Multi-Bunch Dynamics

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Overview

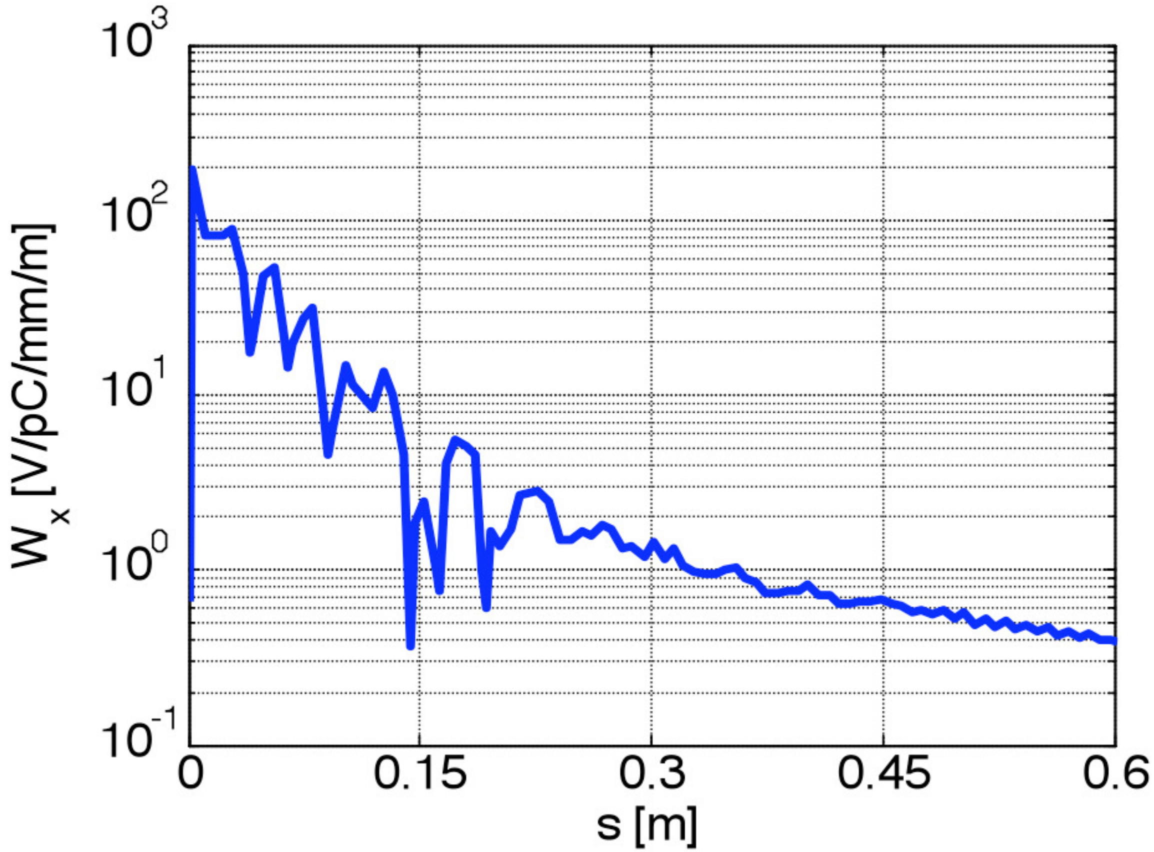
- Multi-bunch wakefields
- An simple semi-analytical treatment
- Some examples

Remember: Multi-Bunch Wakefields

- Long-range transverse wakefields have the form

$$W_\perp(z) = \sum_i^\infty 2k_i \sin\left(2\pi\frac{z}{\lambda_i}\right)\exp\left(-\frac{\pi z}{\lambda_i Q_i}\right)$$

- In practice need to consider only a limited number of modes
- There impact can be reduced by detuning and damping



Remember: Two-Particle Wakefield Model

- Assume bunch can be represented by two particles and constant $K(s) = 1/\beta^2$
  - second particle is kicked by transverse wakefield
  - initial oscillation

$$x_1'' + \frac{1}{\beta^2}x_1 = 0 \qquad x_2'' + \frac{1}{\beta^2}x_2 = \frac{Ne^2W_\perp}{P_Lc}x_1$$

$$x_1 = x_0 \cos\left(\frac{s}{\beta}\right)$$

$$x_2'' + \frac{1}{\beta^2}x_2 = x_0\frac{Ne^2W_\perp}{P_Lc}\cos\left(\frac{s}{\beta}\right)$$

- Solution is simple with an ansatz

$$x_2 = x_0 \cos\left(\frac{s}{\beta}\right) + \left(\frac{x_0Ne^2W_\perp\beta}{2E}s\right)\sin\left(\frac{s}{\beta}\right)$$

$\Rightarrow$ Amplitude of second particle oscillation is growing

$\Rightarrow$ The bunch charge and length matter as well as the lattice

$\Rightarrow$ Have a closer look into wakefields

Direct Bunch-to-Bunch Effect

- We assume that $W(s)$, $\beta(s)$ or $E(s)$ depend on s

  $\Rightarrow$ analytical full solution of differential equation is tough

  $\Rightarrow$ use a perturbative approach

- Assume that the effect per betatron oscillation is small
- Complex normalised oscillation amplitude with no wake

$$y_1(s) = y_{1,0} \exp\left(-i \int_0^s \frac{1}{\beta(s')} ds'\right)$$

at a following bunch

$$\begin{aligned} y_2(s) \approx & \left(y_{2,0} + \int_0^s i \frac{y_{1,0} W_\perp(z,s') N e^2 \beta(s')}{2E(s')} ds'\right) \\ & \exp\left(-i \int_0^s \frac{1}{\beta(s')} ds'\right) \end{aligned} \tag{1}$$

- In the following, we will ignore the phase factor and only consider the normalised complex oscillation amplitudes y_i

Matrix Formalism

- If more than two bunches we need to account for indirect effects
- Define a_{j-k} to be the direct change of the final amplitude $y_{j,f}$ of bunch j that is induced by the initial offset y_k of bunch k
- a_{j-k} is given by integrating to the end of the main linac $\hat{s}$

$$a_{j-k} = \int_0^{\hat{s}} \frac{W(z_j - z_k, s) N e^2 \beta(s)}{2E(s)} ds \qquad (2)$$

