Linear beam dynamics overview

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Outline – Transverse transfer matrices

- Hill’s equations
  - Derivation
  - Harmonic oscillator
- Transport Matrices
  - Matrix formalism
  - Drift
  - Thin lens
  - Quadrupoles
  - Dipoles
    - Sector magnets
    - Rectangular magnets
  - Doublet
  - FODO

Equations of motion – Linear fields

- Consider *s*-dependent fields from dipoles and normal quadrupoles $B_y = B_0(s) - g(s)x \ , \ \ B_x = -g(s)y$
- The total momentum can be written $P = P_0(1 + \frac{\Delta P}{P})$
- With magnetic rigidity $B_0\rho = \frac{P_0}{q}$ and normalized gradient

$k(s) = \frac{g(s)}{B_0\rho}$ the equations of motion are

$$x'' - \left(k(s) - \frac{1}{\rho(s)^2}\right) x = \frac{1}{\rho(s)} \frac{\Delta P}{P}$$
$$y'' + k(s)\, y = 0$$

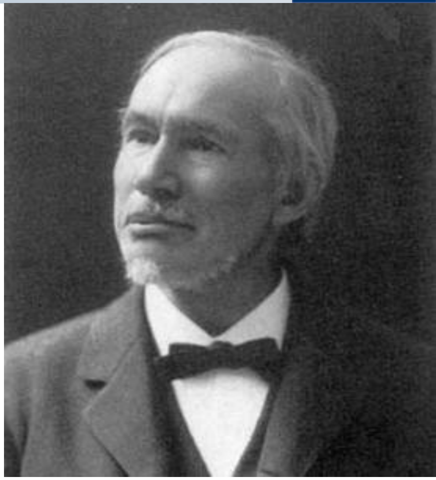
- Inhomogeneous equations with *s*-dependent coefficients
- Note that the term $1/\rho^2$ corresponds to the dipole **week focusing**
- The term $\Delta P/(P\rho)$ represents **off-momentum** particles

Hill's equations

- Solutions are combination of the ones from the homogeneous and inhomogeneous equations
- Consider particles with the design momentum. The equations of motion become

$$\begin{aligned} x'' + K_x(s)\ x &= 0 \\ y'' + K_y(s)\ y &= 0 \end{aligned}$$

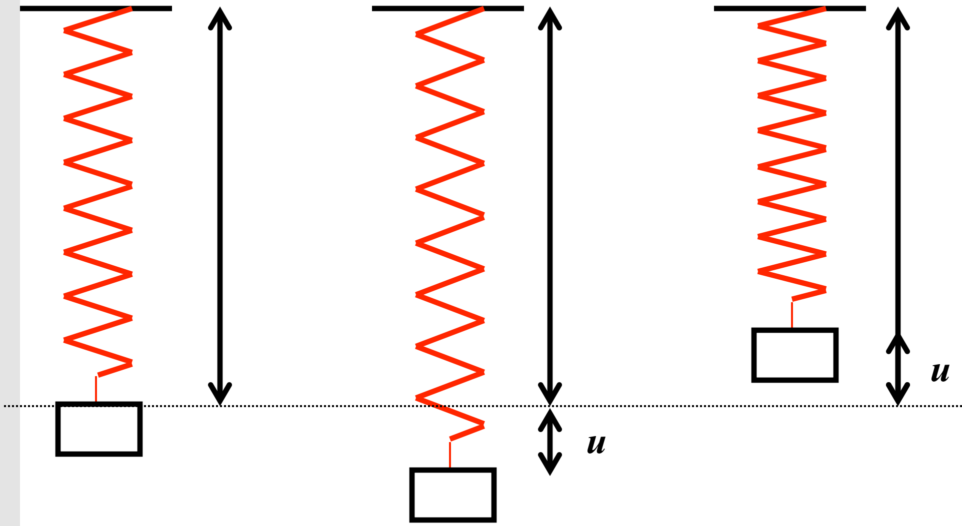
with $K_x(s) = -\left(k(s) - \frac{1}{\rho(s)^2}\right) \ , \quad K_y(s) = k(s)$



George Hill

- **Hill's equations of linear transverse particle motion**
- Linear equations with *s*-dependent coefficients (harmonic oscillator with time dependent frequency)
- In a ring (or in transport line with symmetries), coefficients are periodic $K_x(s) = K_x(s+C) \ , \quad K_y(s) = K_y(s+C)$
- Not straightforward to derive analytical solutions for whole accelerator

Harmonic oscillator – spring



- Consider $K(s) = k_0 = \text{constant}$

$$u'' + k_0\, u = 0$$

- Equations of harmonic oscillator with solution

$$\begin{aligned} u(s) &= C(s)\, u(0) + S(s)\, u'(0) \\ u'(s) &= C'(s) u(0) + S'(s) u'(0) \end{aligned}$$

with

$$C(s) = \cos(\sqrt{k_0}\, s)\ , \quad S(s) = \frac{1}{\sqrt{k_0}} \sin(\sqrt{k_0}\, s) \qquad \text{for } k_0 > 0$$

$$C(s) = \cosh(\sqrt{|k_0|}\, s)\ , \quad S(s) = \frac{1}{\sqrt{|k_0|}} \sinh(\sqrt{|k_0|}\, s) \qquad \text{for } k_0 < 0$$

- Note that the solution can be written in **matrix** form

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$$

Matrix formalism

- General **transfer matrix** from s_0 to s

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0) \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$

- Note that $\det(\mathcal{M}(s|s_0)) = C(s|s_0)S'(s|s_0) - S(s|s_0)C'(s|s_0) = 1$ which is always true for conservative systems
- Note also that $\mathcal{M}(s_0|s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}$
- The accelerator can be build by a series of matrix multiplications

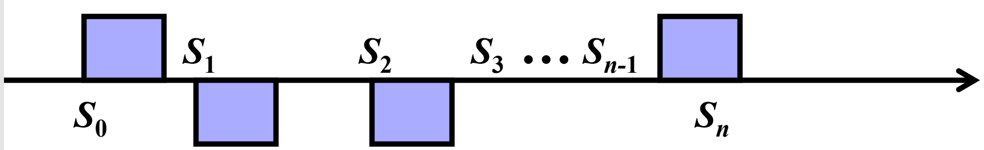
$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1}) \ldots \mathcal{M}(s_3|s_2) \cdot \mathcal{M}(s_2|s_1) \cdot \mathcal{M}(s_1|s_0)$$

from s_0 to s_1

from s_0 to s_2

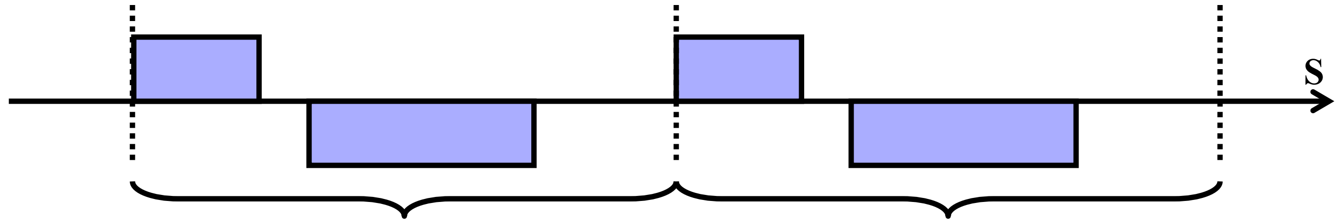
from s_0 to s_3

from s_0 to s_n



Symmetric lines

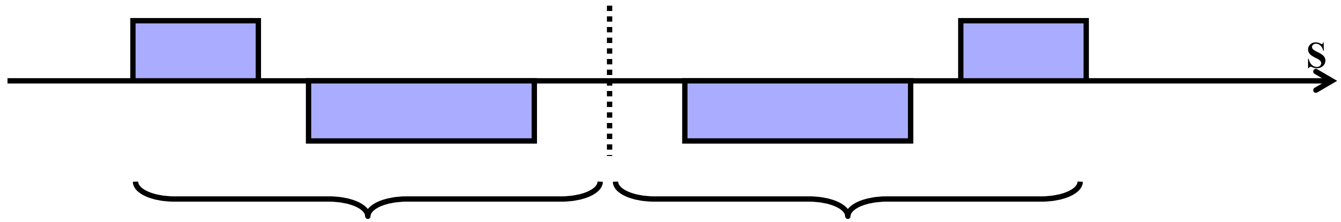
- System with normal symmetry



$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathcal{M}_{\text{tot}} = \mathcal{M} \cdot \mathcal{M} = \mathcal{M}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

- System with mirror symmetry



$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathcal{M}_r = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_r \cdot \mathcal{M} = \begin{pmatrix} ad + bc & 2bd \\ 2ac & ad + cb \end{pmatrix}$$

4x4 Matrices

- Combine the matrices for each plane

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) \\ C'_x(s) & S'_x(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_y(s) & S_y(s) \\ C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

to get a total 4x4 matrix

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) & 0 & 0 \\ C'_x(s) & S'_x(s) & 0 & 0 \\ 0 & 0 & C_y(s) & S_y(s) \\ 0 & 0 & C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

Uncoupled motion

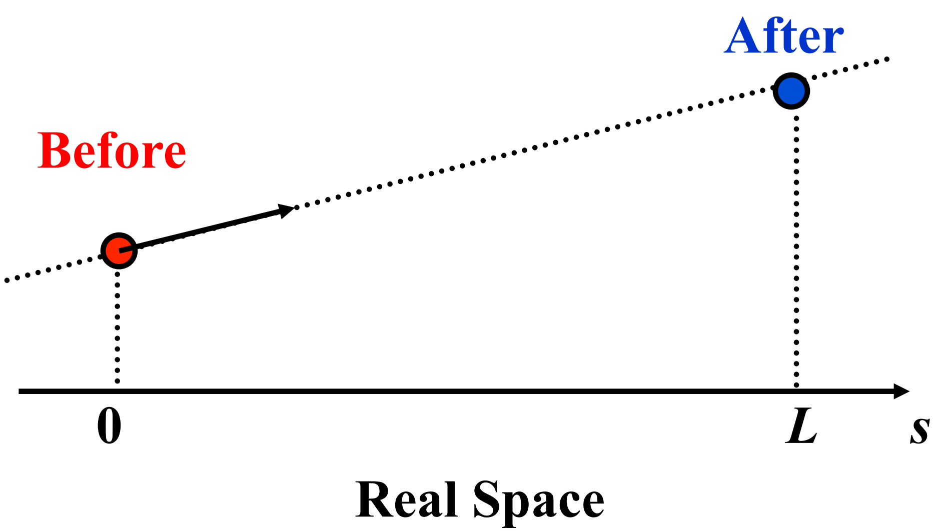
- Consider a drift (no magnetic elements) of length L=s-s_0

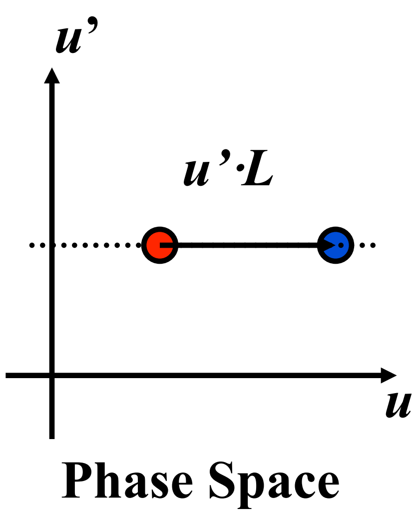
$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & s-s_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

$$\mathcal{M}_{\mathrm{drift}}(s|s_0) = \begin{pmatrix} 1 & s-s_0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} u(s) &= u_0 + \overbrace{(s-s_0)}^{L} u'_0 = u_0 + L u'_0 \\ u'(s) &= u'_0 \end{aligned}$$

- Position changes if particle has a slope which remains unchanged.





