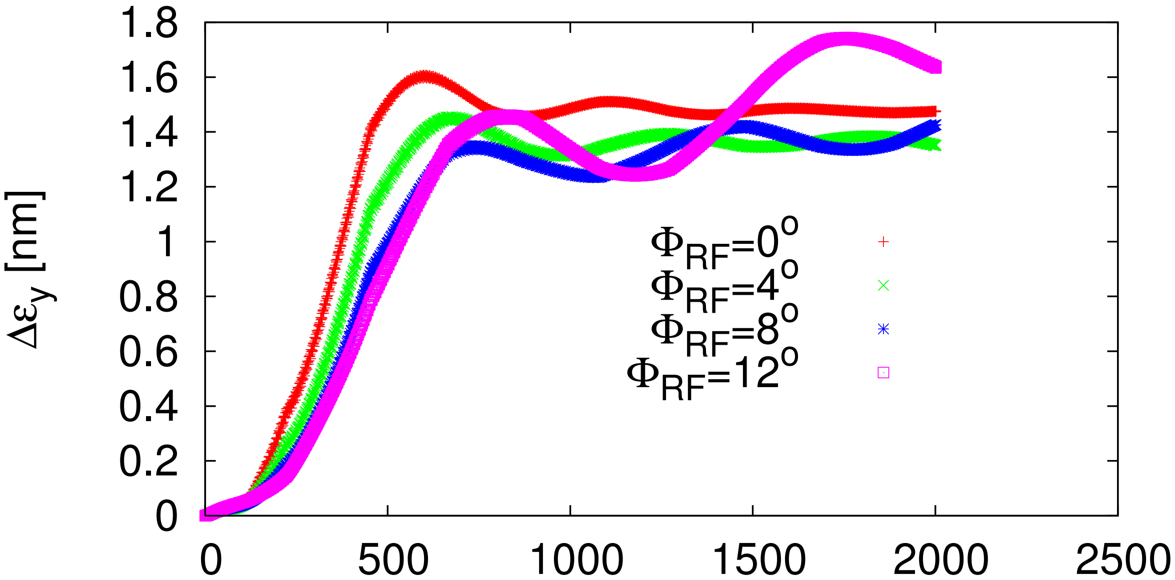
- Trade-off in fixed lattice
  - large energy spread is more stable
  - small energy spread is better for alignment

⇒ Beam with $N = 3.7 \times 10^9$ can be stable



⇒ Tolerances are not a unique number



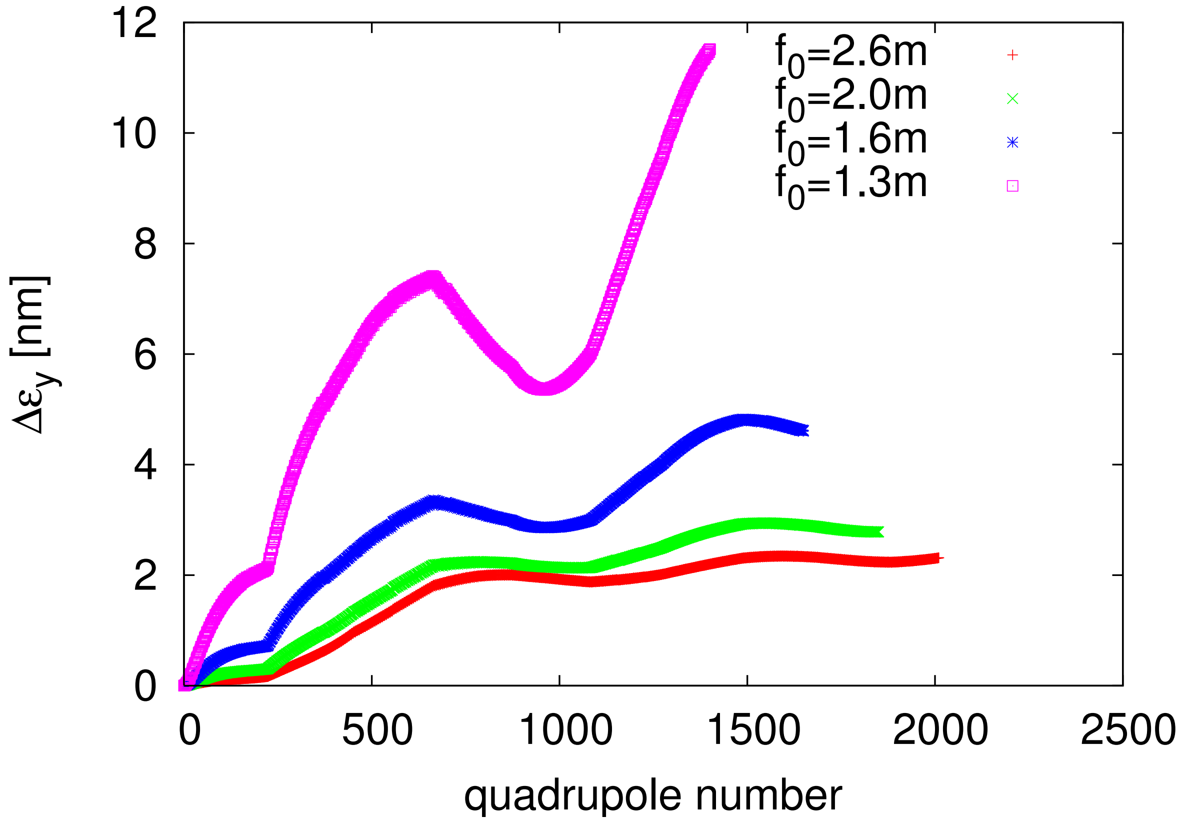


Lattice Strength

- We try different lattices
  - all scale $f = f_0\sqrt{E/E_0}$ and $L = L_0\sqrt{E/E_0}$ with $L_0 = 1.15 f_0$