- For n bunches define matrix a

$$a_{jk} = i a_{j-k} \quad \text{for} \quad j > k$$

  and otherwise $a_{jk} = 0$

- This matrix describes the direct impact of the initial offset of each of the n bunches on the final offset of each other bunch

$$\vec{y}_f = (1 + a)\vec{y}_i$$

Full Bunch-to-Bunch Effect

- Define A the matrix that gives the impact of each bunch on each other including indirect effects
$$\vec{y}_f = A\vec{y}_i$$
- We can develop it iteratively
$$\vec{y}_f = \lim_{m\to\infty}\left(1+\frac{a}{m}\right)^m \vec{y}_i$$
hence
$$A = \lim_{m\to\infty}\left(1+\frac{a}{m}\right)^m$$
$$\Rightarrow A = \exp(a)$$
note that one can develop
$$A = \exp(a) = \sum_{k=0}^{\infty}\frac{a^k}{k!} = \sum_{k=0}^{n-1}\frac{a^k}{k!}$$
Here, we use $a^n = 0$ since $a_{jk} = 0$ for $j \leq k$.

Example for Geometric Wakefields

- Use CLIC long-range wakefield model from optimisation
  - field at second bunch $6.6\,\mathrm{kV/pCm^2}$
  - afterwards field is zero
- Calculate factors a_k

$$a_k = i\sum_j \frac{L_j\beta_j}{2E_j} W(z_k)Ne^2 \approx 380\,\mathrm{m^2GeV^{-1}}W(z_k)Ne^2$$

yields

$$a_1 \approx 1.5 \qquad a_{k\neq 1} = 0$$

for $j \geq k$ one finds

$$A_{jk} = \frac{(ia_1)^{(j-k)}}{(j-k)!} \tag{3}$$

Some Detail

- Example with four bunches
  - direct effects matrix

$$a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ ia_1 & 0 & 0 & 0 \\ ia_2 & ia_1 & 0 & 0 \\ ia_3 & ia_2 & ia_1 & 0 \end{pmatrix}$$

  - development for indirect effects (only kick on second bunch)

$$1 + \frac{a}{m} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ ia_1/m & 1 & 0 & 0 \\ 0 & ia_1/m & 1 & 0 \\ 0 & 0 & ia_1/m & 1 \end{pmatrix}$$

  - development for Taylor series

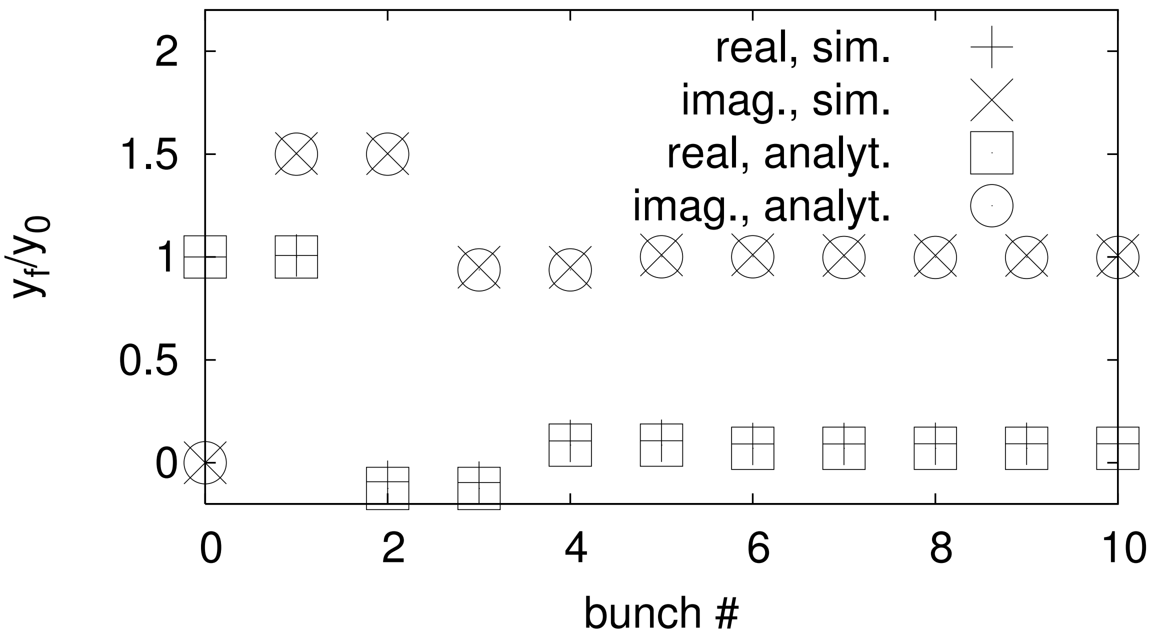
$$a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ ia_1 & 0 & 0 & 0 \\ 0 & ia_1 & 0 & 0 \\ 0 & 0 & ia_1 & 0 \end{pmatrix} \qquad a^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ -a_1^2 & 0 & 0 & 0 \\ 0 & -a_1^2 & 0 & 0 \end{pmatrix} \qquad a^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ -ia_1^3 & 0 & 0 & 0 \end{pmatrix}$$

Comparison to Simulation

- We calculate and simulate the offset of a train with coherent initial offset at the end of the linac

⇒ The agreement between simulation and calculation is good (no surprise)

⇒ The first few bunches are scattered but on the flat top we observe a phase shift



Understanding the Result

- We had calculated the coefficients of the matrix A

$$A_{jk} = \frac{(ia_1)^{(j-k)}}{(j-k)!} \tag{4}$$

for a coherent offset the amplitude of bunch j is given by

$$\sum_{k=0}^{j} A_{jk} = \sum_{k=0}^{j} \frac{(ia_1)^{(j-k)}}{(j-k)!}$$

$$\Rightarrow \sum_{k=0}^{j} A_{jk} = \sum_{k=0}^{j} \frac{(ia_1)^{(k)}}{(k)!}$$

this is obviously the definition of the exp $\exp(ia_1)$, with good convergence already after only a few bunches

