Beam dynamics with radiation damping

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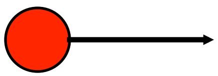
Outline

- Radiation damping
  - Synchrotron oscillations
  - Betatron oscillations
  - Robinson theorem
- Radiation integrals
- Quantum excitation
- Equilibrium emittances

- Up to this point, the transport of a relativistic particle around a ring was treated as a conservative process
- The particle change of momentum (acceleration) results in emission of synchrotron radiation
- It turns out that this is much more important in circular then linear accelerators
- The emission of synchrotron radiation results in energy lost by the particle and the damping of oscillations, called **radiation damping**
- This energy lost is recovered by the RF accelerating cavities in the longitudinal direction but not in the transverse

Why circular machines?



$$\mathbf{p} = m_0 \mathbf{v} \qquad v << c$$

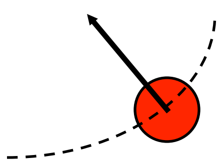
$$P_s = \frac{e^2}{6\pi\varepsilon_0 m_0^2 c^3} \left(\frac{d\mathbf{p}}{dt}\right)^2$$

Larmor Power radiated by non-relativistic particles is very small

$$\mathbf{p} = \gamma m_0 \mathbf{v} \qquad v \approx c$$

$$P_s = \frac{e^2}{6\pi\varepsilon_0 m_0^2 c^3} \left(\frac{dp}{dt}\right)^2$$

Power radiated by relativistic particle in linear accelerator is negligible



$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 (m_0 c^2)^4} \frac{E^4}{\rho^2}$$

Power radiated by relativistic particle in circular accelerator is very strong (Liénard, 1898)

Lienard's Paper

- "Electric and Magnetic Field produced by an electric charge concentrated at a point and travelling on an arbitrary path"

Prophetically published in the french journal "The Electric Light"

L'Éclairage Électrique

REVUE HEBDOMADAIRE D'ÉLECTRICITÉ

DIRECTION SCIENTIFIQUE

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CHAMP ÉLECTRIQUE ET MAGNÉTIQUE

PRODUIT PAR UNE CHARGE ÉLECTRIQUE CONCENTRÉE EN UN POINT ET ANIMÉE D'UN MOUVEMENT QUELCONQUE

Admettons qu'une masse électrique en mouvement de densité ρ et de vitesse u en chaque point produit le même champ qu'un courant de conduction d'intensité $u\rho$. En conservant les notations d'un précédent article (¹) nous obtiendrons pour déterminer le champ, les équations

$$\frac{1}{4\pi}\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right)=\rho u_x+\frac{df}{dt} \quad (1)$$

$$V^2\left(\frac{dh}{dy}-\frac{dg}{dz}\right)=-\frac{1}{4\pi}\frac{d\alpha}{dt} \quad (2)$$

avec les analogues déduites par permutation tournante et en outre les suivantes

$$\rho=\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right) \quad (3)$$

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0. \quad (4)$$

De ce système d'équations on déduit facilement les relations

$$\left(V^2\Delta-\frac{d^2}{dt^2}\right)f=V^2\frac{d\rho}{dx}+\frac{d}{dt}(\rho u_x) \quad (5)$$

$$\left(V^2\Delta-\frac{d^2}{dt^2}\right)\alpha=4\pi V^2\left[\frac{d}{dz}(\rho u_y)-\frac{d}{dy}(\rho u_z)\right] \quad (6)$$

(¹) La théorie de Lorentz, *L'Éclairage Électrique*, t. XIV, p. 417. α, β, γ, sont les composantes de la force magnétique et f, g, h, celles du déplacement dans l'éther.