$\Rightarrow$ We need $f_0 \leq 2\,\mathrm{m}$

- But would like to have some reserve



Magnet Considerations

- The maximum strength of a focusing magnet is limited
  - for a normal conducting design rule of thumb is $1\,\mathrm{T}$ at the pole-tip

$\Rightarrow$ Required integrated magnet strength is

$$\frac{\mathrm{T}}{\mathrm{m}}\frac{E}{0.3\,\mathrm{GeV}}\frac{\mathrm{m}}{f}$$

- For CLIC poletip radius is given by practical considerations of magnet design $a \approx 5\,\mathrm{mm}$ yielding a gradient of $200\,\mathrm{T/m}$
- We chose about 10% of the machine to be quadrupoles

  $\Rightarrow$ fill factor is $\approx 80\%$
  - 10% are lost for flanges (mainly on structures)
- Use $L_0 = 1.5\,\mathrm{m}$ and $f_0 = 1.3\,\mathrm{m}$ yields

$$\eta_q = \frac{E_0}{0.3\,\mathrm{GeV}}\frac{\mathrm{T/m}}{200\,\mathrm{T/m^2}}\frac{\mathrm{m}}{f_0}\frac{1}{L_0}$$

$$\Rightarrow \eta_q \approx 7.7\%$$

- We use discrete lengths hence we loose a bit more

Sectors in CLIC

- For practical reasons we do not change the lattice continuously but in steps
- To go from the periodic lattice of one sector to the periodic lattice of the next we need to perform matching
  - we change the strength of seven magnets to achieve a transfer matrix M with

$$\begin{pmatrix} \beta_{x,2} & -\alpha_{x,2} & 0 & 0 \\ -\alpha_{x,2} & \gamma_{x,2} & 0 & 0 \\ 0 & 0 & \beta_{y,2} & -\alpha_{y,2} \\ 0 & 0 & -\alpha_{y,2} & \gamma_{y,2} \end{pmatrix} = M \begin{pmatrix} \beta_{x,1} & -\alpha_{x,1} & 0 & 0 \\ -\alpha_{x,1} & \gamma_{x,1} & 0 & 0 \\ 0 & 0 & \beta_{y,1} & -\alpha_{y,1} \\ 0 & 0 & -\alpha_{y,1} & \gamma_{y,1} \end{pmatrix} M^T$$

  here $\gamma = (1+\alpha^2)/\beta$ is the third Twiss parameter is used, in spite of my promise

  we require that a similar equation holds true for off-energy particles

Warning

- We found that the jittering beam should be most stable for smallest beta-functions
- But we still have to make sure that the imperfections will not make this solution impossible

  $\Rightarrow$ have to come back to this topic

Imperfections

Introduction

- We also have to be able to express imperfections in the matrix model
- Assume that the transfer-matrix for a beam line is

$$M = M_2 M_1$$

the perturbation at the location between M_2 and M_1 can be written as

$$\vec{x}_f = M_2 M_1 \vec{x}_0 \quad \rightarrow \vec{x}_f = M_2(M_1 \vec{x}_0 + \vec{\delta})$$

hence we can write for many imperfections

$$\vec{x}_f = M\vec{x}_0 + \sum_i M_{i\rightarrow f}\vec{\delta}_i$$

with the transfer matrices $M_{i\rightarrow f}$ from imperfection i to the end

Kick Induced by a Misplaced Element

- Assume that element i with transfer matrix M_i is offset by $\vec{y}_i$

$$\vec{\delta}_i = M_i(M_{0\to i}\vec{x}_0 - \vec{y}) + \vec{y} - M_i M_{0\to i}\vec{x}_0$$

  we transform the beam into the system of the element track through the element and transform back

- Note: in some cases one needs to transfer into the element system by also multiplying with a matrix (e.g. rotated elements)

- We simplify the term

$$\vec{\delta}_i = -(M_i - 1)\vec{y}_i$$

- The additional effect at the end of the lattice Δ_i is given by

$$\vec{\Delta}_i = -M_{i\to f}(M_i - 1)\vec{y}_i$$

- The total effect of all elements is then

$$\vec{x}_f = M\vec{x}_0 - \sum_i M_{i\to f}(M_i - 1)\vec{y}_i$$

Examples

$$\vec{\Delta}_i = -M_{i\rightarrow f}(M_i - 1)\vec{y}_i \qquad \vec{\delta}_i = -(M_i - 1)\vec{y}_i$$