Impact of Beam Jitter

- Coherent jitter of the whole train is important
  - RMS final offsets of the bunches is

$$F_c = \frac{\vec{y}_f \vec{y}_f^*}{\vec{y}_i \vec{y}_i^*}$$

$$\Rightarrow F_c = \frac{(A\vec{y}_0)(A\vec{y}_0)^*}{\vec{y}_i \vec{y}_i^*}$$

$$\Rightarrow F_c = \frac{1}{n} \sum_k \left| \sum_j A_{kj} \right|^2$$

  In our case $F_c \approx 1$

- Random bunch-to-bunch jitter

$$F_{rms} = \frac{\Sigma_{k=0}^{n-1} \Sigma_{j=1}^{k} A_{j,k} A_{j,k}^*}{n}$$

  In our case $F_{rms} \approx \left(1 + \Sigma_{i=0}^{n-1} A_{i,0}^2\right) \approx 4.9$

  $\Rightarrow$ at the limit of what is acceptable

Some Remarks

- The impact of two wakefields is given by

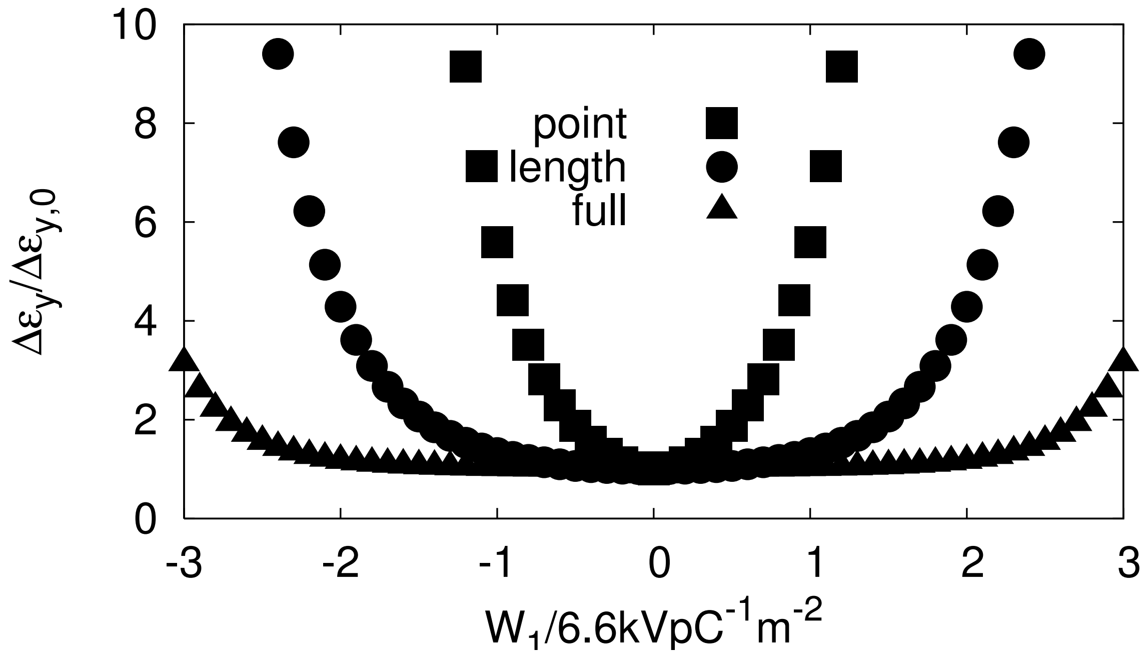
$$A = A_1 A_2$$

$\Rightarrow$ The order of the kicks does not matter

- It is usually sufficient to know the first column, the other follow

Bunches with Energy Spread

- If bunches are not point-like the results change
  - an energy spread leads to a more stable case
- Simulations show
  - point-like bunches
  - bunches with energy spread due to bunch length
  - including also initial energy spread



$\Rightarrow$ Point-like bunches is a pessimistic assumption for the dynamic effects

Multi-Bunch Effects in ILC

Average beta-function is

$$\bar{\beta} = \left(\frac{2}{3} + \frac{L}{6f}\right)\check{\beta} + \frac{1}{3}\hat{\beta}$$

for ILC we find $\bar{\beta} \approx 66\,\mathrm{m}$
Effective gradient is $G \approx 30\,\mathrm{MV/m}$

$$\int_{E_0}^{E_f} \frac{\bar{\beta}}{2E}\frac{dE}{eG} = \frac{\bar{\beta}}{2eG}\left(\ln(E_f) - \ln(E_0)\right)$$

using $E_0 = 15\,\mathrm{GeV}$ and $E_f = 250\,\mathrm{GeV}$

$$a = 3660\,\mathrm{m}^2/\mathrm{GeV}$$

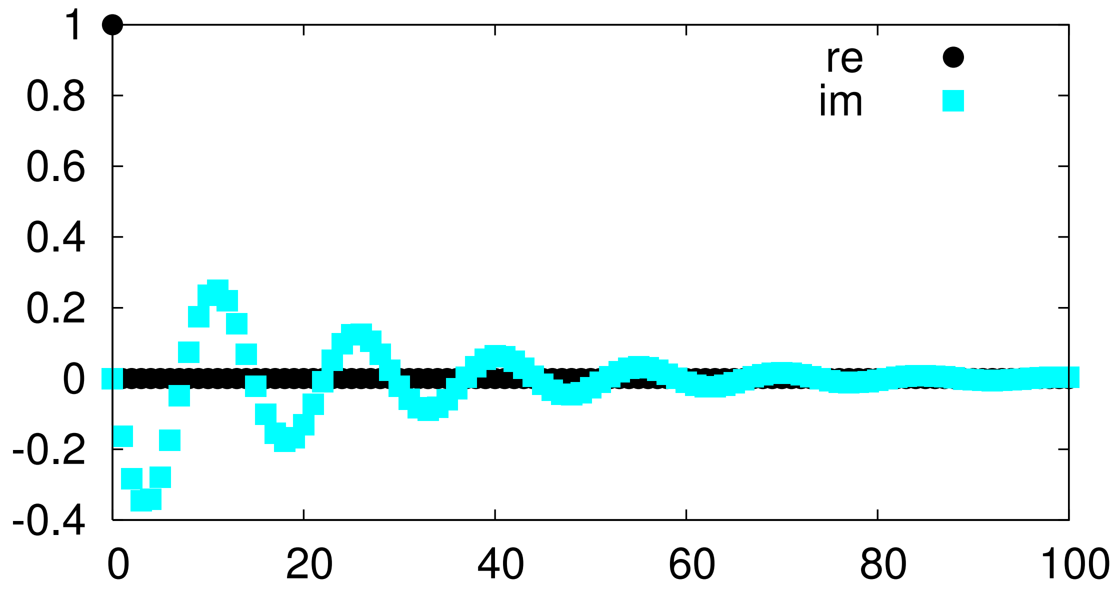
Long-Range Wakefields (ILC)