(De)focusing thin lens

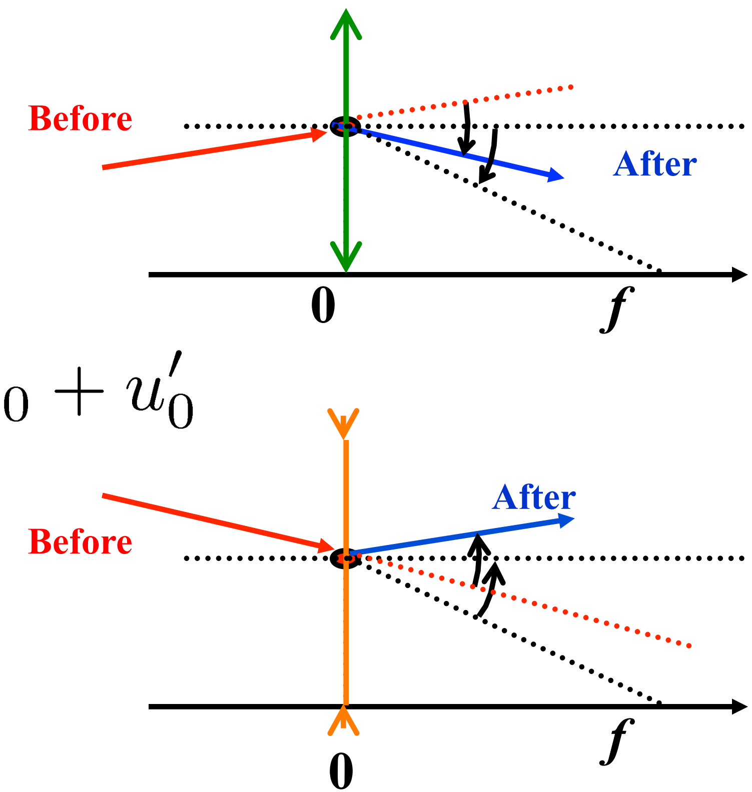
- Consider a lens with focal length $\pm f$

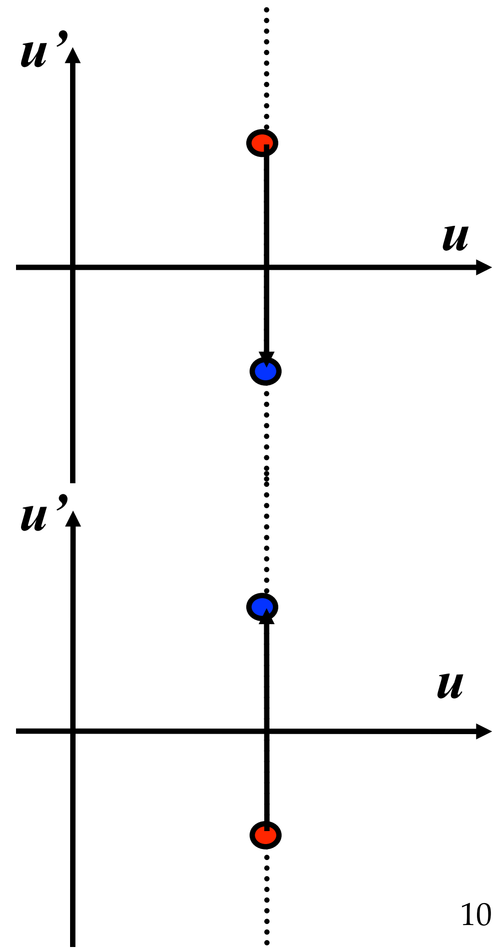
$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

$$\mathcal{M}_{\mathrm{lens}}(s|s_0) = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix}$$

- Slope **diminishes** (focusing) or **increases** (defocusing) for positive position, which remains unchanged.

$$\begin{aligned} u(s) &= u_0 \\ u'(s) &= \mp\frac{1}{f}u_0 + u'_0 \end{aligned}$$

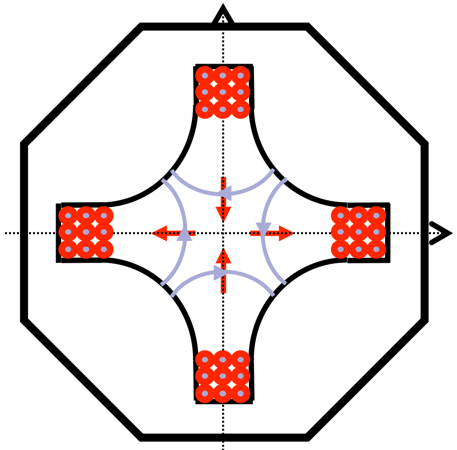




Quadrupole

- Consider a quadrupole magnet of length $L = s\text{-}s_0$. The field is

$$B_y = -g(s)x \ , \quad B_x = -g(s)y$$

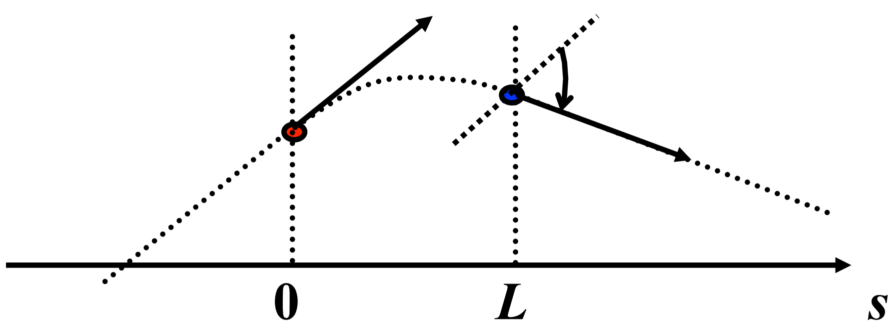


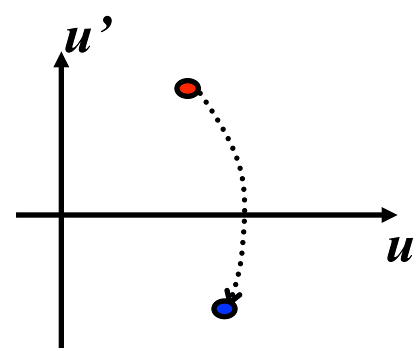
- with normalized quadrupole gradient (in $\mathbf{m}^{-2}$)

$$k(s) = \frac{g(s)}{B_0\rho}$$

The transport through a quadrupole is

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k}(s-s_0)) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}(s-s_0)) \\ \sqrt{k}\sin(\sqrt{k}(s-s_0)) & \cos(\sqrt{k}(s-s_0)) \end{pmatrix} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}$$





(De)focusing Quadrupoles

- For a focusing quadrupole (k>0)

$$\mathcal{M}_{\rm QF} = \begin{pmatrix} \cos(\sqrt{k}L) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}L) \\ -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) \end{pmatrix}$$

- For a defocusing quadrupole (k<0)

$$\mathcal{M}_{\rm QD} = \begin{pmatrix} \cosh(\sqrt{|k|}L) & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}L) \\ \sqrt{|k|}\sinh(\sqrt{|k|}L) & \cosh(\sqrt{|k|}L) \end{pmatrix}$$

- By setting $\sqrt{k}L \to 0$

$$\mathcal{M}_{\rm QF,QD} = \begin{pmatrix} 1 & 0 \\ -kL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \mathcal{M}_{\rm lens}$$

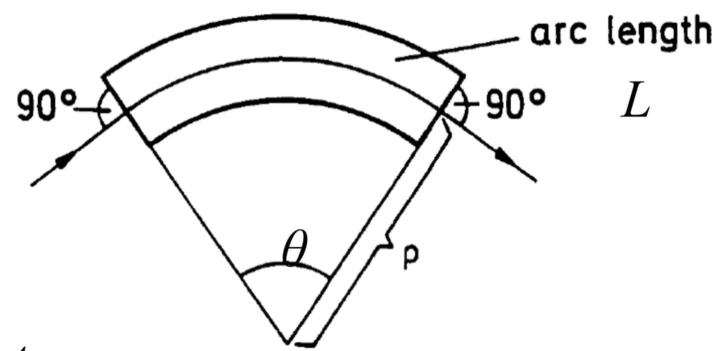
- **Note** that the **sign** of k or f is now absorbed inside the symbol
- In the other plane, focusing becomes defocusing and vice versa

Sector Dipole

- Consider a dipole of (arc) length L.
- By setting in the focusing quadrupole matrix $k = \frac{1}{\rho^2} > 0$ the transfer matrix for a sector dipole becomes

$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}$$

with a bending radius $\theta = \frac{L}{\rho}$

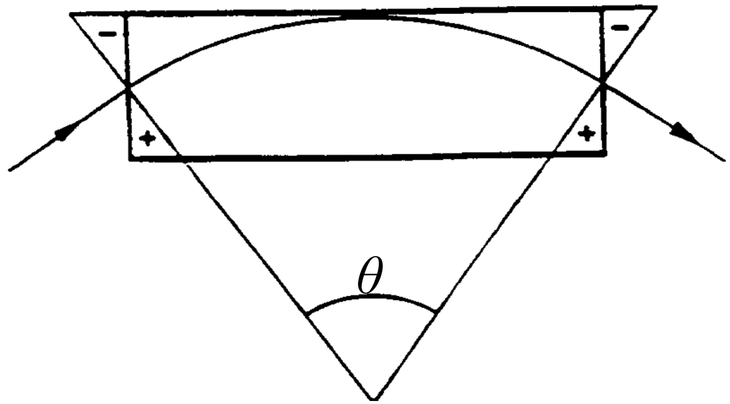


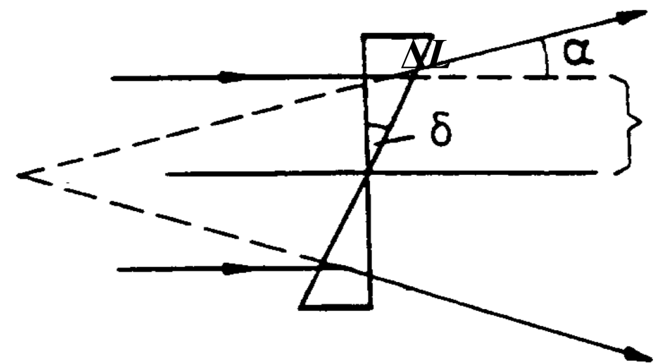
- In the non-deflecting plane $\frac{1}{\rho} \to 0$

$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \mathcal{M}_{\text{drift}}$$

- This is a **hard-edge** model. In fact, there is some **edge focusing** in the vertical plane
- Matrix generalized by adding gradient (**synchrotron magnet**)

Rectangular Dipole

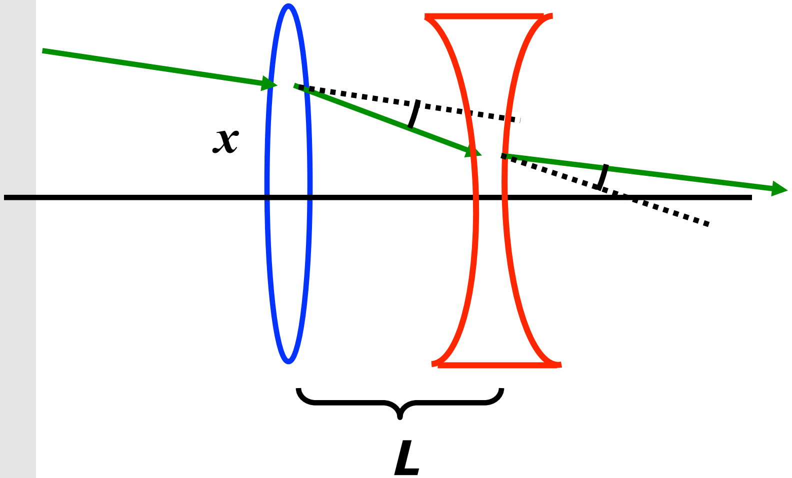




- Consider a rectangular dipole with bending angle θ. At each edge of length ΔL, the deflecting angle is changed by
$$\alpha = \frac{\Delta L}{\rho} = \frac{\theta \tan\delta}{\rho}$$
i.e., it acts as a thin defocusing lens with focal length $\frac{1}{f} = \frac{\tan\delta}{\rho}$
- The transfer matrix is $\mathcal{M}_{\text{rect}} = \mathcal{M}_{\text{edge}} \cdot \mathcal{M}_{\text{sector}} \cdot \mathcal{M}_{\text{edge}}$ with
$$\mathcal{M}_{\text{edge}} = \begin{pmatrix} 1 & 0 \\ \frac{-\tan(\delta)}{\rho} & 1 \end{pmatrix}$$
- For **θ<<1**, **δ=θ/2**
- In deflecting plane (like **drift**), in non-deflecting plane (like **sector**)

$$\mathcal{M}_{x;\text{rect}} = \begin{pmatrix} 1 & \rho\sin\theta \\ 0 & 1 \end{pmatrix} \quad \mathcal{M}_{y;\text{rect}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}$$