Soient maintenant quatre fonctions ψ, F, G, H définies par les conditions

$$\left(V^2\Delta-\frac{d^2}{dt^2}\right)\psi=-4\pi V^2\rho. \quad (7)$$

$$\left.\begin{aligned}\left(V^2\Delta-\frac{d^2}{dt^2}\right)F&=-4\pi V^2\rho u_x\\ \left(V^2\Delta-\frac{d^2}{dt^2}\right)G&=-4\pi V^2\rho u_y\\ \left(V^2\Delta-\frac{d^2}{dt^2}\right)H&=-4\pi V^2\rho u_z\end{aligned}\right\} \quad (8)$$

On satisfera aux conditions (5) et (6) en prenant

$$4\pi f=-\frac{d\psi}{dx}-\frac{1}{V^2}\frac{dF}{dt} \quad (9)$$

$$\alpha=\frac{dH}{dy}-\frac{dG}{dz} \quad (10)$$

Quant aux équations (1) à (4), pour qu'elles soient satisfaites, il faudra que, en plus de (7) et (8), on ait la condition

$$\frac{d\psi}{dt}+\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=0. \quad (11)$$

Occupons-nous d'abord de l'équation (7). On sait que la solution la plus générale est la suivante :

$$\psi=\int\frac{\rho\left[x',y',z',t-\frac{r}{V}\right]}{r}\,d\varpi \quad (12)$$

Why electrons?

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 (m_0 c^2)^4} \frac{E^4}{\rho^2}$$

Power inversely proportional to 4th power of rest mass (proton **2000 times** heavier than electron) On the other hand, for multi TeV hadron colliders (LHC) synchrotron radiation is an important issue (protection with absorbers)

$$\Delta E = \frac{e^2}{3\varepsilon_0 (m_0 c^2)^4} \frac{E^4}{\rho}$$

By integrating around one revolution, the **energy loss per turn** is obtained**.** For the ILC DR it is around 4.5 MeV/turn. On the other hand, for LEPII (**120 GeV**) it was 6GeV/turn, i.e. circular electron machines of 100s of GeV become not very practical

Synchrotron Radiation formulas for e⁻/e⁺

- The power radiated by relativistic electron can be rewritten as

$$P_\gamma = \frac{cC_\gamma E^4}{2\pi\rho} = \frac{e^2c^3}{2\pi}C_\gamma E^2 B^2 \quad \text{with } C_\gamma = 8.85\times 10^{-5}\frac{\text{m}}{(\text{GeV})^3}$$

- The energy loss per turn can be expressed as

$$U_0 = \frac{C_\gamma E^4}{2\pi}\oint\frac{ds}{\rho^2} \quad \text{with } \mathcal{I}_2 = \oint\frac{ds}{\rho^2}$$ the 2nd **radiation integral**

- For a lattice with uniform bending radius (iso-magnetic) this yields:

$$U_0[keV] = 88.5\frac{E^4}{\rho}\frac{[\text{GeV}]^4}{[\text{m}]}$$

- If this energy were not recovered , particles would gradually spiral inward until lost on vacuum chamber wall
- RF cavities replace this lost energy by providing momentum kicks to the beam in the longitudinal direction

Damping of synchrotron oscillations

- Consider the differential equation of the energy for longitudinal motion

$$\Delta\ddot{E} + \alpha_s \Delta\dot{E} + \Omega\Delta E = 0$$

with damping coefficient $\alpha_s = \frac{1}{2T_0}\frac{dU}{dE}$

where U is the energy requirement per turn of the particle, and T_0 the revolution period and the synchrotron frequency

$$\Omega^2 = \frac{e\alpha_c \omega_{RF} V_0 \cos\phi_s}{ET_0}$$

- The solution can be written as a damped oscillation in energy and time with respect to the ideal synchromous particle

$$\Delta E(t) = A_E e^{-\alpha_s t}\cos(\Omega t - \varphi_s)$$

$$\tau(t) = \frac{-\alpha_c A_E}{E_0\Omega} e^{-\alpha_s t}\sin(\Omega t - \varphi_s)$$

- Note that the synchrotron motion is damped towards the motion of the synchronous particle
- The damping coefficient is dependent on the energy of the particle through the radiated power but also through the revolution period. In the following, we try to establish this relationship
- A particle with energy spread follows a dispersive trajectory with dispersion D

$$ds' = (1 + \frac{\Delta x}{\rho})ds = (1 + \frac{D}{\rho}\frac{\Delta E}{E})ds$$