- Analytic calculations show that some modes can be dangerous

  $\Rightarrow$ would like to mitigate the effect

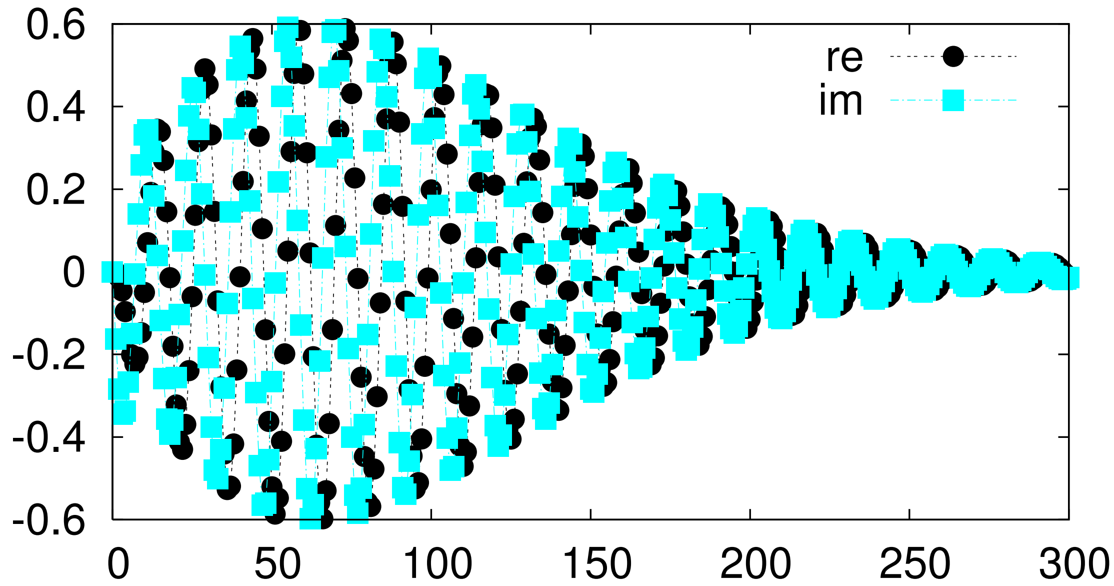
f [GHz]	$2k\,\mathrm{V/pCm^2}$	Q	F_{rms}
1.6506	19.98	7e4	0.0008
1.6991	301.86	5e4	0.29
1.7252	423.41	2e4	0.127
1.7545	59.86	2e4	0.002
1.7831	49.2	7.5e3	0.0004
1.7949	21.70	1e4	0.0001
1.8342	13.28	5e4	0.0002
1.8509	11.26	2.5e4	—
1.8643	191.56	5e4	0.06
1.8731	255.71	7e4	0.3
1.8795	50.8	1e5	0.007
2.5630	42.41	1e5	0.003
2.5704	20.05	1e5	0.0007
2.5751	961.28	5e4	15.7

Example: Bad Mode



- One of the modes is particularly bad
  - upper plot shows only direct effect $(1 + a)$
  - lower plot shows full effect (A)

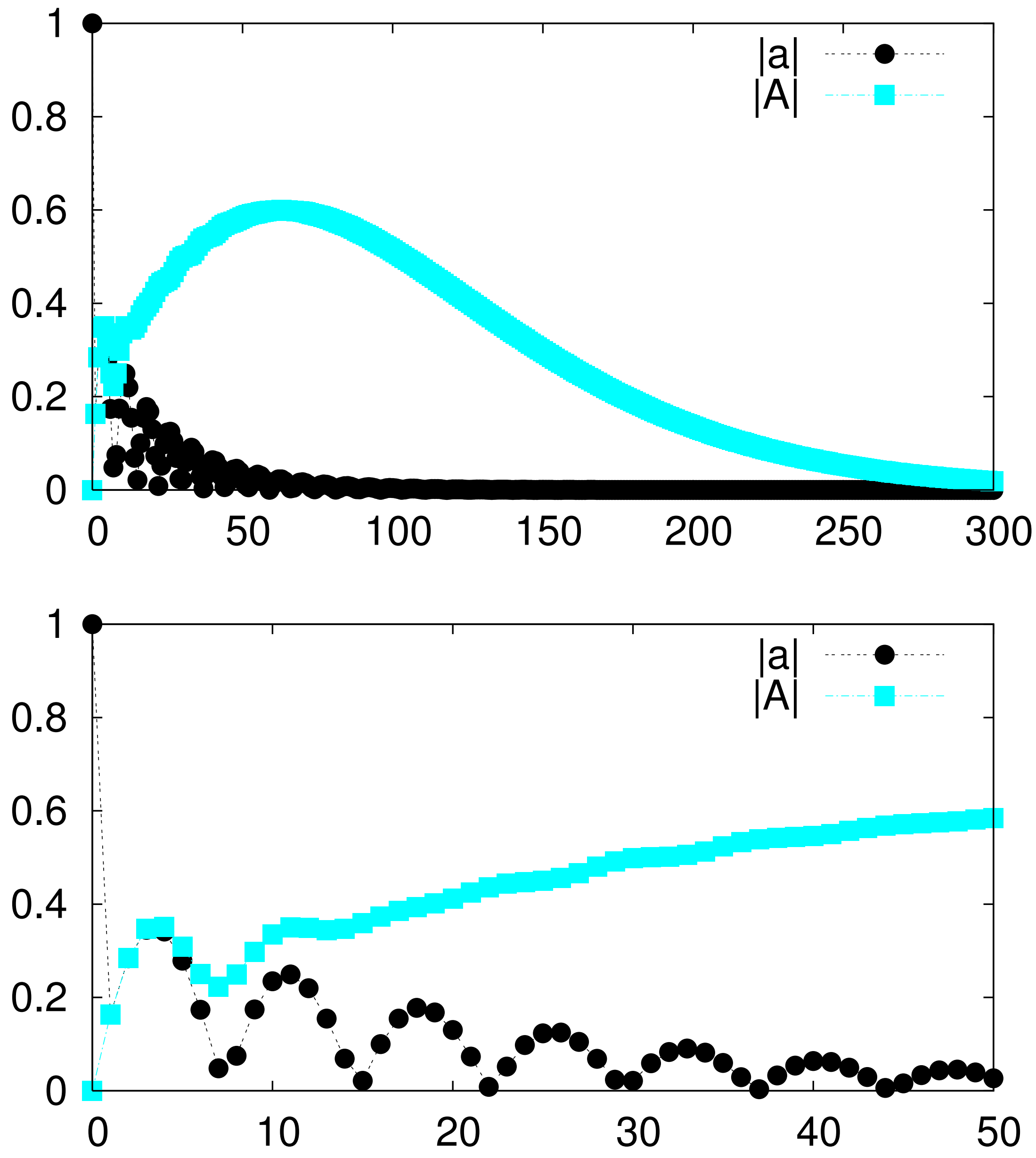
$\Rightarrow$ The indirect effect is very important