Quadrupole doublet

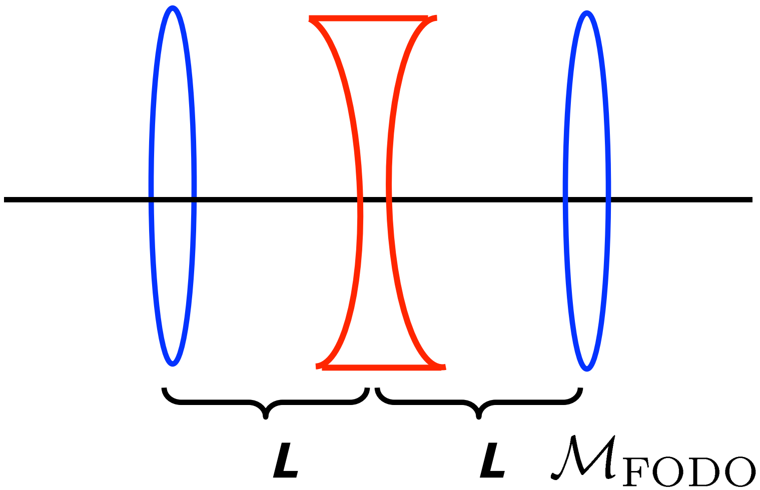


- Consider a quadrupole doublet, i.e. two quadrupoles with focal lengths f_1 and f_2 separated by a distance L.
- In thin lens approximation the transport matrix is

$$\mathcal{M}_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_1} & L \\ -\frac{1}{f^\star} & 1 - \frac{L}{f_2} \end{pmatrix}$$

with the **total focal length** $\frac{1}{f^\star} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1\ f_2}$

- Setting $f_1 = -f_2 = f$ $\quad \frac{1}{f^\star} = \frac{L}{f^2}$
- **Alternating gradient focusing** seems overall focusing
- This is only valid in thin lens approximation



- Consider defocusing quad "sandwiched" by two focusing quads with focal lengths $\pm f$.
- Symmetric transfer matrix from center to center of focusing quads

$$\mathcal{M}_{\mathrm{FODO}} = \mathcal{M}_{\mathrm{HQF}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{QD}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{HQF}}$$

with the transfer matrices

$$\mathcal{M}_{\mathrm{HQF}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}, \quad \mathcal{M}_{\mathrm{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \quad \mathcal{M}_{\mathrm{QD}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

- The total transfer matrix is

$$\mathcal{M}_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ -\frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}$$

- General solutions of Hill's equations
  - Floquet theory
- Betatron functions
- Transfer matrices revisited
  - General and periodic cell
- General transport of betatron functions
  - Drift
  - Beam waist

Solution of Betatron equations

- Betatron equations are linear

$$\begin{aligned} x'' + K_x(s)\,x &= 0 \\ y'' + K_y(s)\,y &= 0 \end{aligned}$$

with periodic coefficients

$$K_x(s) = K_x(s+C)\;,\quad K_y(s) = K_y(s+C)$$

- **Floquet theorem** states that the solutions are

$$u(s) = Aw(s)\cos(\psi(s)+\psi_0)$$

where $w(s)$, $\psi(s)$ are periodic with the same period

$$w(s) = w(s+C)\;,\quad \psi(s) = \psi(s+C)$$

- Note that solutions resemble the one of harmonic oscillator

$$u(s) = A\cos(\psi(s)+\psi_0)$$

- Substitute solution in Betatron equations

$$u'' + K(s)\,u = A\underbrace{(2w'\psi' + w\psi'')}_{0}\sin(\psi+\psi_0) + A\underbrace{(w'' - w\psi'^2 + Kw)}_{0}\cos(\psi+\psi_0) = 0$$

Betatron functions

- By multiplying with w the coefficient of sin

$$2w'w\psi' + w^2\psi'' = (w^2\psi')' = 0$$

- Integrate to get $\psi = \int \frac{ds}{w^2(s)}$

- Replace ψ' in the coefficient of cos and obtain

$$w^3(w'' + K_x w) = 1$$

- Define the **Betatron** or t**wiss** or **lattice functions** (Courant-Snyder parameters)

$$\begin{aligned} \beta(s) &\equiv w^2(s) \\ \alpha(s) &\equiv -\frac{1}{2}\frac{d\beta(s)}{ds} \\ \gamma(s) &\equiv \frac{1+\alpha^2(s)}{\beta(s)} \end{aligned}$$

Betatron motion

- The on-momentum linear betatron motion of a particle is described by

$$u(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s)+\psi_0)$$

with $\alpha,\ \beta,\ \gamma$ the twiss functions $\alpha(s) = -\frac{\beta(s)'}{2},\quad \gamma = \frac{1+\alpha(s)^2}{\beta(s)}$

ψ the **betatron phase** $\psi(s) = \int \frac{ds}{\beta(s)}$

and the **beta function** β is defined by the **envelope equation**

$$2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4$$

- By differentiation, we have that the **angle** is

$$u'(s) = \sqrt{\frac{\epsilon}{\beta(s)}}\left(\sin(\psi(s)+\psi_0) + \alpha(s)\cos(\psi(s)+\psi_0)\right)$$

- Eliminating the angles by the position and slope we define the **Courant-Snyder invariant**

$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = \epsilon$$

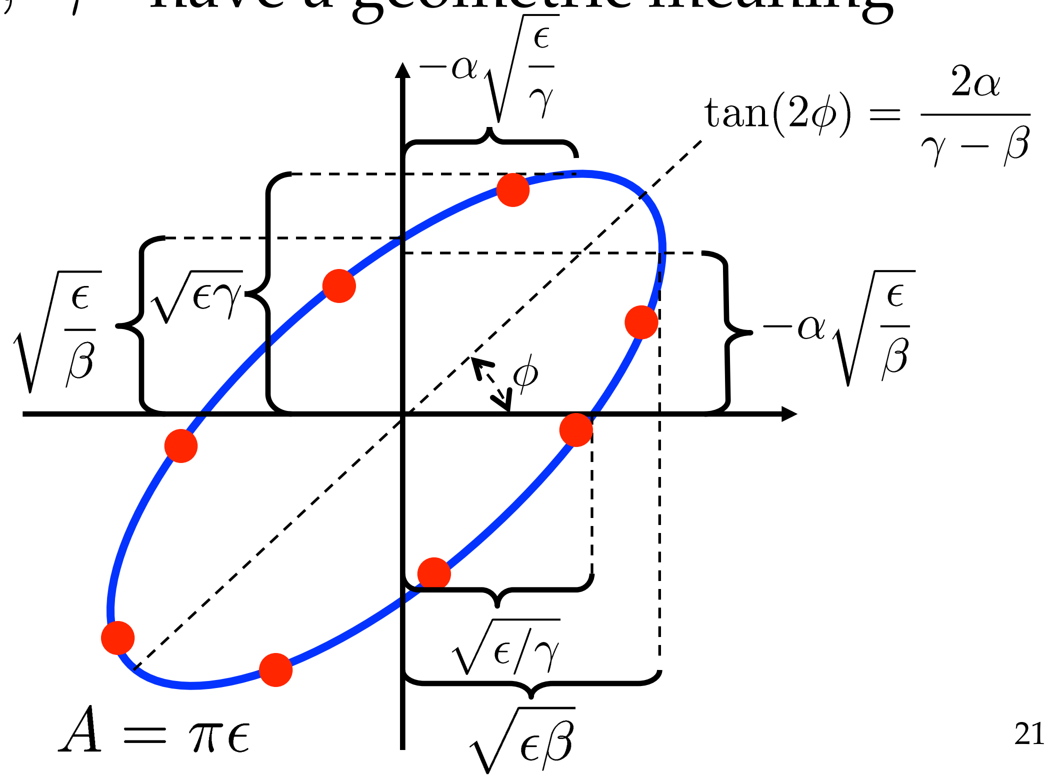
- This is an ellipse in phase space with area **πε**
- The twiss functions $\alpha,\ \beta,\ \gamma$ have a geometric meaning

- The beam envelope is

$$E(s) = \sqrt{\epsilon\beta(s)}$$

- The beam divergence

$$A(s) = \sqrt{\epsilon\gamma(s)}$$



General transfer matrix

- From equation for position and angle we have

$$\cos(\psi(s)+\psi_0) = \frac{u}{\sqrt{\epsilon\beta(s)}}\ , \quad \sin(\psi(s)+\psi_0) = \sqrt{\frac{\beta(s)}{\epsilon}}u' + \frac{\alpha(s)}{\sqrt{\epsilon\beta(s)}}u$$

- Expand the trigonometric formulas and set $\psi(0)=0$ to get the transfer matrix from location 0 to s

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \mathcal{M}_{0\to s} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

with

$$\mathcal{M}_{0\to s} = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}(\cos\Delta\psi + \alpha_0 \sin\Delta\psi) & \sqrt{\beta(s)\beta_0}\sin\Delta\psi \\ \frac{(a_0-a(s))\cos\Delta\psi-(1+\alpha_0\alpha(s))\sin\Delta\psi}{\sqrt{\beta(s)\beta_0}} & \sqrt{\frac{\beta_0}{\beta(s)}}(\cos\Delta\psi - \alpha_0\sin\Delta\psi) \end{pmatrix}$$

and $\Delta\psi = \int_0^s \frac{ds}{\beta(s)}$ the **phase advance**

Periodic transfer matrix

- Consider a periodic cell of length C
- The optics functions are $\beta_0 = \beta(C) = \beta\ , \quad \alpha_0 = \alpha(C) = \alpha$

  and the phase advance $\mu = \int_0^C \frac{ds}{\beta(s)}$

- The transfer matrix is

$$\mathcal{M}_C = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix}$$

- The cell matrix can be also written as

$$\mathcal{M}_C = \mathcal{I} \cos\mu + \mathcal{J} \sin\mu$$

with $\mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the **Twiss matrix** $\mathcal{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$

Stability conditions

- From the periodic transport matrix $\mathrm{Trace}(\mathcal{M}_C) = 2\cos\mu$ and the following stability criterion

$$|\mathrm{Trace}(\mathcal{M}_C)| < 2$$

- From transfer matrix for a cell

$$\mathcal{M}_C = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

we get

$$\cos\mu = \frac{1}{2}(m_{11} + m_{22}),\ \beta = \frac{m_{12}}{\sin\mu},\ \alpha = \frac{m_{11} - m_{22}}{2\sin\mu},\ \gamma = -\frac{m_{21}}{\sin\mu}$$

Tune and working point

- In a ring, the **tune** is defined from the 1-turn phase advance
$$Q_{x,y} = \frac{1}{2\pi}\oint \frac{ds}{\beta_{x,y}(s)} = \frac{\nu_{x,y}}{2\pi}$$
i.e. number betatron oscillations per turn
- Taking the average of the betatron tune around the ring we have in **smooth approximation**
$$\nu = 2\pi Q = \frac{C}{\langle\beta\rangle} \rightarrow Q = \frac{R}{\langle\beta\rangle}$$
- Extremely useful formula for deriving scaling laws
- The position of the tunes in a diagram of horizontal versus vertical tune is called a **working point**
- The tunes are imposed by the choice of the quadrupole strengths
- One should try to avoid **resonance conditions**

- For a general matrix between position 1 and 2

$$\mathcal{M}_{s_1\to s_2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \text{and the inverse } \mathcal{M}_{s_2\to s_1} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

- Equating the invariant at the two locations

$$\epsilon = \gamma_{s_2} {u_{s_2}}^2 + 2\alpha_{s_2} u_{s_2} u'_{s_2} + \beta_{s_2} u'^2_{s_2} = \gamma_{s_1} {u_{s_1}}^2 + 2\alpha_{s_1} u_{s_1} u'_{s_1} + \beta_{s_1} u'^2_{s_1}$$

and eliminating the transverse positions and angles

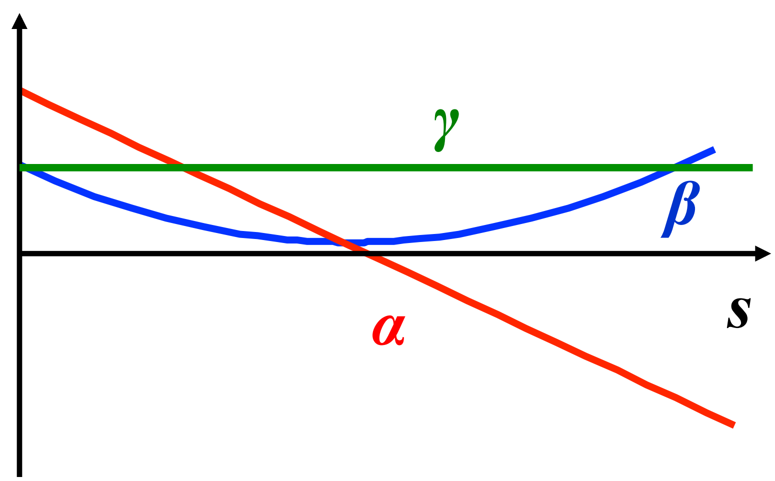
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22}+m_{12}m_{21} & -m_{22}m_{12} \\ m_{21}^2 & 2m_{22}m_{21} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_1}$$