- The energy requirement per turn can be obtained by the integral of the radiated power in one revolution

$$U = \oint P_s dt = \oint P_s ds'/c = \frac{1}{c}\oint P_s(1 + \frac{D}{\rho}\frac{\Delta E}{E})ds$$

- Differentiating with respect to the energy

$$\frac{dU}{dE} = \frac{1}{c}\oint \left[\frac{dP_s}{dE} + \frac{D}{\rho}(\frac{dP_s}{dE}\frac{\Delta E}{E} + \frac{P_s}{E})\right] ds$$

Damping of synchrotron oscillations II

- Taking into account that the average energy spread around the ring should be zero the previous integral is written:

$$\frac{dU}{dE} = \frac{1}{c}\oint\left(\frac{dP_s}{dE} + \frac{D}{\rho}\frac{P_s}{E}\right)ds$$

- Setting $\mathcal{C} = \dfrac{e^4c^3}{6\pi\varepsilon_0(m_0c^2)^4}$ and taking into account the definition of the magnetic rigidity, the expression of the radiation power is written $P_s = \mathcal{C}E^2B^2$
- Its derivative with respect to the energy gives

$$\frac{dP_s}{dE} = 2\frac{P_s}{E}(1 + Dk\rho)$$

where we used the identity $\dfrac{dB}{dE} = \dfrac{dB}{dx}\dfrac{dx}{dE} = \dfrac{dB}{dx}\dfrac{D}{E} = Bk\rho\dfrac{D}{E}$

- Replacing in the integral at the top,

$$\frac{dU}{dE} = \frac{2U_0}{E} + \frac{1}{cE}\oint DP_s\left(2k\rho + \frac{1}{\rho}\right)ds$$

Damping of synchrotron oscillations III

- Replacing in the last integral the expression of the power

$$\oint DP_s\left(2k\rho+\frac{1}{\rho}\right)ds=\frac{\mathcal{C}E^4}{e^2c^2}\oint\frac{D}{\rho}\left(2k+\frac{1}{\rho^2}\right)ds$$

and taking into account that $U_0=\frac{1}{c}\oint P_s ds=\frac{\mathcal{C}E^4}{e^2c^3}\oint\frac{ds}{\rho^2}$

the damping of synchrotron motion is written

$$\alpha_s=\frac{1}{2T_0}\frac{dU}{dE}=\frac{U_0}{2ET_0}(2+\mathcal{D})=\frac{U_0}{2ET_0}\mathcal{J}_s$$

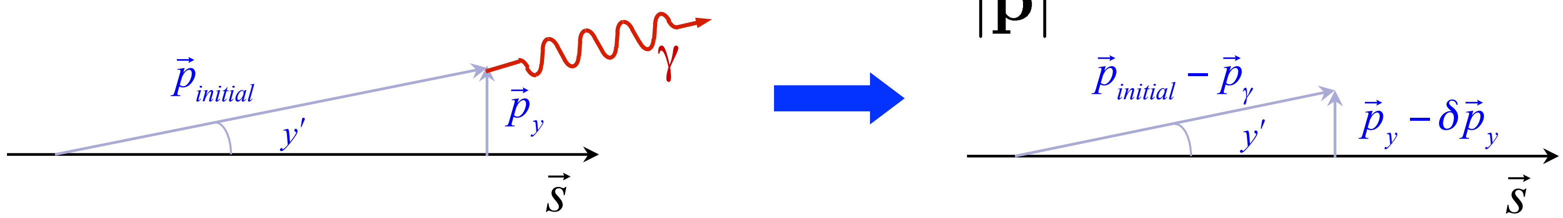
with the damping partition number defined as

$$\mathcal{D}=\frac{\oint\frac{D}{\rho}\left(2k+\frac{1}{\rho^2}\right)ds}{\oint\frac{ds}{\rho^2}}=\frac{\mathcal{I}_4}{\mathcal{I}_2}$$