- Thin quadrupole

$$\vec{\delta}_i = -\left(\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\vec{y}_i$$

$$\vec{\delta}_i = -\begin{pmatrix} 0 & 0 \\ \frac{1}{f} & 0 \end{pmatrix}\vec{y}_i$$

hence

$$\vec{\delta}_i = \begin{pmatrix} 0 \\ -\frac{y}{f} \end{pmatrix}$$

- The additional kick is the same as the one from a thin dipole:

$$\vec{\delta}_i = \begin{pmatrix} 0 \\ a \end{pmatrix}$$

Simple Example

- Particle is kicked with angle δ at s_1
- Go into normalised phase space

$$\begin{pmatrix} \frac{1}{\sqrt{\beta(s_1)}} & 0 \\ \frac{\alpha(s_1)}{\sqrt{\beta(s_1)}} & \sqrt{\beta(s_1)} \end{pmatrix} \begin{pmatrix} 0 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ \delta\sqrt{\beta(s_1)} \end{pmatrix}$$

$\Rightarrow$ a kick is more important at a position with large β

- Phase advance is given by $S = \sin(\phi(s_2) - \phi(s_1))$, $C = \cos(\phi(s_2) - \phi(s_1))$

$$\begin{pmatrix} C & S \\ -S & C \end{pmatrix} \begin{pmatrix} 0 \\ \delta\sqrt{\beta(s_1)} \end{pmatrix} = \begin{pmatrix} S \\ C \end{pmatrix} \delta\sqrt{\beta(s_1)}$$

- Amplitude at s_2 is calcuated by going back to normal phase space

$$\begin{pmatrix} \sqrt{\beta(s_2)} & 0 \\ -\frac{\alpha(s_2)}{\sqrt{\beta(s_2)}} & \frac{1}{\sqrt{\beta(s_2)}} \end{pmatrix} \begin{pmatrix} S \\ C \end{pmatrix} \delta\sqrt{\beta(s_1)}$$

$$= \begin{pmatrix} \sqrt{\beta(s_1)\beta(s_2)}S \\ \alpha(s_2)\sqrt{\frac{\beta(s_1)}{\beta(s_2)}}S + C\sqrt{\frac{\beta(s_1)}{\beta(s_2)}} \end{pmatrix} \delta$$

Imperfections in Normalised Coordinates

- The linac is not the final system

  $\Rightarrow$ we are often not interested in the final position in real coordinates but in normalised coordinates

  - can be easily translated into a beam further downstream

- We saw that imperfections mainly can be understood as applying a kick to the beam, the trajectory does not jump

- Example for a thin quadrupole with offset

$$\vec{\delta}_{N,i} = \sqrt{\gamma} \begin{pmatrix} \frac{1}{\sqrt{\beta(s_1)}} & 0 \\ \frac{\alpha(s_1)}{\sqrt{\beta(s_1)}} & \sqrt{\beta(s_1)} \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \vec{y}_i$$

$$\vec{\delta}_{N,i} = \begin{pmatrix} 0 & 0 \\ -\sqrt{\beta\gamma}\frac{1}{f} & 0 \end{pmatrix} \vec{y}_i = \begin{pmatrix} 0 \\ -\sqrt{\beta\gamma}\frac{1}{f} y_i \end{pmatrix}$$

$\Rightarrow$ sensitivity depends on the local beta-function

Impact on the Emittance

- We consider multi-pulse emittance
- Assume a quadrupole is jittering with RMS value σ_q
- The increase in normalised angle can be calculated as

$$\sigma_{Nx'} = \sqrt{\epsilon + \beta\gamma\left(\frac{\sigma_q}{f}\right)^2}$$

$\Rightarrow$ for small perturbations

$$\sigma_{Nx'} \approx \epsilon\left[1 + \frac{\beta\gamma}{2\epsilon}\left(\frac{\sigma_q}{f}\right)^2\right]$$

$\Rightarrow$ the emittance growth is

$$\Delta\epsilon \approx \frac{\beta\gamma}{2}\left(\frac{\sigma_q}{f}\right)^2$$

$\Rightarrow$ the emittance growth depends on the square of the perturbation

$\Rightarrow$ the emittance growth depends on the beta-function

Kick and Emittance Growth

$$y'^2_{new} = \frac{1}{2}\left((-y' + \delta)^2 + (y' + \delta)^2\right)$$

$$\rightarrow y'^2_{new} = \frac{1}{2}\left((y'^2 - 2y'\delta + \delta^2) + (y'^2 + 2y'\delta + \delta^2)\right)$$

$$\rightarrow y'^2_{new} = y'^2 + \delta^2$$

Calulating the emittance (no correlation)