Example I: Drift

- Consider a drift with length s
- The transfer matrix is $$\mathcal{M}_{\text{drift}} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$
- The betatron transport matrix is $$\begin{pmatrix} 1 & -2s & s^2 \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{pmatrix}$$

from which

$$\begin{aligned} \beta(s) &= \beta_0 - 2s\alpha_0 + s^2\gamma_0 \\ \alpha(s) &= \alpha_0 - s\gamma_0 \\ \gamma(s) &= \gamma_0 \end{aligned}$$



- Consider the beta matrix $\mathcal{B} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$ the matrix

$$\mathcal{M}_{1\to 2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \text{ and its transpose } \mathcal{M}^T_{1\to 2} = \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

- It can be shown that

$$\boxed{\mathcal{B}_2 = \mathcal{M}_{1\to 2} \cdot \mathcal{B}_1 \cdot \mathcal{M}^T_{1\to 2}}$$

- Application in the case of the drift

$$\mathcal{B} = \mathcal{M}_{\text{drift}} \cdot \mathcal{B}_0 \cdot \mathcal{M}^T_{\text{drift}} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$$

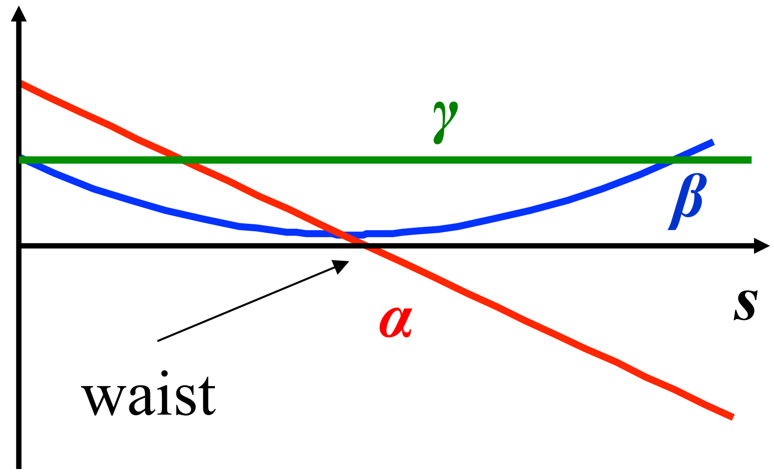
and

$$\mathcal{B} = \begin{pmatrix} \beta_0 - 2s\alpha_0 + s^2\gamma_0 & -\alpha_0 + s\gamma_0 \\ -\alpha_0 + s\gamma_0 & \gamma_0 \end{pmatrix}$$

Example II: Beam waist

- For beam waist α=0 and occurs at s = α_0 / γ_0
- Beta function grows quadratically and is minimum in waist

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$



- The beta at the waste for having beta minimum $\frac{d\beta(s)}{d\beta_0} = 0$

  in the middle of a drift with length L is $\beta_0 = \frac{L}{2}$

- The phase advance of a drift is $\mu = \int_0^{L/2} \frac{ds}{\beta(s)} = \arctan(\frac{L}{2\beta_0})$

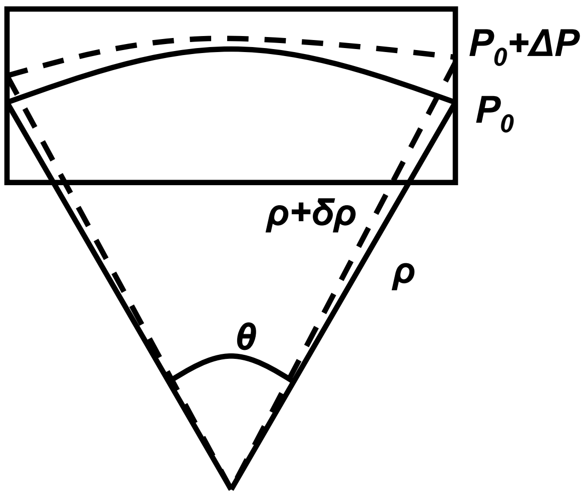
  which is π/2 when $\beta_0 \to \infty$. Thus, for a drift $\mu = \leq \pi$

Outline – Off-momentum dynamics

- Off-momentum particles
  - Effect from dipoles and quadrupoles
  - Dispersion equation
  - 3x3 transfer matrices
- Periodic lattices in circular accelerators
  - Periodic solutions for beta function and dispersion
  - Symmetric solution
  - 3x3 FODO cell matrix

Effect of dipole on off-momentum particles

- Up to now all particles had the same momentum P_0
- What happens for off-momentum particles, i.e. particles with momentum $P_0+\Delta P$?
- Consider a dipole with field B and bending radius ρ
- Recall that the magnetic rigidity $B\rho = \frac{P_0}{q}$ and for off-momentum particles



$$B(\rho + \Delta\rho) = \frac{P_0 + \Delta P}{q} \Rightarrow \frac{\Delta\rho}{\rho} = \frac{\Delta P}{P_0}$$

- Considering the effective length of the dipole unchanged

$$\theta\rho = l_{eff} = \text{const.} \Rightarrow \rho\Delta\theta + \theta\Delta\rho = 0 \Rightarrow \frac{\Delta\theta}{\theta} = -\frac{\Delta\rho}{\rho} = -\frac{\Delta P}{P_0}$$

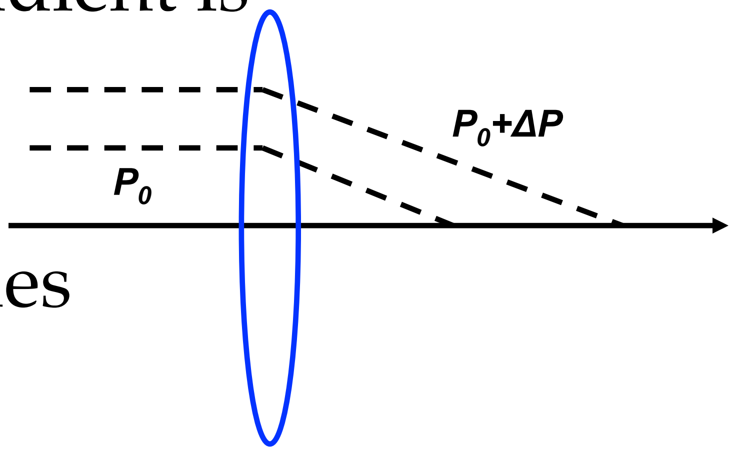
- Off-momentum particles get different deflection (different orbit)

$$\Delta\theta = -\theta\frac{\Delta P}{P_0}$$

Off-momentum particles and quadrupoles

- Consider a quadrupole with gradient ***G***
- Recall that the normalized gradient is

$$K = \frac{q\,G}{P_0}$$



and for off-momentum particles

$$\Delta K = \frac{dK}{dP}\Delta P = -\frac{qG}{P_0}\frac{\Delta P}{P_0}$$

- Off-momentum particle gets different focusing

$$\Delta K = -K\frac{\Delta P}{P_0}$$

- This is equivalent to the effect of **optical lenses** on **light of different wavelengths**

Dispersion equation

- Consider the equations of motion for off-momentum particles

$$x'' + K_x(s)x = \frac{1}{\rho(s)}\frac{\Delta P}{P}$$

- The solution is a sum of the **homogeneous** equation (on-momentum) and the **inhomogeneous** (off-momentum)

$$x(s) = x_H(s) + x_I(s)$$

- In that way, the equations of motion are split in two parts

$$\begin{aligned} x''_H + K_x(s)x_H &= 0 \\ x''_I + K_x(s)x_I &= \frac{1}{\rho(s)}\frac{\Delta P}{P} \end{aligned}$$

- The **dispersion function** can be defined $D(s) = \dfrac{x_I(s)}{\Delta P/P}$
- The dispersion equation is

$$\boxed{D''(s) + K_x(s)\ D(s) = \frac{1}{\rho(s)}}$$

Dispersion solution for a bend

- Simple solution by considering motion through a sector dipole with constant bending radius ρ
- The dispersion equation becomes $$D''(s) + \frac{1}{\rho^2} D(s) = \frac{1}{\rho}$$
- The solution of the homogeneous is harmonic with frequency $1/\rho$
- A particular solution for the inhomogeneous is $D_p = \text{constant}$ and we get by replacing $D_p = \rho$
- Setting $D(0) = D_0$ *and* $D'(0) = D_0'$, the solutions for dispersion are

$$
\begin{aligned}
D(s) &= D_0 \cos(\frac{s}{\rho}) + D_0' \rho \sin(\frac{s}{\rho}) + \rho(1 - \cos(\frac{s}{\rho})) \\
D'(s) &= -\frac{D_0}{\rho} \sin(\frac{s}{\rho}) + D_0' \cos(\frac{s}{\rho}) + \sin(\frac{s}{\rho})
\end{aligned}
$$

General dispersion solution

- General solution possible with perturbation theory and use of Green functions
- For a general matrix $\mathcal{M} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S(s)' \end{pmatrix}$ the solution is

$$D(s) = S(s) \int_{s_0}^{s} \frac{C(\bar{s})}{\rho(\bar{s})} d\bar{s} + C(s) \int_{s_0}^{s} \frac{S(\bar{s})}{\rho(\bar{s})} d\bar{s}$$

- One can verify that this solution indeed satisfies the differential equation of the dispersion (and the sector bend)
- The general betatron solutions can be obtained by 3X3 transfer matrices including dispersion

$$\mathcal{M}_{3\times 3} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix}$$

- Recalling that $x(s) = x_B(s) + D(s)\dfrac{\Delta P}{P}$

$$\begin{pmatrix} x(s) \\ x'(s) \\ \Delta p/p \end{pmatrix} = \mathcal{M}_{3\times 3} \begin{pmatrix} x(s_0) \\ x'(s_0) \\ \Delta p/p \end{pmatrix}$$ and $$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \mathcal{M}_{3\times 3} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}$$

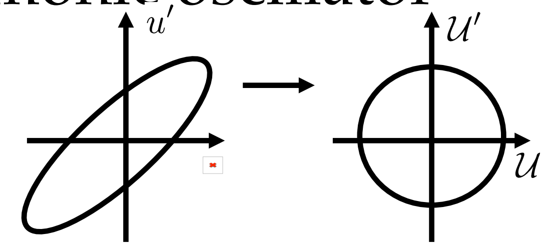
General solution for the dispersion

- Introduce **Floquet variables**

$$\mathcal{U} = \frac{u}{\sqrt{\beta}}\ ,\quad \mathcal{U}' = \frac{d\mathcal{U}}{d\phi} = \frac{\alpha}{\sqrt{\beta}}u + \sqrt{\beta}u'\ ,\quad \phi = \frac{\psi}{\nu} = \frac{1}{\nu}\int\frac{ds}{\beta(s)}$$

- The Hill's equations are written $\frac{d^2\mathcal{U}}{d\phi^2} + \nu^2\mathcal{U} = 0$
- The solutions are the ones of an harmonic oscillator

$$\begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \sqrt{\epsilon}\begin{pmatrix}\cos(\nu\phi)\\ -\sin(\nu\phi)\end{pmatrix}$$



- For the dispersion solution $\mathcal{U} = \frac{D}{\sqrt{\beta}}\frac{\Delta P}{P}$, the inhomogeneous equation in Floquet variables is written

$$\frac{d^2D}{d\phi^2} + \nu^2 D = -\frac{\nu^2\beta^{3/2}}{\rho(s)}$$

- This is a forced harmonic oscillator with solution

$$D(s) = \frac{\sqrt{\beta(s)}\nu}{2\sin(\pi\nu)}\oint\frac{\sqrt{\beta(\sigma)}}{\rho(\sigma)}\cos[\nu(\phi(s)-\phi(\sigma)+\pi)]d\sigma$$

- Note the **resonance conditions** for integer tunes!!!

- For **drifts** and **quadrupoles** which do not create dispersion the 3x3 transfer matrices are just

$$\mathcal{M}_{\text{drift,quad}} = \begin{pmatrix} \mathcal{M}_{2\times 2} & & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & 0 & 1 \end{pmatrix}$$

- For the deflecting plane of a **sector bend** we have seen that the matrix is

$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

and in the non-deflecting plane is just a drift.