Example: Bad Mode

- Amplitude of the oscillation is shown
  - upper plot shows only direct effect $(1+a)$
  - lower plot shows full effect (A)
- When the amplitude of the direct effect decays the indirect effect still grows

$\Rightarrow$ indirect effects are indeed important



Detuning

To make our life simple we neglect damping

We split the wakefield $W(z) = a\sin(kz)$ into two modes

$$W(z) = W_0 \frac{\sin((k+\Delta)z) + \sin((k-\Delta)z)}{2}$$

the resulting amplitude is

$$W(z) = W_0 \sin(kz)\cos(\Delta z)$$

integrating over a Gaussian distribution yields

$$W(z) = W_0 \sin(kz) \int_0^\infty \frac{2}{\sqrt{2\pi}\sigma_\Delta} \exp\left(-\frac{\Delta^2}{2\sigma_\Delta^2}\right) \cos(\Delta z) d\Delta$$

$$\Rightarrow W(z) = W_0 \sin(kz) \exp\left(-\frac{(z\Delta)^2}{2}\right)$$

$\Rightarrow$ The bad mode is reduced to $F_{rms} = 0.14$ already for a detuning of 10^{-4}

Details of the Calculation

Use

$$\sin((k+\Delta)z) = \sin(kz)\cos(\Delta z) + \cos(kz)\sin(\Delta z)$$

and

$$\sin((k-\Delta)z) = \sin(kz)\cos(\Delta z) - \cos(kz)\sin(\Delta z)$$

$$\Rightarrow \sin((k+\Delta)z)+\sin((k-\Delta)z) = \sin(kz)\cos(\Delta z)+\cos(kz)\sin(\Delta z)+\sin(kz)\cos(\Delta z)-\cos(kz)\sin(\Delta z)$$

$$\Rightarrow \sin((k+\Delta)z) + \sin((k-\Delta)z) = 2\sin(kz)\cos(\Delta z)$$

Also

$$\int_0^\infty \exp(-a^2x^2)\cos(bx) = \frac{\sqrt{\pi}}{2a}\exp\left(-\frac{b^2}{4a^2}\right)$$