$$\epsilon = \sqrt{< y^2 >< y'^2 >}$$

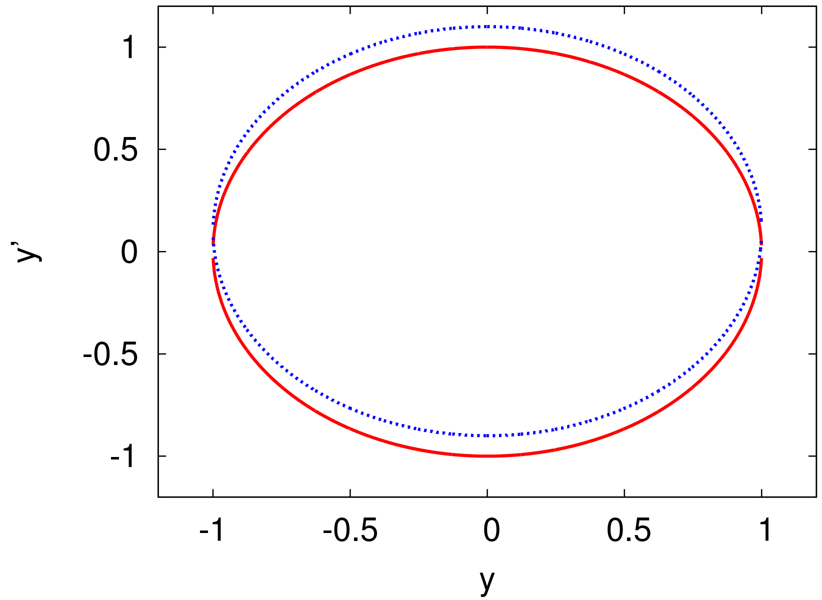
we find

$$\epsilon_{new} = \sqrt{\sigma_y^2(\sigma_{y'}^2 + \delta^2)}$$

$$\frac{\epsilon_{new}}{\epsilon_{old}} = \sqrt{\frac{\sigma_y^2(\sigma_{y'}^2 + \delta^2)}{\sigma_y^2\sigma_{y'}^2}}$$

$$\frac{\epsilon_{new}}{\epsilon_{old}} = \sqrt{\frac{(\sigma_{y'}^2 + \delta^2)}{\sigma_{y'}^2}}$$

$$\frac{\epsilon_{new}}{\epsilon_{old}} \approx 1 + \frac{1}{2}\frac{\delta^2}{\sigma_{y'}^2}$$



Note: after filamentation

$$y'^2_{new} = y'^2 + \frac{1}{2}\delta^2 \qquad y^2_{new} = y^2 + \frac{1}{2}\delta^2$$

Hence

$$\frac{\epsilon_{new}}{\epsilon_{old}} \approx 1 + \frac{1}{2}\frac{\delta^2}{\sigma_{y'}^2}$$

Coupling of the Planes

- A rotated quadrupole couples the two planes
- Example of thin quadrupole

$$M_c = \begin{pmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & \cos\phi & 0 & -\sin\phi \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & \sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/f & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/f & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & \cos\phi & 0 & \sin\phi \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

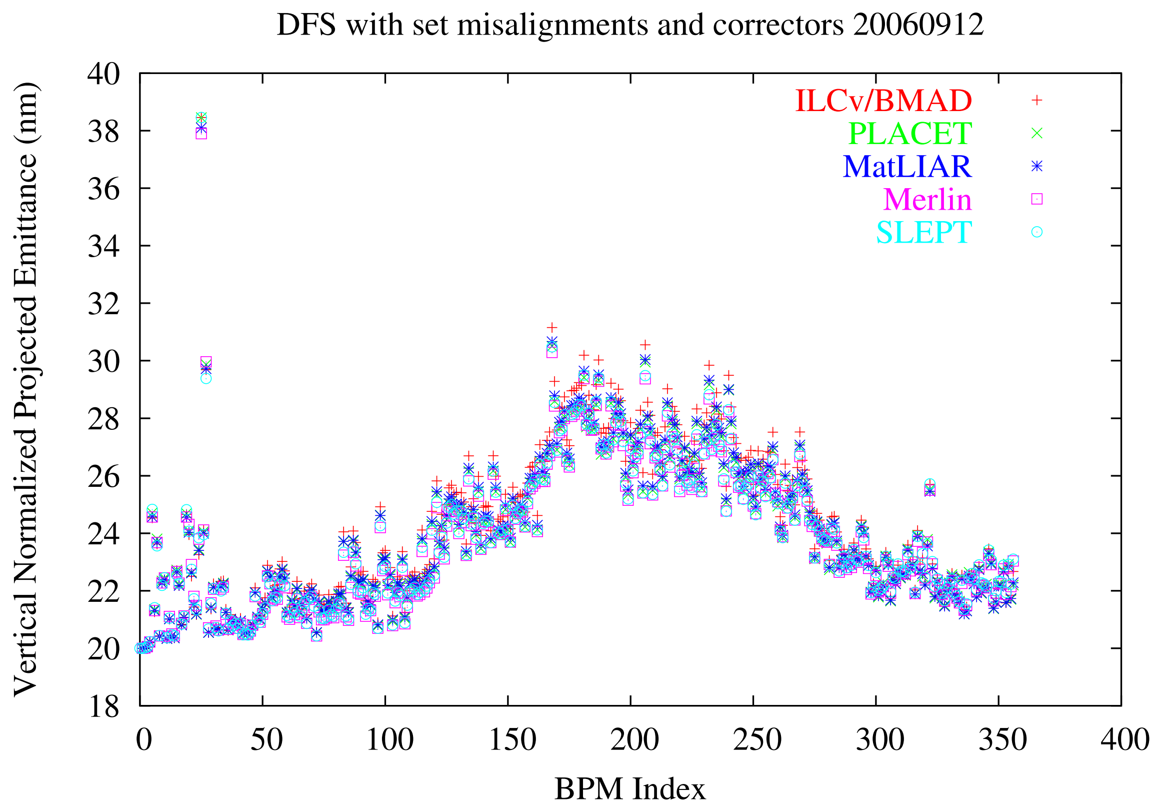
$$M_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ (\cos^2\phi - \sin^2\phi)/f & 1 & 2\sin\phi\cos\phi/f & 0 \\ 0 & 0 & 1 & 0 \\ 2\sin\phi\cos\phi/f & 0 & -(\cos^2\phi - \sin^2\phi)/f & 1 \end{pmatrix}$$