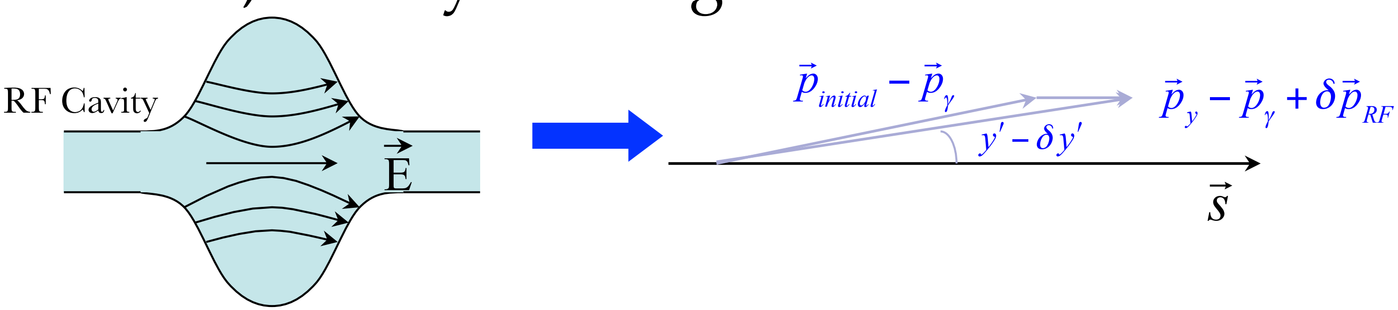
- Entirely defined by the lattice!
- Bending magnets and quads are usually separated and the damping partition number is usually extremely small

Damping of vertical oscillations

- Synchrotron radiation emitted in the direction of motion of electron, whose momentum is reduced
- This reduces the vertical component of the momentum but the angle remains the same $y' = \frac{\delta p_\perp}{|\mathbf{p}|}$



- The key for betatron damping is the energy recovery by the RF cavities, as only the longitudinal momentum is restored



- The change in energy will not affect the vertical position but the angle changes proportionally $\delta y' = y' \frac{\delta E}{E}$

Damping of vertical oscillations II

- Recall now solution of Hill's equations in the vertical plane, assuming that the beta function is slowly varying (i.e. alpha function is zero), for simplicity

$$y = A\cos\phi \;\; \text{and} \;\; y' = -\frac{A}{\beta(s)}\sin\phi$$

- The betatron oscillation amplitude is $A^2 = y^2 + [\beta(s)y']^2$
- The change of the amplitude becomes

$$\delta(A^2) = \beta(s)\delta(y'^2) \Rightarrow A\delta A = -\beta^2(s)y'^2\frac{\delta E}{E}$$

- By averaging over all angles $\langle \beta^2(s)y'^2 \rangle = \frac{A^2}{2\pi}\int_0^{2\pi}\sin^2\phi d\phi = \frac{A^2}{2}$ and $A\langle\delta A\rangle = -\frac{A^2}{2}\frac{\delta E}{E}$
- Summing up the energy losses for a full turn $\frac{\Delta A}{A} = -\frac{U_0}{2E}$
- Thus, in one turn the amplitudes are damped with a constant

$$\alpha_y = -\frac{\Delta A}{A\Delta t} = \frac{U_0}{2ET_0}$$

- The vertical betatron amplitude is thus exponentially decaying
$$A(t) = A(0)e^{-\alpha_y t}$$
- Equivalently, the damping of the vertical emittance is given by
$$\epsilon_y = \epsilon_y(0)e^{-2\alpha_y t}$$

- This means that the vertical emittance in the absence of dispersion or coupling will be reduced to zero
- Actually, due to radiation emission, the vertical oscillations are not reduced to zero
- This gives a "quantum limit", beyond which the vertical emittance cannot be further reduced

Damping of horizontal oscillations

- The horizontal motion is described by

$$x = x_\beta + x_e = A\cos\phi + D\frac{\Delta E}{E} \quad \text{and} \quad x' = x'_\beta + x'_e = -\frac{A}{\beta(s)}\sin\phi + D'\frac{\Delta E}{E}$$