- Synchrotron magnets have focusing and bending included in their body.
- From the solution of the sector bend, by replacing *1/ρ* with

$$\sqrt{K} = \sqrt{\frac{1}{\rho^2} - k}$$

- For *K*>0 $$\mathcal{M}_{\mathrm{syF}} = \begin{pmatrix} \cos\psi & \frac{\sin\psi}{\sqrt{K}} & \frac{1-\cos\psi}{\rho K} \\ -\sqrt{K}\sin\psi & \cos\psi & \frac{\sin\psi}{\rho\sqrt{K}} \\ 0 & 0 & 1 \end{pmatrix}$$

- For *K*<0 $$\mathcal{M}_{\mathrm{syD}} = \begin{pmatrix} \cosh\psi & \frac{\sinh\psi}{\sqrt{|K|}} & -\frac{1-\cosh\psi}{\rho|K|} \\ \sqrt{|K|}\sinh\psi & \cosh\psi & \frac{\sinh\psi}{\rho\sqrt{|K|}} \\ 0 & 0 & 1 \end{pmatrix}$$

with $$\psi = \sqrt{|k + \frac{1}{\rho^2}|}\,l$$

3x3 transfer matrices - Rectangular magnet

- The end field of a rectangular magnet is simply the one of a quadrupole. The transfer matrix for the edges is

$$\mathcal{M}_{\text{edge}} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan(\theta/2) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- The transfer matrix for the body of the magnet is like for the sector bend $\mathcal{M}_{\text{rect}} = \mathcal{M}_{\text{edge}} \cdot \mathcal{M}_{\text{sect}} \cdot \mathcal{M}_{\text{edge}}$
- The total transfer matrix is

$$\mathcal{M}_{\text{rect}} = \begin{pmatrix} 1 & \rho\sin\theta & \rho(1-\cos\theta) \\ 0 & 1 & 2\tan(\theta/2) \\ 0 & 0 & 1 \end{pmatrix}$$

Periodic solutions

- Consider two points s_0 and s_1 for which the magnetic structure is repeated.
- The optical function follow periodicity conditions

$$\beta_0 = \beta(s_0) = \beta(s_1)\ , \quad \alpha_0 = \alpha(s_0) = \alpha(s_1)$$
$$D_0 = D(s_0) = D(s_1)\ , \ D'_0 = D'(s_0) = D'(s_1)$$

- The beta matrix at this point is $\mathcal{B}_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}$
- Consider the transfer matrix from s_0 to s_1 $\quad \mathcal{M}_{1\to 2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$

$$\mathcal{B}_0 = \mathcal{M}_{0\to 1}\cdot\mathcal{B}_0\cdot\mathcal{M}^T_{0\to 1} \Rightarrow \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

- The solution for the optics functions is

$$\beta_0 = \frac{2m_{12}}{\sqrt{2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2}}$$
$$\alpha_0 = \frac{m_{11} - m_{22}}{\sqrt{2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2}}$$

with the condition $2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2 > 0$

Periodic solutions for dispersion

- Consider the 3x3 matrix for propagating dispersion between s_0 and s_1

$$\begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix}$$

- Solve for the dispersion and its derivative to get

$$D_0' = \frac{m_{21}m_{13} + m_{23}(1 - m_{11})}{2 - m_{11} - m_{22}}$$

$$D_0 = \frac{m_{12}D_0' + m_{13}}{1 - m_{11}}$$

with the conditions $m_{11} + m_{22} \neq 2$ and $m_{11} \neq 1$

Symmetric solutions

- Consider two points s_0 and s_1 for which the lattice is mirror symmetric
- The optical function follow periodicity conditions

$$\alpha(s_0) = \alpha(s_1) = 0$$
$$D'(s_0) = D'(s_1) = 0$$

- The beta matrices at s_0 and s_1 are $\mathcal{B}_0 = \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix}$ $\mathcal{B}_1 = \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix}$
- Considering the transfer matrix between s_0 and s_1

$$\mathcal{B}_1 = \mathcal{M}_{0\to1} \cdot \mathcal{B}_0 \cdot \mathcal{M}_{0\to1}^T \Rightarrow \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

- The solution for the optics functions is

$$\beta_0 = \sqrt{-\frac{m_{12}m_{22}}{m_{21}m_{11}}} \quad \text{and} \quad \beta_1 = -\frac{1}{\beta_0}\frac{m_{12}}{m_{21}}$$

with the condition $\frac{m_{12}}{m_{21}} < 0$ and $\frac{m_{22}}{m_{11}} > 0$

Symmetric solutions for dispersion

- Consider the 3x3 matrix for propagating dispersion between s_0 and s_1

$$\begin{pmatrix} D(s_1) \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_0) \\ 0 \\ 1 \end{pmatrix}$$

- Solve for the dispersion in the two locations

$$\begin{aligned} D(s_0) &= -\frac{m_{23}}{m_{21}} \\ D(s_1) &= -\frac{m_{11}m_{23}}{m_{21}} + m_{13} \end{aligned}$$

- Imposing certain values for beta and dispersion, quadrupoles can be adjusted in order to get a solution

- Consider a general periodic structure of length $2L$ which contains **N** cells. The transfer matrix can be written as

$$\mathcal{M}(s+N\cdot 2L|s) = \mathcal{M}(s+2L|s)^N$$

- The periodic structure can be expressed as

$$\mathcal{M} = \mathcal{I}\cos\mu + \mathcal{J}\sin\mu$$

with $\mathcal{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$.

- Note that because $\det(\mathcal{M}) = 1 \rightarrow \beta\gamma - a^2 = 1$
- Note also that $\mathcal{J}^2 = -\mathcal{I}$
- By using **de Moivre's formula**

$$\mathcal{M}^N = (\mathcal{I}\cos\mu + \mathcal{J}\sin\mu)^N = \mathcal{I}\cos(N\mu) + \mathcal{J}\sin(N\mu)$$

- We have the following general stability criterion

$$|\mathrm{Trace}(\mathcal{M}^N)| = 2\cos(N\mu) < 2$$

3X3 FODO cell matrix

- Insert a sector dipole in between the quads and consider $\theta = L/\rho << 1$
- Now the transfer matrix is $\mathcal{M}_{\mathrm{HFODO}} = \mathcal{M}_{\mathrm{HQF}} \cdot \mathcal{M}_{\mathrm{sector}} \cdot \mathcal{M}_{\mathrm{HQD}}$ which gives

$$\mathcal{M}_{\mathrm{HFODO}} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{L^2}{2\rho} \\ 0 & 1 & \frac{L}{\rho} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and after multiplication

$$\mathcal{M}_{\mathrm{HFODO}} = \begin{pmatrix} 1 - \frac{L}{f} & L & \frac{L^2}{(2\rho)} \\ -\frac{L}{f^2} & 1 + \frac{L}{f} & \frac{L}{\rho}(1 + \frac{L}{2f}) \\ 0 & 0 & 1 \end{pmatrix}$$

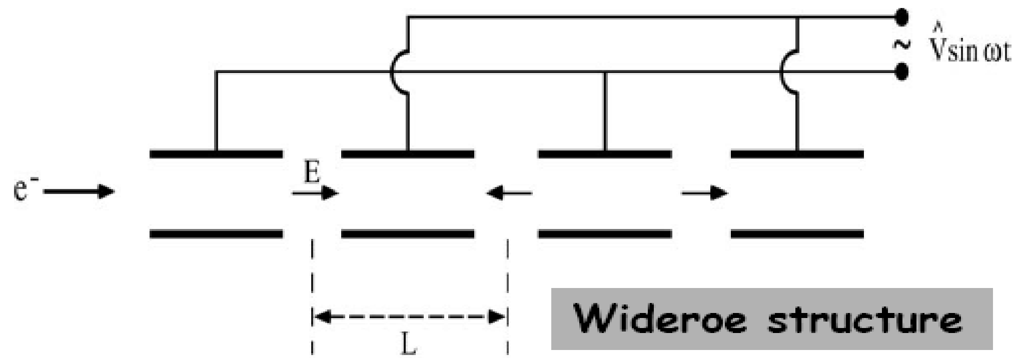
Longitudinal dynamics

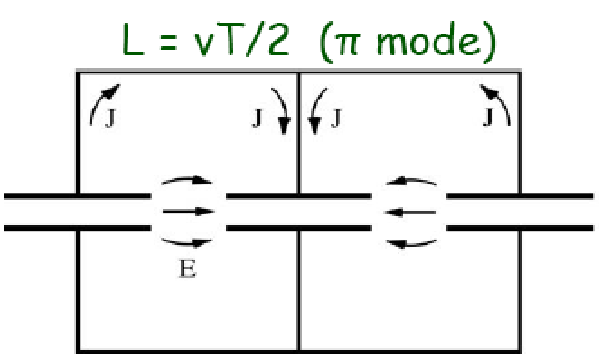
- RF acceleration
- Energy gain and phase stability
- Momentum compaction and transition
- Equations of motion
  - Small amplitudes
  - Longitudinal invariant
- Separatrix
- Energy acceptance
- Stationary bucket
- Adiabatic damping

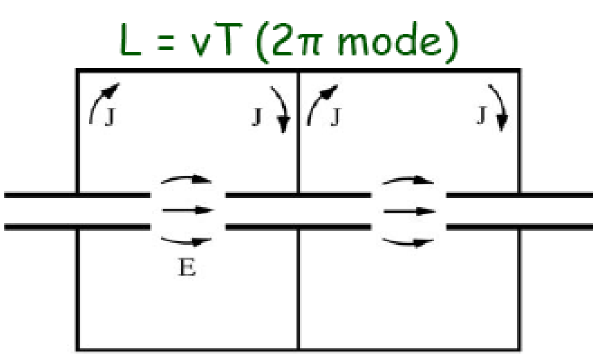
RF acceleration

- The use of RF fields allows an arbitrary number of accelerating steps in gaps and electrodes fed by RF generator
- The electric field is not longer continuous but sinusoidal alternating half periods of acceleration and deceleration
- The synchronism condition for RF period T_{RF} and particle velocity v

$$L = vT_{RF}/2 = \beta c \frac{\pi}{\omega_{RF}} = \beta\lambda/2$$







Energy gain

Assuming a sinusoidal electric field $E_z = E_0 \cos(\omega_{RF} t + \phi_s)$ where the synchronous particle passes at the middle of the gap g, at time $t = 0$, the energy is

$$W(r,t) = q\int E_z dz = q\int_{-g/2}^{g/2} E_0 \cos(\omega_{RF}\frac{z}{v} + \phi_s)dz$$

And the energy gain is $\Delta W = qE_0 \int_{-q/2}^{g/2} \cos(\omega_{RF}\frac{z}{v})dz$

and finally $\Delta W = qV\frac{\sin\Theta/2}{\Theta/2} = qV\,\mathrm{T}$ with the transit time factor defined as

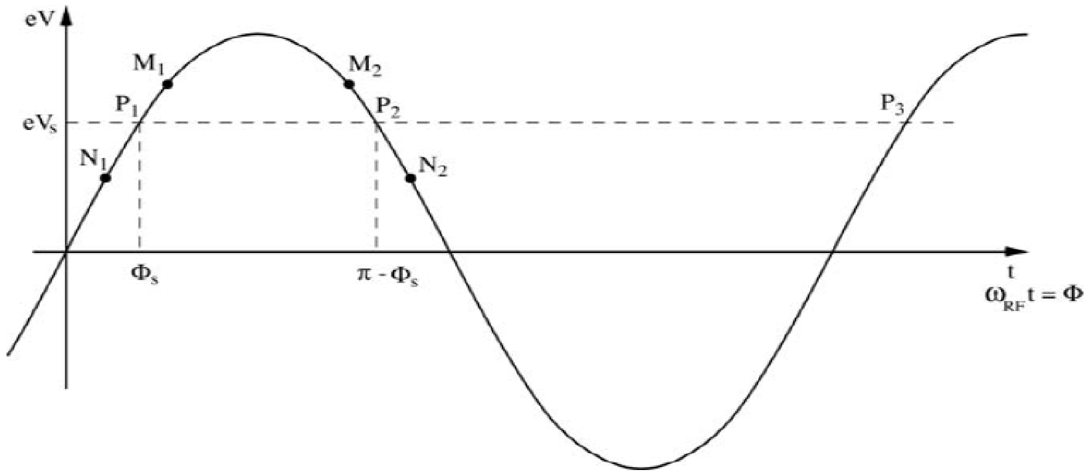
$$T = \frac{\sin(\omega g/2v)}{\omega g/2v}$$