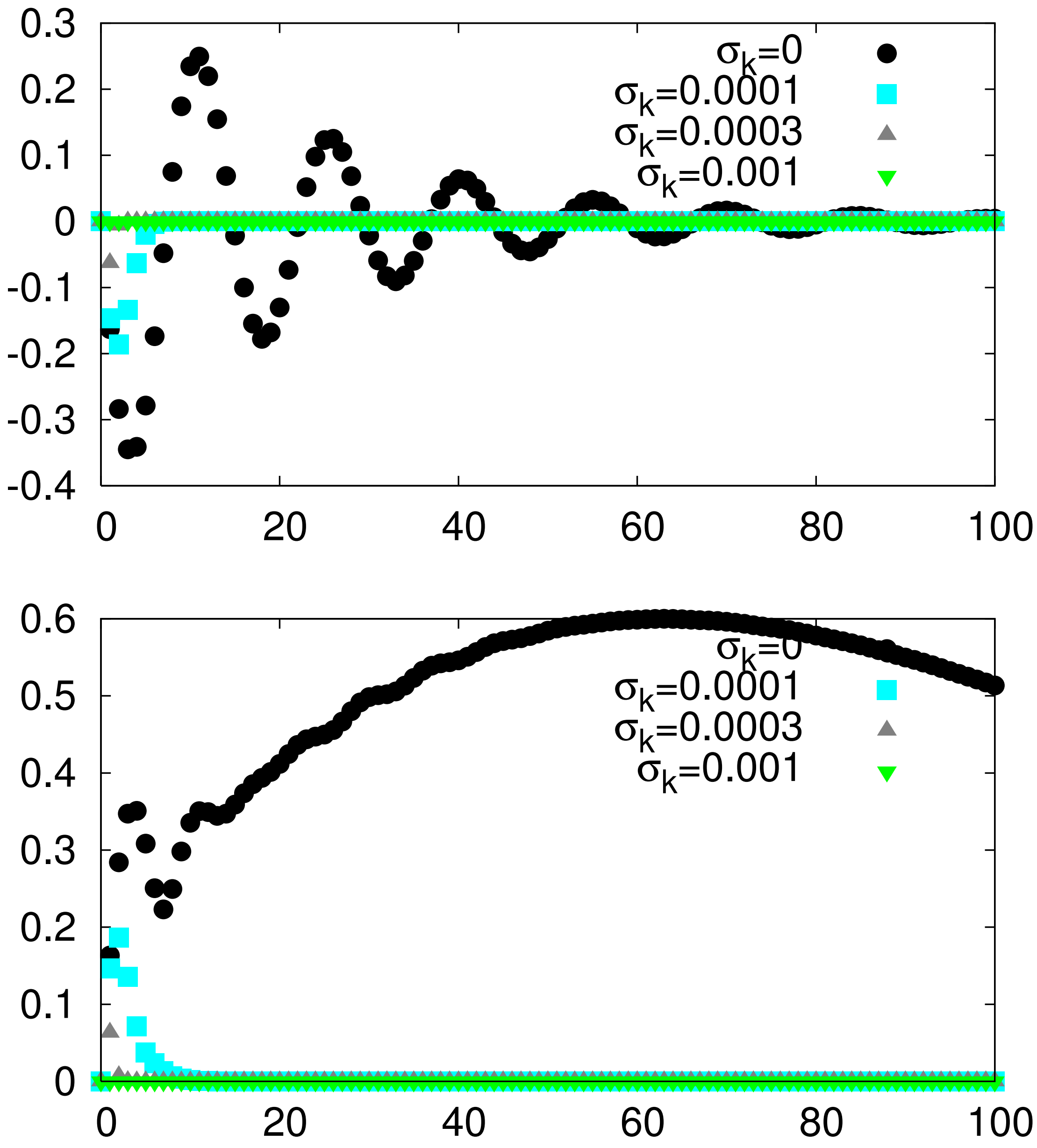
Example: Bad Mode

- Due to the large bunch distance even little detuning is efficient

  $\Rightarrow$ the mode completely disappears for the assumed detuning of 0.1%

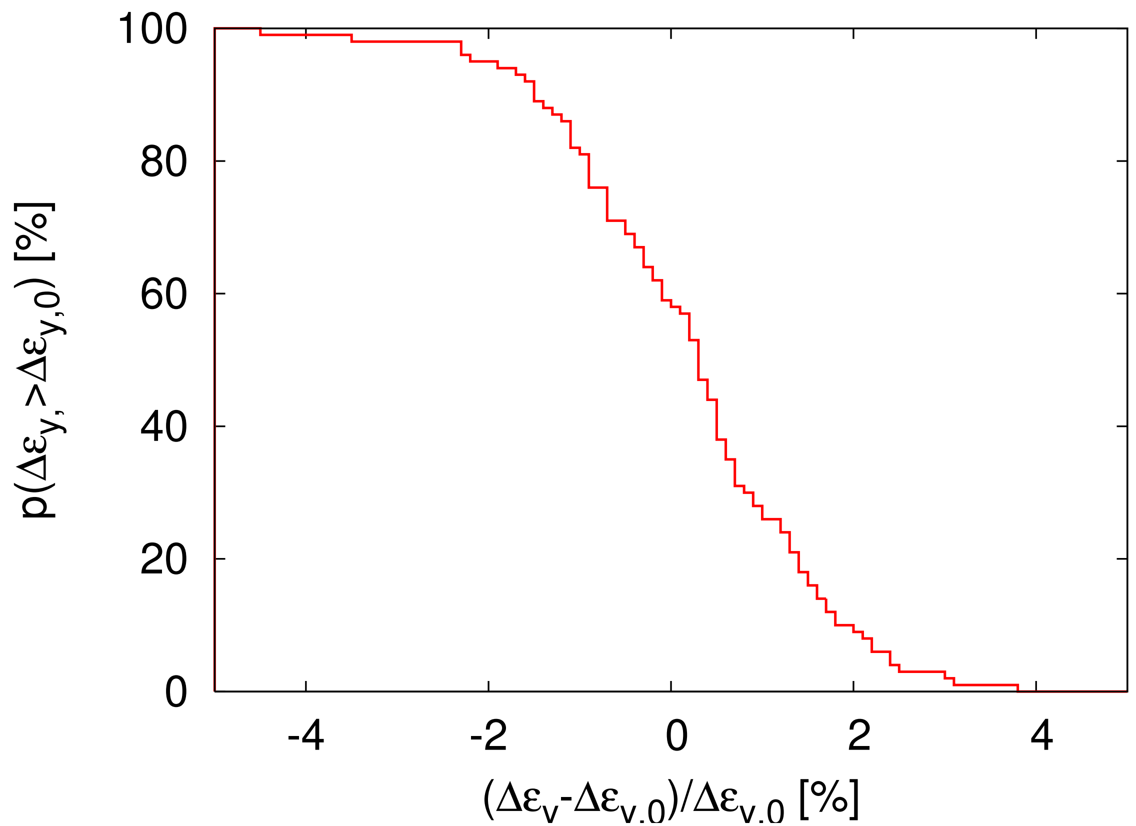
- But the detuning is random from cavity to cavity