- Energy change u due to photon emission results in a change of the dispersive part but not of the total coordinates so that

$$\delta x_\beta = -\delta x_e = D\frac{u}{E} \quad \text{and} \quad x\delta x'_\beta = -\delta x'_e = D'\frac{u}{E}$$

- The change of the Betatron amplitude $A^2 = x_\beta^2 + \left[\beta(s)x'_\beta\right]^2$ becomes $A\delta A = -(Dx_\beta + \beta^2(s)D'x'_\beta)\frac{u}{E}$
- The energy loss in an element *dl* is written

$$u = -\frac{P_s(x_\beta)}{c}dl = -\frac{1}{c}\left(P_s + 2\frac{P_s}{B}\frac{dB}{dx}x_\beta\right)\left(1 + \frac{x_\beta}{\rho}\right)ds$$

- Substituting to the change in amplitude and averaging over the angles (and some patience…)

$\frac{\Delta A}{A} = -(1-\mathcal{D})\frac{U_0}{2E}$ and the damping coefficient $\alpha_x = \frac{U_0}{2ET_0}(1-\mathcal{D})$

Robinson theorem

- Grouping the damping constants and introducing the three damping times and damping partition numbers

$$\alpha_s = \frac{1}{\tau_s} = \frac{U_0}{2ET_0}(2+\mathcal{D}) = \frac{U_0}{2ET_0}\mathcal{J}_s$$

$$\alpha_y = \frac{1}{\tau_y} = \frac{U_0}{2ET_0} = \frac{U_0}{2ET_0}\mathcal{J}_y$$

$$\alpha_x = \frac{1}{\tau_x} = \frac{U_0}{2ET_0}(1-\mathcal{D}) = \frac{U_0}{2ET_0}\mathcal{J}_x$$

- The **Robinson** theorem (1958) states that the sum of the damping partition number is an invariant

$$\mathcal{J}_x + \mathcal{J}_y + \mathcal{J}_s = 4$$

- In storage ring with separated function magnets, $\mathcal{D} \ll 1$ and

$$\mathcal{J}_x = 1\,, \quad \mathcal{J}_y = 1\,, \quad \mathcal{J}_s = 2$$

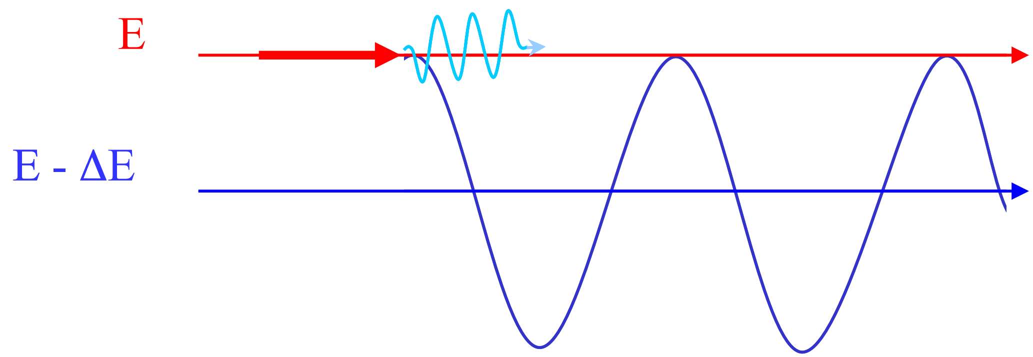
- The longitudinal damping occurs at twice the rate of the damping in the two transverse dimensions

Transverse emittances

- Radiation damping provides a direct mechanism to take hot injected beams and reduce the equilibrium parameters to a regime useful for high luminosity colliders and high brightness light sources.
- At the same time, the radiated power plays a dominant role in the design of the associated hardware and its protection
- If the only effect was radiation damping, the transverse emittances would be damped to zero.
- Photons are emitted in energy bursts in localized areas and horizontal betatron oscillations are excited as well (quantum fluctuations)
- Vertical emittance can become very small and only excited by coupling with the horizontal or residual vertical dispersion
- Electrons are influenced by this stochastic effect and eventually loose memory (unlike hadrons)