- Coupling is important since the horizontal emittance is much larger than the vertical

Simulations

Simulation Procedure and Benchmarking

- All simulation studies are performed with different codes
  - based on 100 different machines
- Benchmarking of tracking codes is essential
- Comparisons performed in ILC framework
  - tracking with errors
  - alignment methods



Integrated Simulations

- Integration of different systems is necessary
  - include correlations in the beam
  - feedback in different areas need to work together
  - tuning and alignment applied in one system are affected by noise generated in another
  - we sometimes need one system to tune and align the other

    e.g. main linac dispersion correction with bumps in bunch compressor and BDS

    luminosity tuning
- Integration of different time-scales is necessary
  - have intra-pulse and pulse-to-pulse feedback
  - tuning takes time and can interfere with feedback
  - alignment can be be sensitive to dynamic effects
  - dynamic effects can be sensitive to tuning and alignment
- Different codes are being developed and are quite mature

  BMAD/ILCv, CHEF, MATLIAR, LUCRETIA, MERLIN, PLACET, SLEPT...

The Banana Effect

At large disruption, correlated offsets in the beam can lead to instability

The emittance growth in the beam leads to correlation of the mean y position to z

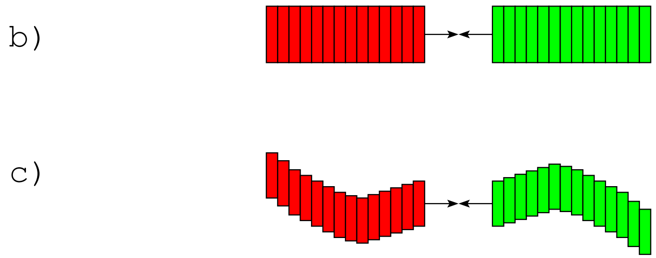
a) shows development of beam in the main linac

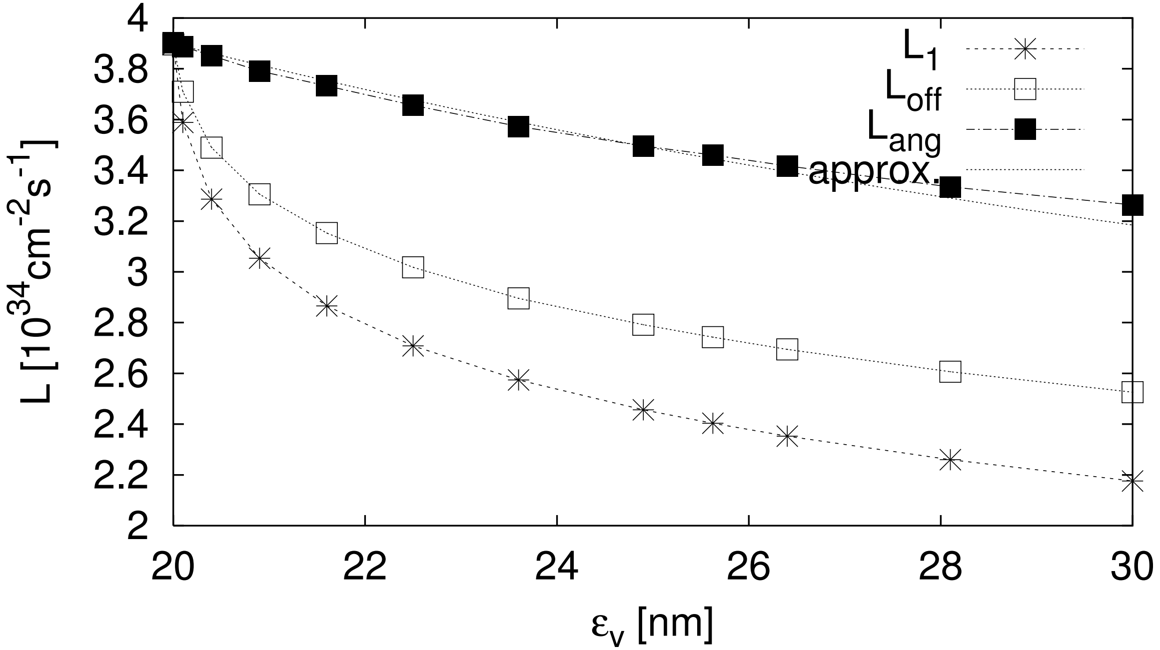
b) simplified beam-beam calculation using projected emittances