  $\Rightarrow$ effective detuning will be less

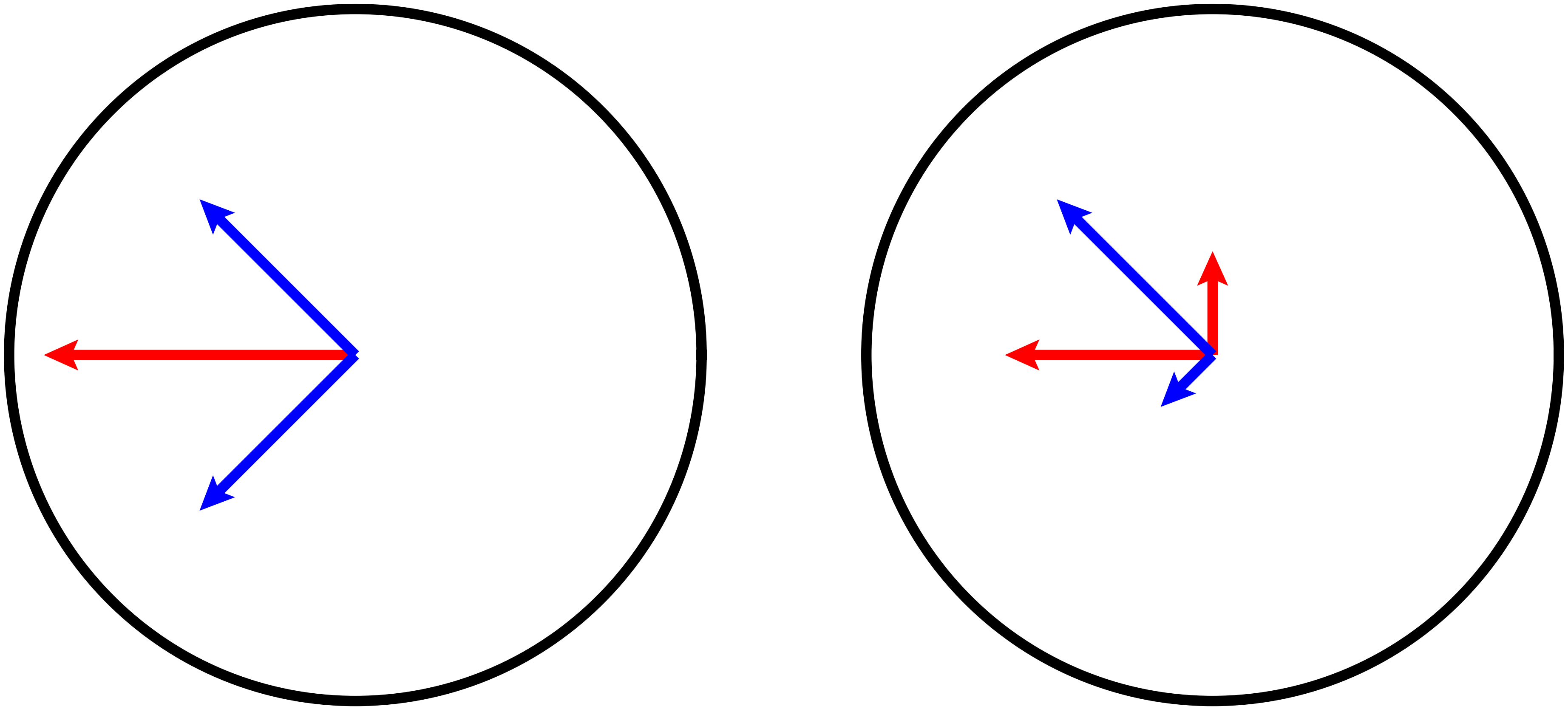


Beam Jitter

- Perfect machines used
- 100 machines simulated
  - TESLA wakefields with 0.1% RMS frequency spread
  - beam set to an offset
  - 5% bunch-to-bunch charge variations in uncorrected test beam
  - additional relative emittance growth due to multi-bunch is determined



Uneven Damping



- We had realised that ILC uses different phase advance in the two planes
- One of the reasons are long-range wakefields
- The beam jitters much more in the horizontal plane(due to the larger emittance)
- If the damping is not in the horizontal and vertical plane and is uneven this may lead to vertical wakefield kicks due to the horizontal oscillation

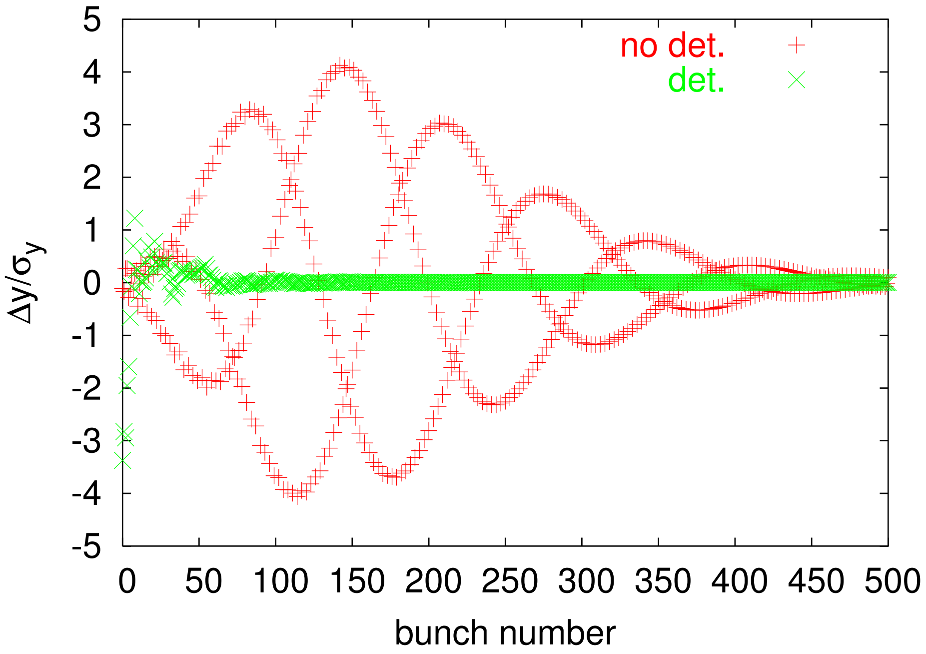
Reminder: Static Multi-Bunch Effects (ILC)

- Simulation of long-range transverse wakefield effects
  - with no detuning
  - with random detuning from cavity to cavity

⇒ Cavity detuning is essential

⇒ Need to ensure that this detuning is present
  - it does happen naturally
  - but also if you depend on it?

- Note results depend on exact frequency of transverse modes
  - some uncertainty in the prediction
  - but not a worry with detuning



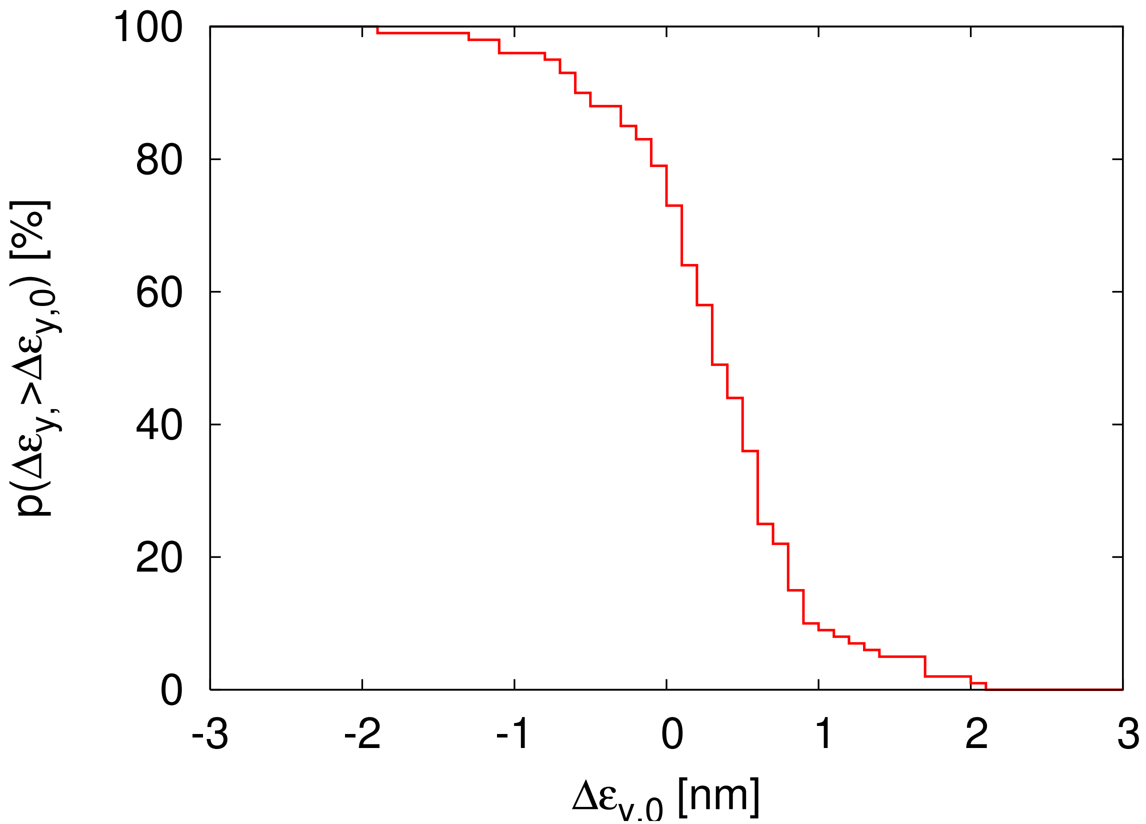
All main linac cavities are scattered by 500 $\mu\mathrm{m}$