Equilibrium Beam Properties

- The emission of photons by the beam is a random process around the ring
- Photons are emitted within a cone around the direction of the beam particle with a characteristic angle *1/γ*
- This quantized process excites oscillations in each dimension



- In the absence of resonance or collective effects, which also serve to **heat** the beam, the balance between quantum excitation and radiation damping results in the equilibrium beam properties that are characteristics of a given ring lattice

Quantum Excitation - Longitudinal

- For the very short timescales corresponding to photon emission, we can take the equations of motion we previously obtained for synchrotron motion and write:

$$\Delta E^2(t) + \frac{E^2\Omega^2}{\alpha_c^2}\tau^2(t) = A_E^2$$

where A_E is a constant of the motion.

- The change in A_E due to the emission of photons should be estimated
- The emission of an individual photon will not affect the time variable, however, it will cause an instantaneous change in the value of ΔE

- From the solution of the synchrotron equation of motion, the energy difference is

$$\delta(\Delta E) = A_0 \cos\Omega\left(t - t_0\right) - \frac{u}{E}\cos\Omega\left(t - t_1\right) = A_1 \cos\Omega\left(t - t_1\right)$$

where u is the energy radiated at time t_1. Thus

$$A_1^2 = A_0^2 + \left(\frac{u}{E}\right)^2 - \frac{2A_0 u}{E}\cos\Omega\left(t_1 - t_0\right)$$

and $$\Delta A^2 = \left\langle A^2 - A_0^2 \right\rangle = \frac{u^2}{E^2}$$

- Considering the rate of photon emission $\mathcal{N}$, the average change in synchrotron amplitude due to photon emission is

$$\frac{d\left\langle A^2 \right\rangle}{dt} = \mathcal{N}\left(\frac{u}{E_0}\right)^2$$

- By including, the radiation damping term, the net change in the synchrotron amplitude can be written as:

$$\frac{d\left\langle A^2\right\rangle}{dt} = -2\alpha_E\left\langle A^2\right\rangle + \mathcal{N}\frac{u^2}{E^2}$$

- The equilibrium properties of a bunch are obtained when the rate of growth from quantum excitation and the rate of damping from radiation damping are equal
- For an ensemble of particles where the RMS energy amplitude is represented by the RMS energy spread, the equilibrium condition are written as

$$\sigma_\delta^2 = \left(\frac{\sigma_E}{E}\right)^2 = \frac{\left\langle A^2\right\rangle}{2} = \frac{\left\langle \mathcal{N}\left\langle u^2\right\rangle\right\rangle_s}{4\alpha_E E^2}$$

Photon Emission

- The term $\left\langle \mathcal{N}\left\langle u^2\right\rangle\right\rangle_s$ is the ring-wide average of the photon emission rate, $\mathcal{N}$, times the mean square energy loss associated with each emission

$$\mathcal{N} = \int_0^\infty n(u)du \quad \text{and} \quad \mathcal{N}\left\langle u^2\right\rangle = \int_0^\infty u^2 n(u)du$$

where $n(u)$ is the photon emission rate at energy u,

$$\left\langle \mathcal{N}\left\langle u^2\right\rangle\right\rangle_s = \frac{1}{C}\oint \mathcal{N}\left\langle u^2\right\rangle ds$$

with C is the ring circumference.

- The derivation of the photon spectrum emitted in a magnetic field is quite lengthy and we just quote the result

$$\mathcal{N}\left\langle u^2\right\rangle = 2C_q\gamma^2\frac{E\,P_\gamma}{\rho} \quad \text{where} \quad C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc} = 3.84\times 10^{-13}m$$

Energy Spread and Bunch Length

- Integrating around the ring then yields the RMS beam energy spread

$$\sigma_\delta^2 = \left(\frac{\sigma_E}{E}\right)^2 = C_q \gamma^2 \frac{I_3}{J_s I_2} = C_q \gamma^2 \frac{I_3}{2I_2 + I_4} \quad \text{where} \quad I_3 = \oint \frac{ds}{|\rho|^3}$$