It can be shown that in general

$$\mathrm{T} = \frac{\int_{-g/2}^{g/2} E(0,z)\cos\omega t(z)dz}{\int_{-g/2}^{g/2} E(0,z)dz}$$

Phase stability

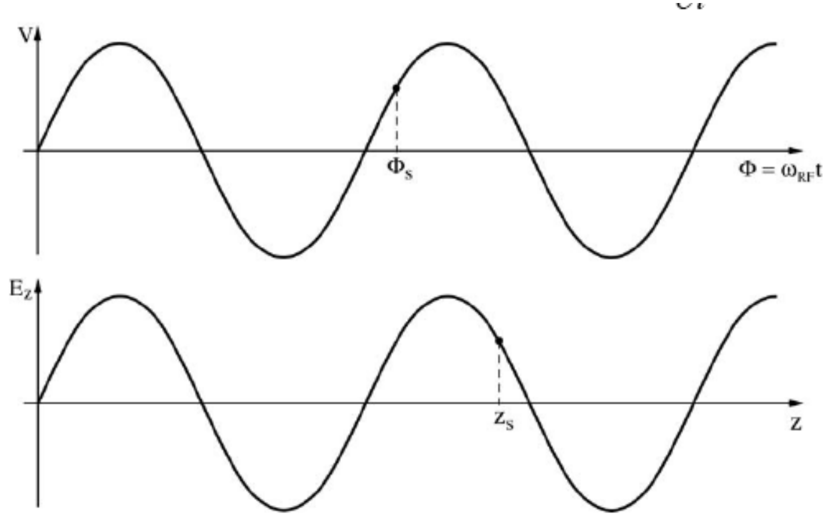
- Assume that a synchronicity condition is fulfilled at the phase ϕ_s and that energy increase produces a velocity increase
- Around point P_1, that arrives earlier (N_1) experiences a smaller accelerating field and slows down
- Particles arriving later (M1) will be accelerated more
- A restoring force that keeps particles oscillating around a stable phase called the synchronous phase ϕ_s
- The opposite happens around point P2 at π-$\phi_{s,}$ i.e. M2 and N2 will further separate



RF de-focusing

In order to have stability, the time derivative of the Voltage and the spatial derivative of the electric field should satisfy

$$\frac{\partial V}{\partial t} > 0 \Rightarrow \frac{\partial E}{\partial z} < 0$$



In the absence of electric charge the divergence of the field is given by Maxwell's equations

$$\nabla \vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \Rightarrow \frac{\partial E_x}{\partial x} > 0$$

where x represents the generic transverse direction.

External focusing is required by using quadrupoles or solenoids

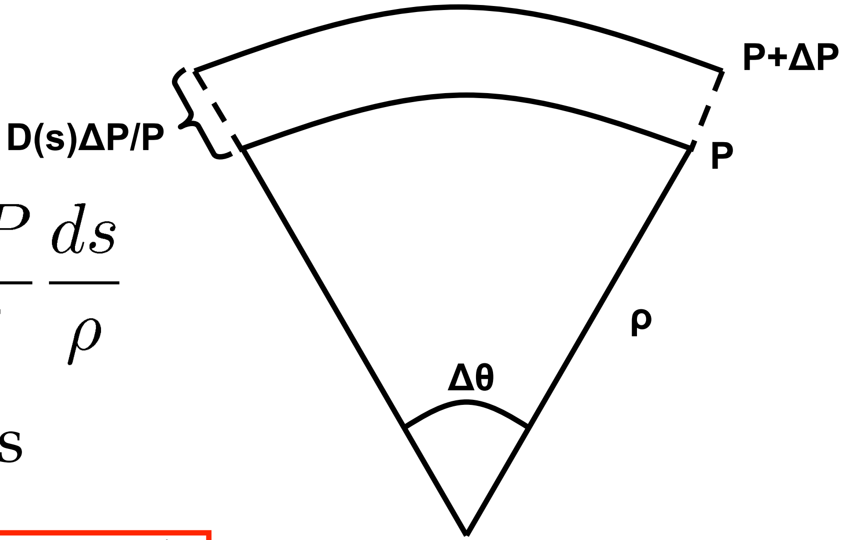
Momentum compaction

- Off-momentum particles on the dispersion orbit travel in a different path length than on-momentum particles
- The change of the path length with respect to the momentum spread is called **momentum compaction**

$$\alpha_c = \frac{\Delta C}{C} / \frac{\Delta P}{P}$$

- The change of circumference is

$$\Delta C = \oint D \frac{\Delta P}{P} d\theta = \oint D \frac{\Delta P}{P} \frac{ds}{\rho}$$



- So the momentum compaction is

$$\alpha_c = \frac{1}{C} \oint \frac{D(s)}{\rho(s)} ds = \left\langle \frac{D(s)}{\rho(s)} \right\rangle$$

Transition energy

- The revolution frequency of a particle is $f = \frac{v}{2\pi\rho} = \frac{\beta c}{2\pi\rho}$
- The change in frequency is $\frac{\Delta f}{f} = \frac{\Delta\rho}{\rho} - \frac{\Delta\beta}{\beta}$
- From the relativistic momentum $Pc = \beta E$ we have

$$\frac{\Delta P}{P} = \frac{\Delta\beta}{\beta} + \frac{\Delta E}{E} \rightarrow \beta^2 \frac{\Delta P}{P}$$ for which $$\frac{\Delta\beta}{\beta} = \frac{1}{\gamma^2}\frac{\Delta P}{P}$$

and the revolution frequency $$\frac{\Delta f}{f} = \left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{\Delta P}{P}$$

The slippage factor is given by $$\eta = \frac{1}{\gamma^2} - \alpha_c$$

For vanishing slippage factor, the transition energy is defined

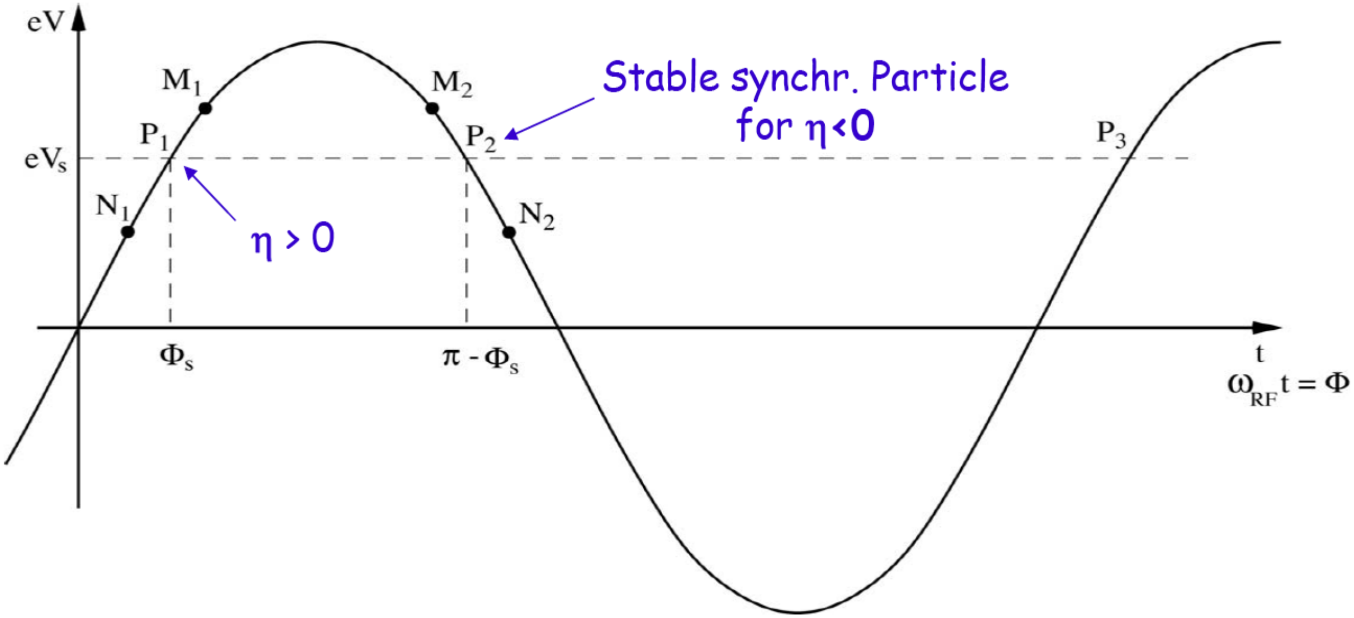
$$\gamma_t = \frac{1}{\sqrt{\alpha_c}}$$

Synchrotron

- Frequency modulated but also B-field increased **synchronously** to match energy and keep revolution radius constant.
- The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference.
- Each synchronous particle has the nominal energy and follow the nominal trajectory
- Magnetic field increases with momentum and the per turn change of the momentum is

$$(\Delta p)_{turn} = e\rho\, B' T_r = \frac{2\pi\, e\rho\, RB'}{v}$$

Phase stability on electron synchrotrons



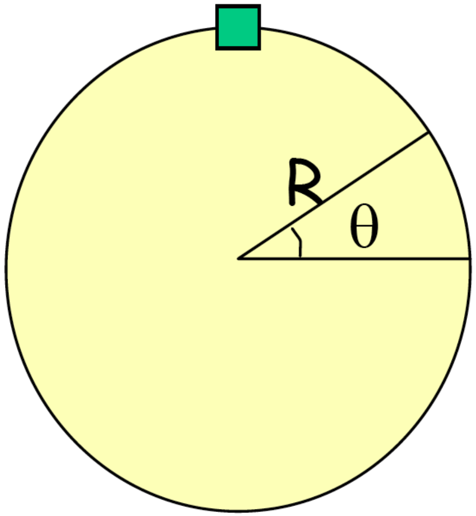
- For electron synchrotrons, the relativistic γ is very large and

$$\eta = \frac{1}{\gamma^2} - \alpha_c \approx -\alpha_c < 0$$

as momentum compaction is positive in most cases

- Above transition, an increase in energy is followed by lower revolution frequency
- A delayed particle with respect to the synchronous one will get closer to it (gets a smaller energy increase) and phase stability occurs at the point P2 ($\pi - \phi_s$)

Energy and phase relation



- The RF frequency and phase are related to the revolution ones as follows

$$f_{RF} = hf_r \quad \Rightarrow \quad \Delta\phi = -h\Delta\theta \quad with \quad \theta = \int \omega_r dt$$

and $\Delta\omega_r = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$

- From the definition of the momentum compaction and for electrons

$$\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega_r}{dp}\right)_s = \frac{E_s}{\omega_{rs}}\left(\frac{d\omega_r}{dE}\right)_s \cong -\alpha_c$$

- Replacing the revolution frequency change, the following relation is obtained between the energy and the RF phase time derivative

$$\frac{\Delta E}{E_s} = \frac{1}{\omega_{rs}\alpha_c h}\frac{d\phi}{dt} = \frac{R}{c\alpha h}\dot{\phi}$$

Longitudinal equations of motion

- The energy gain per turn with respect to the energy gain of the synchronous particle is

$$(\Delta E)_{turn} = e\hat{V}(\sin\phi - \sin\phi_s)$$

- The rate of energy change can be approximated by

$$\frac{d(\Delta E)}{dt} \cong (\Delta E)_{turn} f_{rs} = \frac{c}{2\pi R} e\hat{V}(\sin\phi - \sin\phi_s)$$

- The second energy phase relation is written as

$$\frac{d}{dt}\left(\frac{\Delta E}{E_s}\right) = \frac{ce\hat{V}}{2\pi R E_s}(\sin\phi - \sin\phi_s)$$

- By combining the two energy/phase relations, a 2nd order differential equation is obtained, similar the pendulum

$$\frac{d}{dt}\left(\frac{R}{c\alpha_c h}\frac{d\phi}{dt}\right) + \frac{ce\hat{V}}{2\pi R E_s}(\sin\phi - \sin\phi_s) = 0$$

Small amplitude oscillations

- Expanding the harmonic functions in the vicinity of the synchronous phase

$$\sin\phi - \sin\phi_s = \sin(\phi_s + \Delta\phi) - \sin\phi_s \cong \cos\phi_s \Delta\phi$$

- Considering also that the coefficient of the phase derivative does not change with time, the differential equation reduces to one describing an harmonic oscillator

$$\ddot{\phi} + \Omega_s^2 \Delta\phi = 0 \quad \text{with frequency} \quad \Omega_s^2 = -\frac{c^2 e \alpha_c h \hat{V} \cos\phi_s}{R^2 2\pi E_s}$$