c) beam-beam calculation with full correlation

⇒ Luminosity loss increased

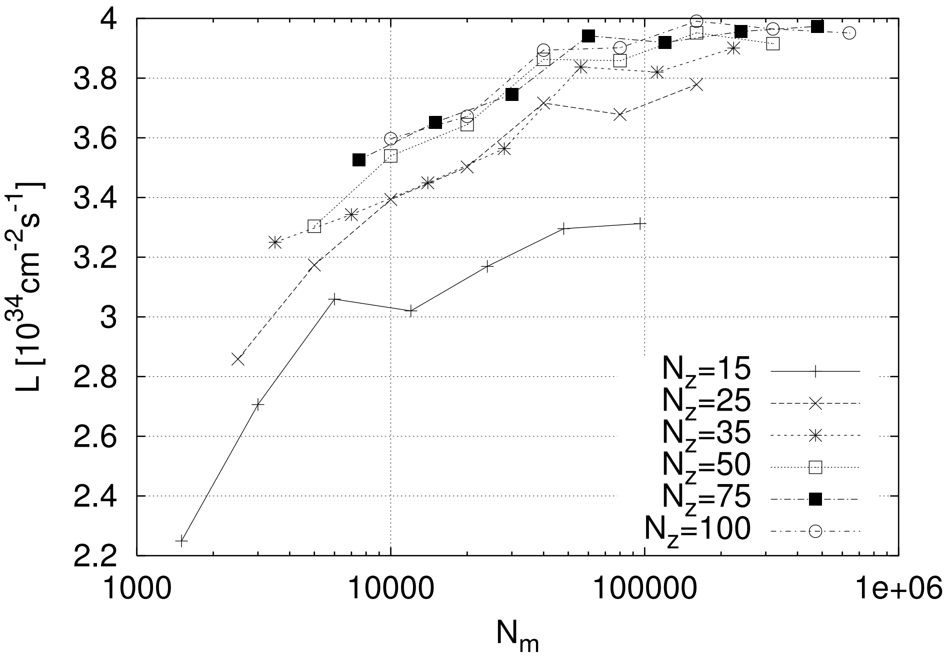
⇒ Cure exists





Computing Time Needed

- Beam-beam requires $\mathcal{O}(10^5)$ particles
- Typical full simulation of one bunch takes $\approx 2 \times 5\,\text{minutes}$
  - ⇒ tracking one train of 2820 bunches takes 20 days
  - ⇒ to track 1000 pulses one would need more than fifty years
- CPUs seem not to become that much faster any more
- But they contain more than one core



TESLA example

⇒ take short cuts, e.g single bunch simulations

⇒ would likely profit from parallel codes in the long term (but normally will run 100 seeds)

- some care needs to be taken for wakefields and the beam-beam interaction
  - wakefields need to be calculated at least in each cavity, i.e. ≈ 8000 times

Main Linac Simulations

- Can track many point-like macro-particles
- Or used particles with sizes
  - the main linac dynamics is largely linear
  - can use ellipses to describe the beam
- Cut the beam into slices
  - remember particles stay in their slice
  - RF curvature and wakefields
- Each slice is represented by a few ellipses
  - incoherent energy spread in the beam
- Need to track the centre and the shape of the ellipses

Curved Main Linac

Introduction

Two main reasons why one might want to have a tunnel that follows the earth curvature

- one can stay close to the surface everywhere (but site dependent)

- in ILC, the helium level will follow the equipotential of the gravity

But there are some problems for the beam dynamics

- one needs to guide the beam on a curved orbit this requires introduction of dispersion

- the dispersion makes the machine operation more difficult

In ILC the arguments for the cryogenics where considered important, so a curved tunnel is chosen
In CLIC there was no benefit to go to a curved tunnel, so the laser-straight option is preferred.

Dispersion

- We deflect a particle of energy E_1 with a dipole corrector (offsetting a quadrupole has exactly the same effect)
  the resulting deflection angle is
  $$\delta_1' \approx 0.3 \frac{\mathrm{GeV}}{\mathrm{Tm}^2} \frac{BL}{E_1}$$
  If we have a second particle at a different energy E_2 it is deflected differently
  $$\delta_2' \approx 0.3 \frac{\mathrm{GeV}}{\mathrm{Tm}^2} \frac{BL}{E_2}$$
  so the two particles will take different trajectories
  The different is described by the dispersion $D_{x,y}$ with
  $$D_x = \frac{\partial x}{\partial \delta} \qquad D_y = \frac{\partial y}{\partial \delta}$$

In a transport line with acceleration there is no clearly defined dispersion Have spurious dispersion from imperfections