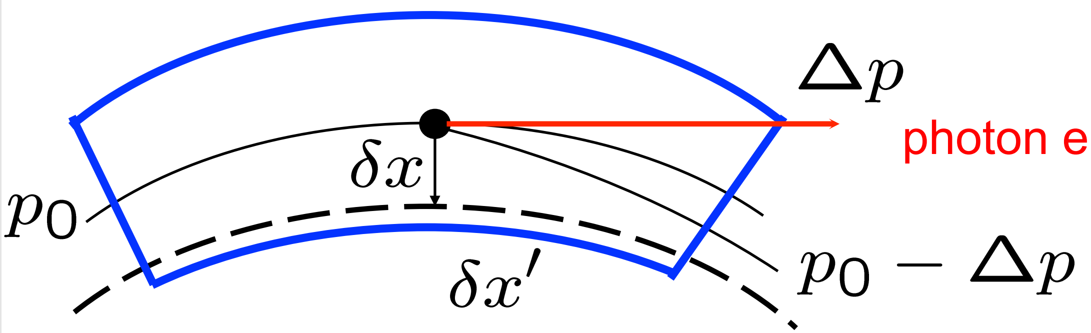
- Using this expression with the synchrotron equations of motion, the bunch length is related to the energy spread by

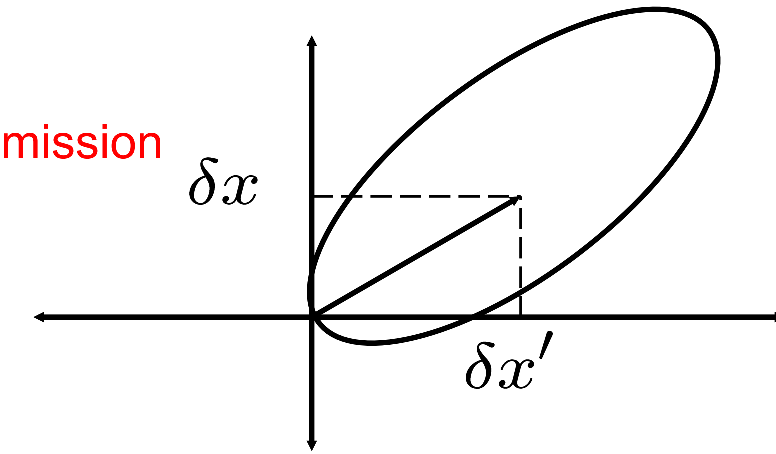
$$\sigma_z = \sigma_\delta \sqrt{\frac{\alpha_c C^2 \gamma m c^2}{2\pi h e V_0 |\cos\phi_s|}}$$

, with the harmonic number

$$h = \frac{f_{\mathrm{RF}} C}{c}$$

- The bunch length scales inversely with the square root of the RF voltage.





- Assume electron along nominal momentum orbit with initially negligible emittance
- After photon emission with momentum *Δp*, electron's momentum becomes p_0-Δp and the trajectory becomes

$$\delta x = D \frac{\Delta p}{p} \quad \text{and} \quad \delta x' = D' \frac{\Delta p}{p}$$

Dispersion emittance

- Recall that the emittance of the betatron ellipse in phase space is

$$\varepsilon_x = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s)$$

- Taking into account the change of the position and angle due to the photon emission, the change of the emittance is

$$\delta\varepsilon_x = \left(\gamma D^2 + 2\alpha D D' + \beta D'^2\right)\left(\frac{\delta p}{p}\right)^2 = \mathcal{H}(s)\left(\frac{\delta p}{p}\right)^2$$

with the "**dispersion**" **emittance** (or curly H-function)

$$\mathcal{H}(s) = \beta(s)D(s)'^2 + 2\alpha(s)D(s)D'(s) + \gamma(s)D(s)^2$$

Horizontal equilibrium emittance

- Averaging over all photon energies and emission probabilities, the **equilibrium emittance** is derived as