- For stability, the square of the frequency should positive and real, which gives the same relation for phase stability when particles are above transition

$$\cos\phi_s < 0 \Rightarrow \pi/2 < \phi_s < \pi$$

- For large amplitude oscillations the differential equation of the phase is written as

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

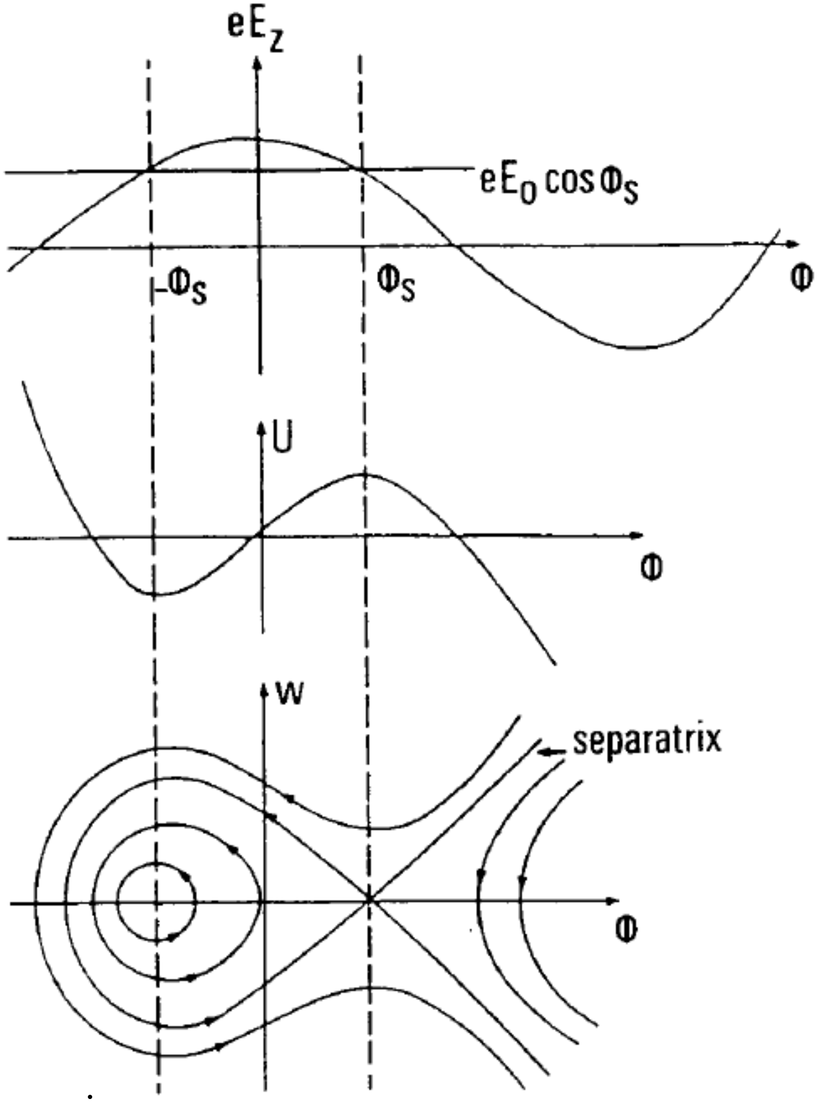
- Multiplying by the time derivative of the phase and integrating, an invariant of motion is obtained

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

reducing to the following expression, for small amplitude oscillations

$$\frac{\dot{\phi}^2}{2} + \frac{\Omega_s^2}{2}\Delta\phi = I$$

Separatrix



- In the phase space (energy change versus phase), the motion is described by distorted circles in the vicinity of ϕ_s (stable fixed point)
- For phases beyond $\pi - \phi_s$ (unstable fixed point) the motion is unbounded in the phase variable, as for the rotations of a pendulum
- The curve passing through $\pi - \phi_s$ is called the **separatrix** and the enclosed area **bucket**

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s\right)$$

Energy acceptance

- The time derivative of the RF phase (or the energy change) reaches a maximum (the second derivative is zero) at the synchronous phase
- The equation of the separatrix at this point becomes

$$\dot{\phi}^2_{\max} = 2\Omega_s^2 \left(2 + (2\phi_s - \pi)\tan\phi_s\right)$$

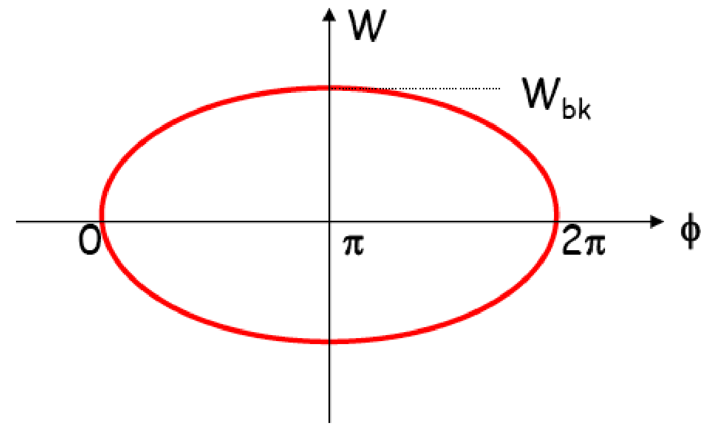
- Replacing the time derivative of the phase from the first energy phase relation

$$\left(\frac{\Delta E}{E_s}\right)_{\max} = \mp\sqrt{\frac{q\hat{V}}{\pi h \alpha_c E_s}\left(2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s\right)}$$

- This equation defines the energy acceptance which depends strongly on the choice of the synchronous phase. It plays an important role on injection matching and influences strongly the electron storage ring lifetime

Stationary bucket

- When the synchronous phase is chosen to be equal to 0 (below transition) or π (above transition), there is no acceleration. The equation of the separatrix is written



$$\frac{\dot{\phi}^2}{2} = 2\Omega_s^2 \sin^2\frac{\phi}{2}$$

- Using the (canonical) variable $W = 2\pi\frac{\Delta E}{\omega_{rs}} = 2\pi\frac{E_s R}{h\,\alpha_c\,\omega_{rs}}\dot{\phi}$ and replacing the expression for the synchrotron frequency

$$W = \pm 2\frac{C}{c}\sqrt{\frac{q\hat{V}E_s}{2\pi\,h\,\alpha_c}}\sin\frac{\phi}{2}$$

. For $\phi = \pi$, the bucket **height** is

$$W_{bk} = 2\frac{C}{c}\sqrt{\frac{e\hat{V}E_s}{2\pi\,h\,\alpha_c}}$$

and the **area** $A_{bk} = 2\int_0^{2\pi} W\,d\phi = 8W_{bk}$

Adiabatic damping

- The longitudinal oscillations can be damped directly by acceleration itself. Consider the equation of motion when the energy of the synchronous particle is not constant

$$\frac{d}{dt}\left(E_s\dot{\phi}\right) = -\Omega_s^2 E_s \Delta\phi$$

- From this equation, we obtain a 2nd order differential equation with a damping term

$$\ddot{\phi} + \frac{\dot{E}_s}{E_s}\dot{\phi} + \Omega_s^2 \Delta\phi = 0$$

- From the definition of the synchrotron frequency the damping coefficient is

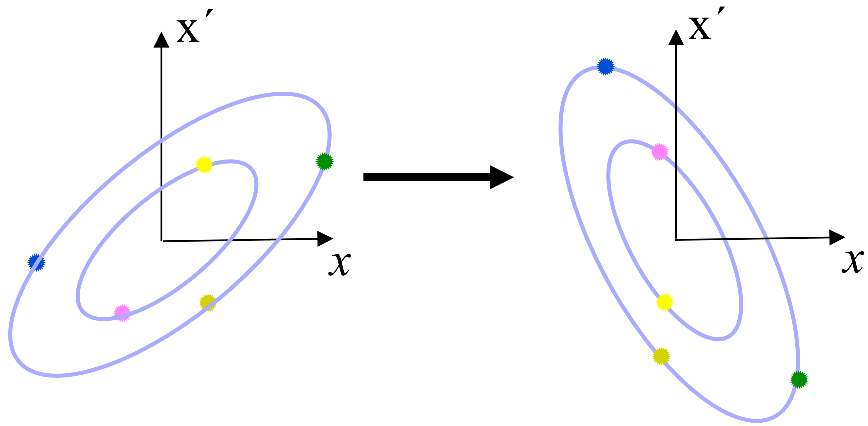
$$\frac{\dot{E}_s}{E_s} = -2\frac{\dot{\Omega}_s}{\Omega_s}$$

Outline – Phase space concepts

- Transverse phase space and Beam representation
- Beam emittance
- Liouville and normalised emittance
- Beam matrix
- RMS emittance
- Betatron functions revisited
- Gaussian distribution

Transverse Phase Space

- Under linear forces, any particle moves on ellipse in phase space (x,x′), (y,y’).
- Ellipse rotates and moves between magnets, but its area is preserved.
- The area of the ellipse defines the **emittance**

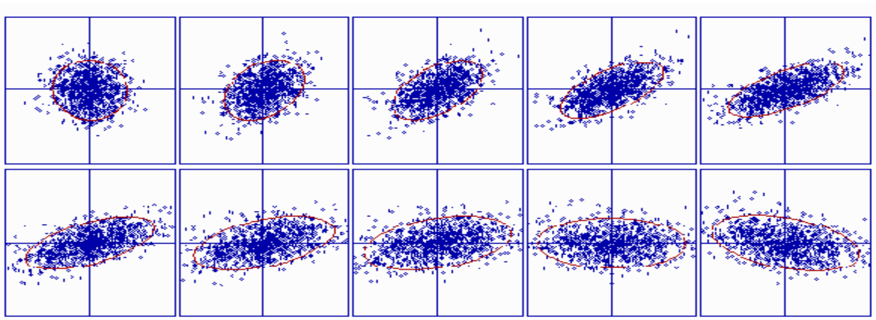


- The equation of the ellipse is

$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = \epsilon$$

with α,β,γ, the twiss parameters

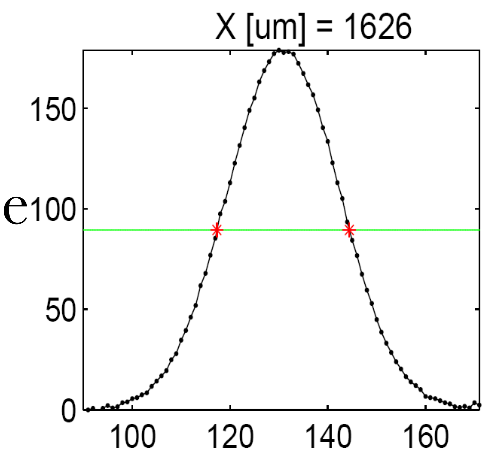
- Due to large number of particles, need of a statistical description of the beam, and its size