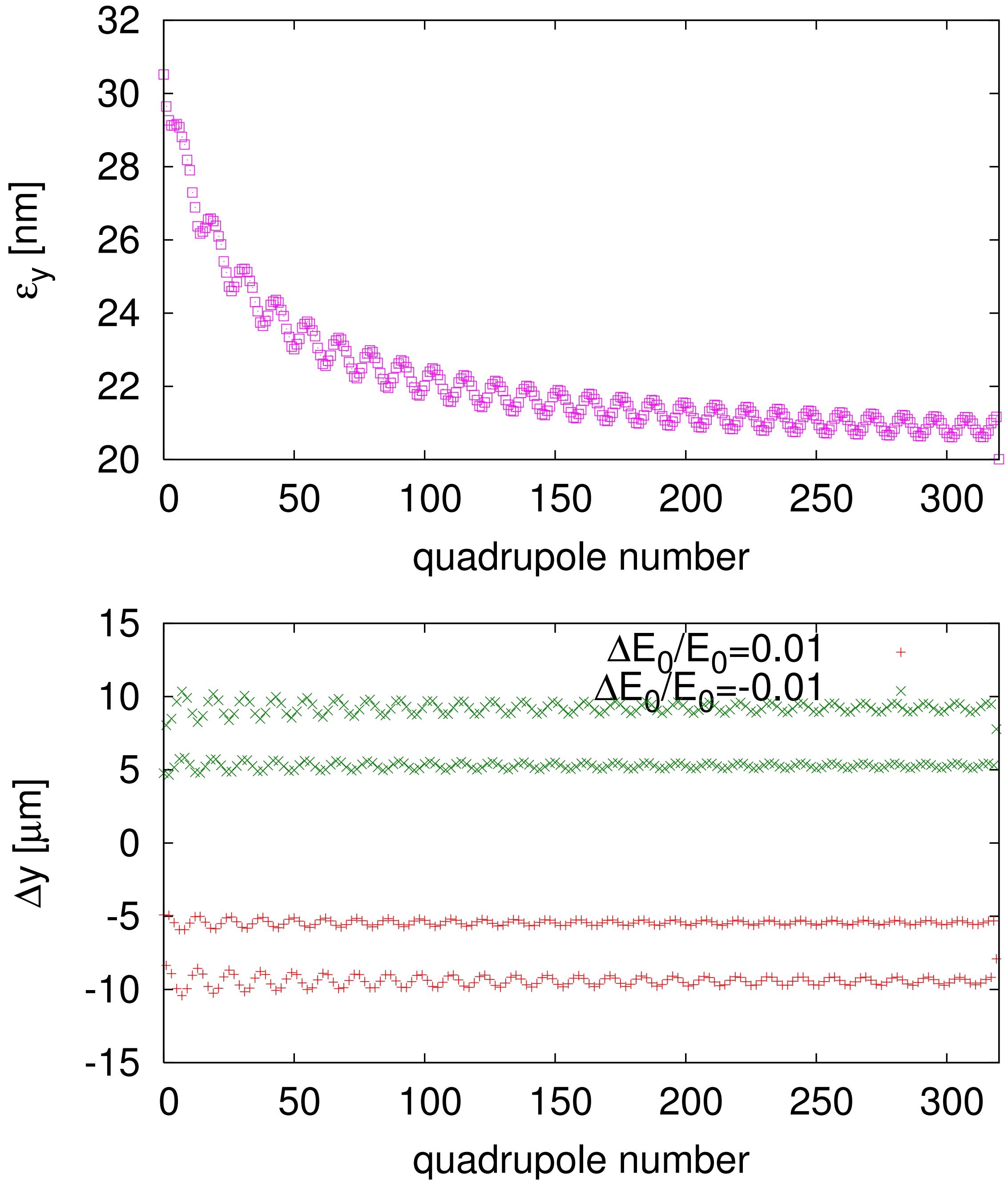
Dispersion in ILC

- Find a periodic solution for the dispersion

⇒ Projected emittance is varying but final value is good

  - good example of projected emittance

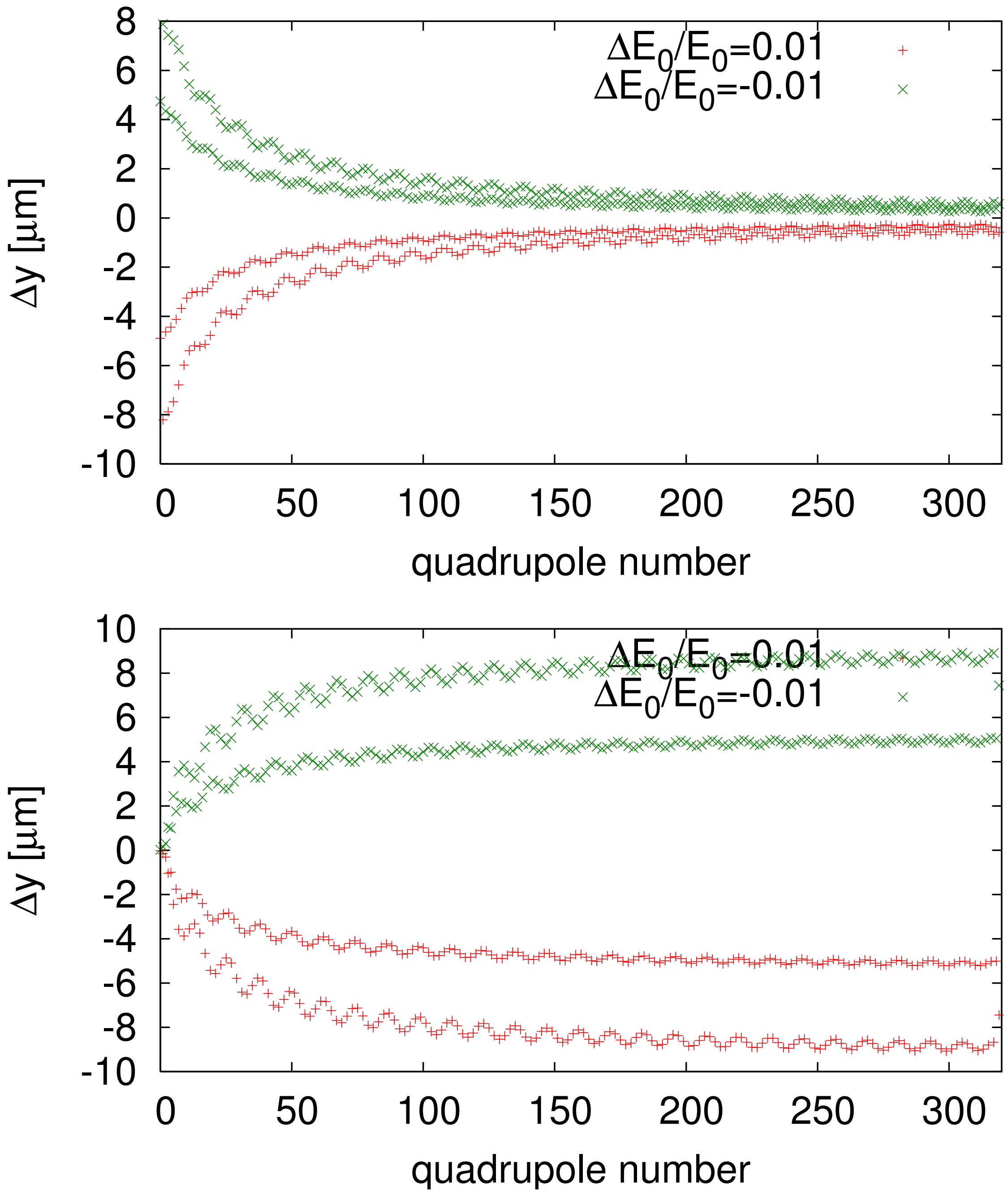
- Particles with constant 1% energy difference shown
- Dispersion is 100 times larger



Initial Energy vs. Gradient

- The incoming beam has an energy spread
- Different longitudinal slices of the beam are accelerated with different gradients

$\Rightarrow$ These path difference need not be the same



Some Comments

Generalised Transfer Matrices

- Mainly to introduce some concepts
- The beam transfer through one element can be described with a simple transfer matrix R

$$\vec{x} = R\vec{x}_0$$

- A number of independent particles (also at different energies) can be tracked by a new matrix R

$$\begin{pmatrix} \vec{x}_{f,1} \\ \vec{x}_{f,2} \\ \ldots \\ \vec{x}_{f,3} \end{pmatrix} = \begin{pmatrix} R_1 & 0 & \ldots & 0 \\ 0 & R_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & R_n \end{pmatrix} \begin{pmatrix} \vec{x}_{i,1} \\ \vec{x}_{i,2} \\ \ldots \\ \vec{x}_{i,n} \end{pmatrix}$$

- A wakefield kick from one particle to the next can be included

$$\begin{pmatrix} \vec{x}_{f,1} \\ \vec{x}_{f,2} \\ \ldots \\ \vec{x}_{f,3} \end{pmatrix} = \begin{pmatrix} R_1 & 0 & \ldots & 0 \\ R_{1,2} & R_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ R_{1,n} & R_{2,n} & \ldots & R_n \end{pmatrix} \begin{pmatrix} \vec{x}_{i,1} \\ \vec{x}_{i,2} \\ \ldots \\ \vec{x}_{i,n} \end{pmatrix}$$