Long-range wakefields are represented by a number of RF modes

$$W_\perp(z) = \sum_{i=0}^{n} a_i \sin\left(\frac{2\pi z}{\lambda_i}\right) \exp\left(-\frac{\pi z}{\lambda_i Q_i}\right)$$

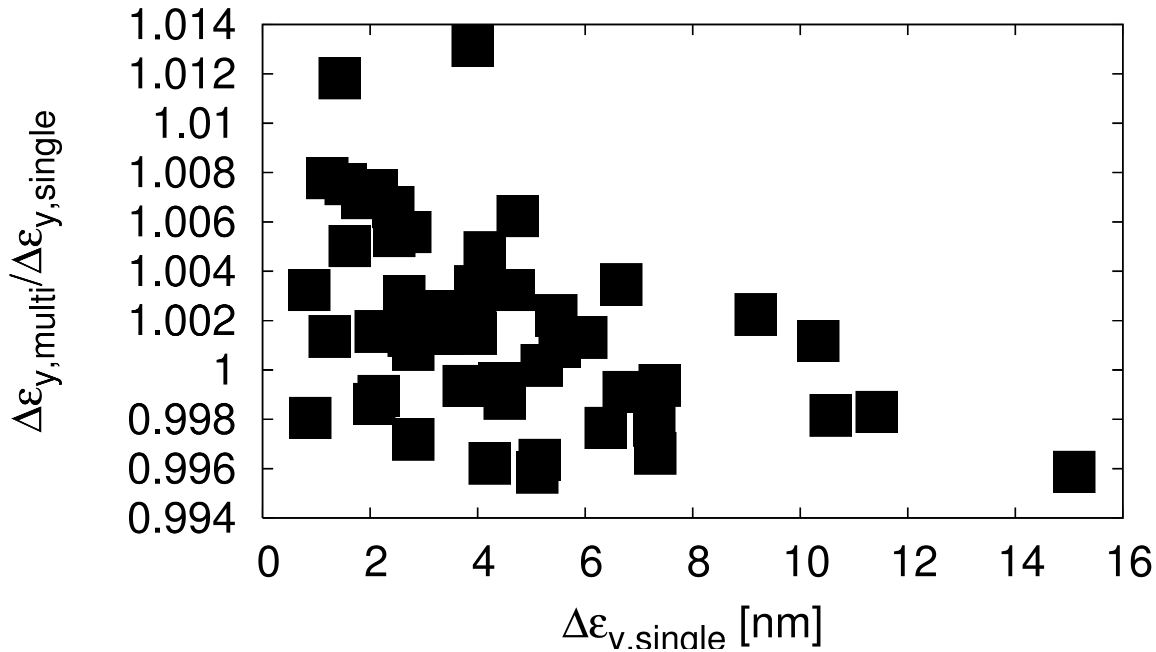
Multi-Bunch Emittance Growth (ILC)

- Standard errors used
- DFS applied
- 100 machines simulated
  - TESLA wakefields with 0.1% RMS frequency spread
  - one-to-one alignment with full train for each machine
  - 5% bunch-to-bunch charge variations in uncorrected test beam



Simple Estimate of Static Imperfections (CLIC)

- In CLIC
  - we misalign all structures
  - perform one-to-one steering with a multi-bunch beam
  - perform one-to-one steering with a single bunch
  - compare the emittance growth



Resistive Wall Wakefield (CLIC)

- The wakefield is given by
$$W(z) = \frac{cZ_0}{\pi b^3}\sqrt{\frac{1}{Z_0 \sigma_r \pi z}} \tag{5}$$
with the impedance of the vacuum Z_0, beam pipe radius b and the conductivity of copper $\sigma_r = 5.8 \cdot 10^7\,\Omega^{-1}\mathrm{m}^{-1}$
- We approximate the structures by averaging the iris radius and length
- Using the matrix formalism we find for initial coherent offset
$$y_{N,f} \approx (1 + 0.02i) y_{all}$$
- We require the effect of the beam pipe in the quadrupoles, drifts and flanges to be smaller
this yields
$$b \geq 3.6\,\mathrm{mm}$$
we choose $b = 4\,\mathrm{mm}$
- In total we then yield
$$y_{N,f} \approx (1 + 0.035i) y_{all}$$

Element Misalignments

Consider only the quadrupole beam pipe misalignment, other contributions should be small.
The calculation is similar to the one for beam jitter, except that the kicks of the elements add in quadrature.
The RMS bunch position scatter F_s at the end of the linac normalised to the beam size can be approximated as

$$F_s = \sum_{j=1}^{n} \frac{1}{n} \sum_i \frac{L_i^2 \beta_i \Delta_i^2}{2E_i} \frac{1}{mc^2\epsilon} \left((W_{j,sum} - \langle W_{sum} \rangle) N e^2 \right)^2 \tag{6}$$

Here, $W_{j,sum}$ is the wakefield at bunch j produced by a coherent offset of all leading bunches and $\langle W_{sum} \rangle$ is the average wakefield. For $\Delta = 100\,\mu\mathrm{m}$ this yields $F_{stat} \approx 0.012$. Simulations with point-like bunches yield the same value. If the geometric wakefields are added the average value remains the same.

Summary

- Multi-Bunch wakefields are important in ILC and CLIC
- Most important source are structre wakefields
- Mitigation using
  - damping
  - detuning
- Simplified calculations can be used to estimate the effects