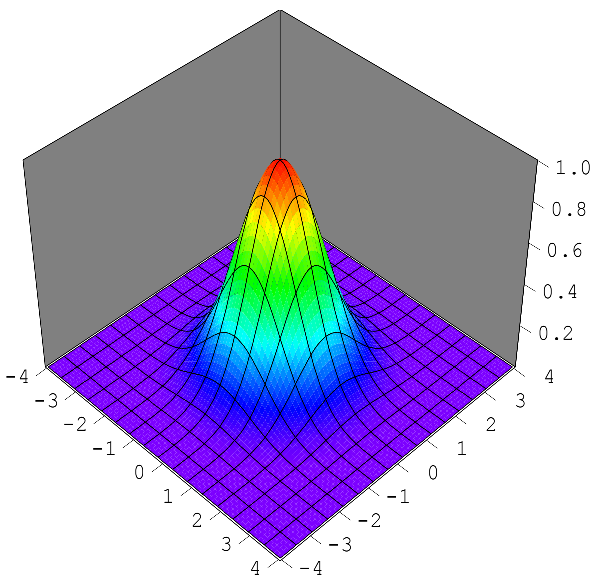


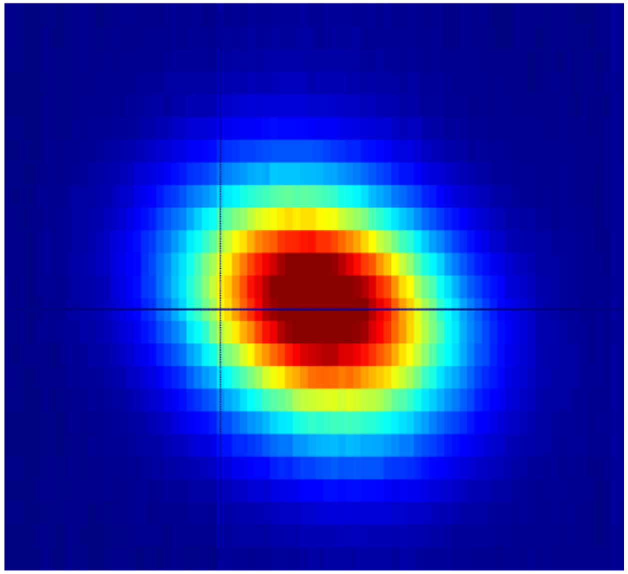
Beam representation

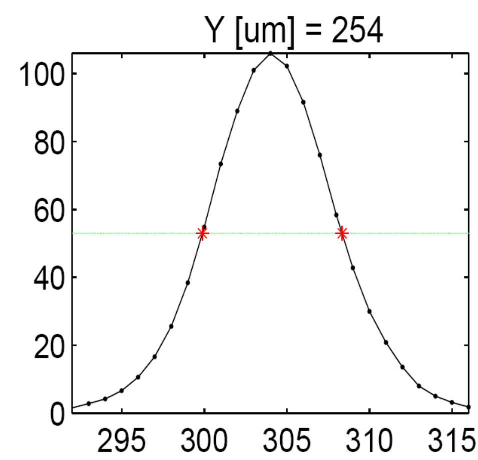
- Beam is a set of millions/billions of particles (N)
- A macro-particle representation models beam as a set of n particles with n<<N
- Distribution function is a statistical function representing the number of particles in phase space between $\mathbf{u} + d\mathbf{u}$, $\mathbf{u}' + d\mathbf{u}'$

$$f(\mathbf{u}, \mathbf{u}')d\mathbf{u}d\mathbf{u}' = \text{number of particles}$$









Liouville emittance

- Emittance represents the phase-space volume occupied by the beam
- The phase space can have different dimensions
  - 2D (**x, x'**) or (**y, y'**) or (**φ, E**)
  - 4D (**x, x' ,y, y'**) or (**x, x' , φ, E**) or (**y, y', φ, E**)
  - 6D (**x, x' , y, y' , φ, E**)
- The resolution of my beam observation is very large compared to the average distance between particles.
- The beam modeled by phase space **distribution function** $f(x, x', y, y', \phi, E)$
- The volume of this function on phase space is the beam **Liouville emittance**

Vlasov and Boltzmann equations

- The evolution of the distribution function is described by **Vlasov** equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{\gamma m_0}\frac{\partial f}{\partial \mathbf{q}} + \mathbf{F}(\mathbf{q})\frac{\partial f}{\partial \mathbf{p}} = 0$$

- Mathematical representation of **Liouville theorem** stating the conservation of phase space volume $(\mathbf{q}, \mathbf{p})$
- In the presence of fluctuations (**radiation**, collisions, etc.) distribution function evolution described by **Boltzmann equation**

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{\gamma m_0}\frac{\partial f}{\partial \mathbf{q}} + \mathbf{F}(\mathbf{q})\frac{\partial f}{\partial \mathbf{p}} = \left.\frac{df}{dt}\right|_{\text{fluct}}$$

- The distribution evolves towards a **Maxwell-Boltzmann statistical equilibrium**

2D and normalized emittance

- When motion is uncoupled, Vlasov equation still holds for each plane individually

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{p_u}{\gamma m_0}\frac{\partial f}{\partial u} + \mathbf{F}(u)\frac{\partial f}{\partial p} = 0$$

- The Liouville emittance in the 2D (u, p_u) phase space is still conserved
- In the case of acceleration, the emittance is conserved in the (u, p_u) but not in the (u, u') (**adiabatic damping**)
- Considering that

$$u' = \frac{du}{ds} = \frac{p_u}{p_s}$$

the beam is conserved in the phase space $(u, u'p_s)$

- Define a **normalised emittance which is conserved during acceleration**

$$\epsilon_n = \beta_r \gamma_r \epsilon$$

- We would like to determine the transformation of the beam enclosed by an ellipse through the accelerator
- Consider a vector **u = (x,x',y,y',...)** in a generalized n-dimensional phase space. In that case the ellipse transformation is

$$\mathbf{u}^T \cdot \mathbf{\Sigma}^{-1} \cdot \mathbf{u} = \mathcal{I}$$

- Application to one dimension gives $\Sigma_{11}u^2 + 2\Sigma_{22}uu' + \Sigma_{22}u'^2 = 1$ and comparing with $\gamma_u u^2 + 2\alpha_u uu' + \beta_u u'^2 = \epsilon_u$ provides the beam matrix $\Sigma_u = \begin{pmatrix} \beta_u & -\alpha_u \\ -\alpha_u & \gamma_u \end{pmatrix} \epsilon_u = \mathcal{B}\epsilon_u$ which can be expanded to more dimensions
- Evolution of the n-dimensional phase space from position 1 to position 2, through transport matrix $\mathcal{M}$

$$\mathcal{M} \cdot \mathbf{\Sigma}_1 \cdot \mathcal{M}^T = \mathbf{\Sigma}_2$$

- The average of a function on the beam distribution defined

$$\langle g(\mathbf{u},\mathbf{u}')\rangle = \frac{1}{n}\sum_{i=1}^{n} g(u_i,u_i') = \frac{1}{N}\iint f(\mathbf{u},\mathbf{u}')g(\mathbf{u},\mathbf{u}')d\mathbf{u}d\mathbf{u}'$$

- Taking the square root, the following **Root Mean Square (RMS)** quantities are defined
  - **RMS beam size**

$$u_{\rm rms} = \sqrt{\sigma_u} = \sqrt{\langle (u - \langle u\rangle)^2\rangle}$$

  - **RMS beam divergence**

$$u'_{\rm rms} = \sqrt{\sigma'_u} = \sqrt{\langle (u' - \langle u'\rangle)^2\rangle}$$

  - **RMS coupling**

$$(uu')_{\rm rms} = \sqrt{\sigma_{uu'}} = \sqrt{\langle (u - \langle u\rangle)(u' - \langle u'\rangle)\rangle}$$

RMS emittance

- Beam modelled as macro-particles
- Involved in processed linked to the statistical size
- The **rms emittance** is defined as

$$\epsilon_{\mathrm{rms}} = \sqrt{\langle u \rangle^2 \langle u' \rangle^2 - \langle u u' \rangle^2}$$

- It is a statistical quantity giving information about the minimum beam size
- For linear forces the rms emittance is conserved in the case of linear forces
- The determinant of the rms beam matrix $det(\Sigma_{\mathrm{rms}}) = \epsilon_{\mathrm{rms}}$
- Including acceleration, the determinant of 6D transport matrices is not equal to 1 but

$$\det(\mathcal{M}_{1\to 2}) = \sqrt{\frac{\beta_{r2}\gamma_{r2}}{\beta_{r1}\gamma_{r1}}}$$

- The best ellipse fitting the beam distribution is

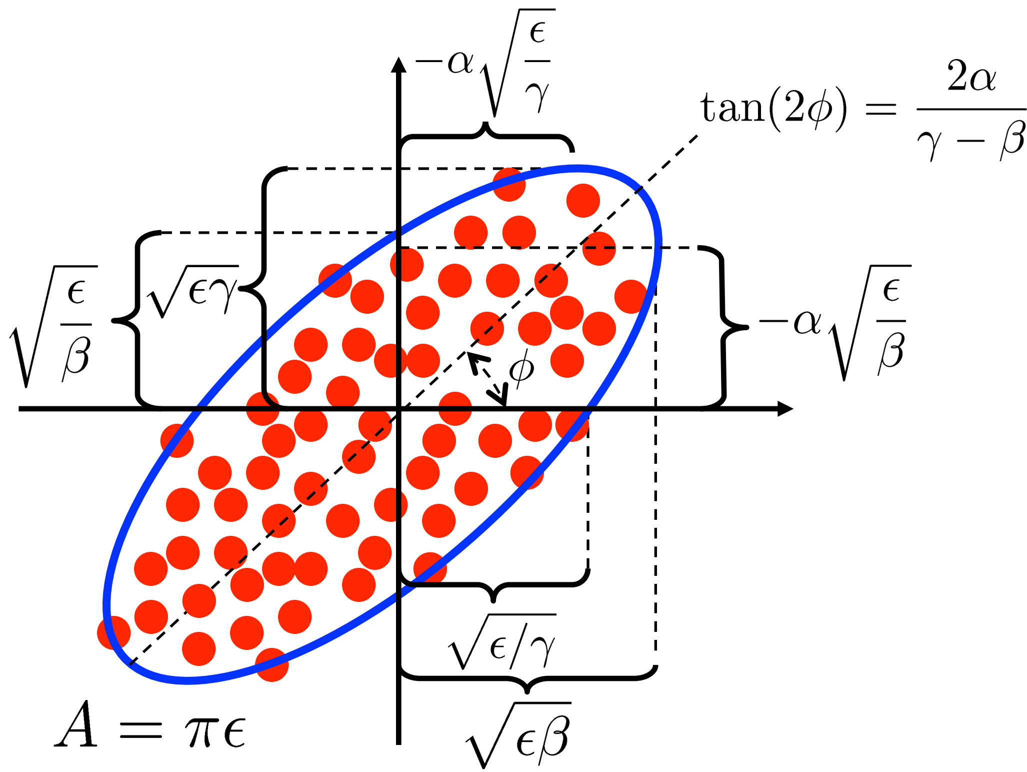
$$\gamma_u u^2 + 2\alpha_u u u' + \beta_u u'^2 = \epsilon_u$$

- The beam betatron functions can be defined through the rms emittance

$$\beta_u = \frac{u^2_{\text{rms}}}{\epsilon_{\text{rms}}} = \frac{\sigma_u}{\epsilon_{\text{rms}}}$$

$$\gamma_u = \frac{u'^2_{\text{rms}}}{\epsilon_{\text{rms}}} = \frac{\sigma'_u}{\epsilon_{\text{rms}}}$$

$$\alpha_u = \frac{(u\ u')_{\text{rms}}}{\epsilon_{\text{rms}}} = \frac{\sigma_{uu'}}{\epsilon_{\text{rms}}}$$



Gaussian distribution

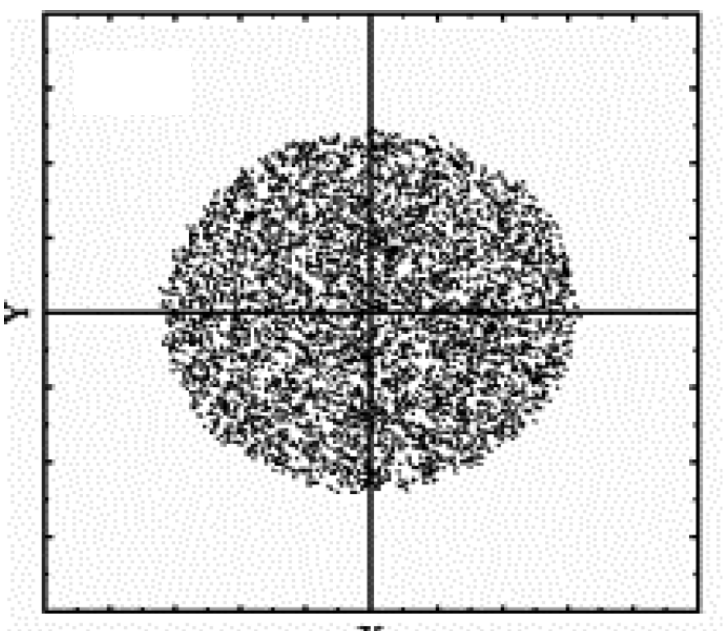
- The **Gaussian distribution** has a gaussian density profile in phase space

$$f(x,x',y,y') = \frac{N}{A} exp\left(-\frac{\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2}{2\epsilon_{x,\mathrm{rms}}} + \frac{\gamma_y y^2 + 2\alpha_y y y' + \beta_y y'^2}{2\epsilon_{y,\mathrm{rms}}}\right)$$

for which $\int f(\mathbf{u},\mathbf{u}')d\mathbf{u}d\mathbf{u}' = N$

- The beam boundary is $\gamma_u u^2 + 2\alpha_u u u' + \beta_u u'^2 = n^2 \epsilon_{u,\mathrm{rms}}$

Uniform (KV)



Gaussian