Example

- In the centre of an accelerating structure, the wakefield kick can be calculated as

$$\begin{pmatrix} \vec{x}_{f,1} \\ \vec{x}_{f,2} \\ \ldots \\ \vec{x}_{f,3} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 & \ldots & 0 \\ \begin{pmatrix} 0 & 0 \\ a_1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \begin{pmatrix} 0 & 0 \\ a_{n-1} & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ a_{n-2} & 0 \end{pmatrix} & \ldots & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \vec{x}_{i,1} \\ \vec{x}_{i,2} \\ \ldots \\ \vec{x}_{i,n} \end{pmatrix}$$

- This works for long- and short-range wakefields
- In simulation codes this is evaluated efficiently using the fact that the matrix is sparse

Some Helpful Model

- The final beam can be described as a vector of slice positions and angles

$$\vec{b}_f = (x_0, \ldots x_{n-1}, x'_0, \ldots x'_{n-1})$$

  this is exactly what we found for a single particle

- The impact of each elements with an offset or angle can be described by a similar vector

$$\vec{b} = \sum_{i=1}^{n} \vec{b}_i \Delta y_i$$

  or

$$\vec{b} = B\vec{\delta}$$

Summary

- You should now have an idea of how to design a lattice that can transport the beam
- To this end we discussed
  - the matrix formalism for beam transport
  - Twiss parameters and normalised phase space
  - wakefield
- We also mentioned imperfections
  - more to come later