$$\epsilon_x = \frac{C_q \gamma^2}{\mathcal{J}_x} \frac{\oint \frac{\mathcal{H}_x(s)}{|\rho_x|^3} ds}{\oint \frac{1}{\rho_x^2} ds} = \frac{C_q \gamma^2}{\mathcal{J}_x} \frac{\mathcal{I}_5}{\mathcal{I}_2}, \text{ with}$$

$$C_q = \frac{55}{32\sqrt{3}} \frac{h}{m_0 c} = 3.83 \times 10^{-13} \text{ m}$$

- For isomagnetic ring with separated function magnets the equilibrium emittance is written

$$\epsilon_x = 1.47 \times 10^{-6} \frac{E^2}{\rho} \frac{1}{l_{\text{bend}}} \int_0^{l_{\text{bend}}} \mathcal{H}_x(s) ds$$

- The integral depends on the optics functions on the bends
- It gets small for small horizontal beta and dispersion, but this necessitates strong quadrupoles
- Smaller bending angle and lower energy reduce equilibrium emittance

Quantum Excitation - Vertical

- In the vertical dimension, assuming an ideal ring with no vertical dispersion, the quantum excitation of the emittance is determined by the opening angle of the emitted photons. The resulting perturbation to the vertical motion can be described as:

$$\delta y = 0 \qquad \delta y' = \frac{\Delta p}{p}\theta_\gamma$$

and the effect to the emittance is $\delta\varepsilon_y = \left(\frac{\Delta p}{p}\theta_\gamma\right)^2 \beta_y$

- Averaging over all photon energies and emission probabilities, the **quantum limit** of the **vertical emittance** is derived as

$$\varepsilon_y \approx \frac{C_q}{2J_y I_2}\oint \frac{\beta_y}{\rho^3} ds$$

Vertical Emittance & Emittance Coupling

- For typical storage ring parameters, the vertical emittance due to quantum excitation is very small
- Assuming a typical β_y values of a few 10's of meters and bending radius of ~100m, the quantum limit is $\varepsilon_y \sim 0.1$ pm.
- The observed sources of vertical emittance are:
  - **emittance coupling** whose source is ring errors which couple the vertical and horizontal betatron motion
  - **vertical dispersion** due to vertical misalignment of the quadrupoles and sextupoles and angular errors in the dipoles
- The vertical and horizontal emittances in the presence of a collection of such errors around a storage ring is commonly described as:

$$\varepsilon_y = \frac{\kappa}{1+\kappa}\varepsilon_0; \quad \varepsilon_x = \frac{1}{1+\kappa}\varepsilon_0 \quad \text{for } 0<\kappa<1$$

ε_0 is the horizontal equilibrium emittance.

Radiation integrals

$$\mathcal{I}_1 = \oint \frac{D}{\rho} ds$$

Momentum compaction factor

$$\alpha_c = \frac{\mathcal{I}_1}{2\pi R}$$

$$\mathcal{I}_2 = \oint \frac{1}{\rho^2} ds$$

Energy loss per turn

$$U_0 = \frac{C_\gamma}{2\pi} E^4 \mathcal{I}_2$$

$$\mathcal{I}_3 = \oint \frac{1}{|\rho|^3} ds$$

Equilibrium energy spread

$$\sigma_\delta^2 = C_q \gamma^2 \frac{\mathcal{I}_3}{2\mathcal{I}_2 + \mathcal{I}_4}$$

$$\mathcal{I}_4 = \oint \frac{D}{\rho^3}(1 + 2k\rho^2) ds$$

$$\mathcal{J}_x = 1 - \frac{\mathcal{I}_4}{\mathcal{I}_2} \;,\quad \mathcal{J}_s = 2 + \frac{\mathcal{I}_4}{\mathcal{I}_2} \;,\quad \mathcal{D} = \frac{\mathcal{I}_4}{\mathcal{I}_2}$$

Damping partition numbers

$$\mathcal{I}_5 = \oint \frac{\mathcal{H}}{|\rho^3|} ds$$

Equilibrium betatron emittance

$$\epsilon_x = C_q \gamma^2 \frac{\mathcal{I}_5}{\mathcal{I}_2 - \mathcal{I}_4}$$