

Using the Hadronic Recoil Cross Section Measurement in Higgs Coupling Fits

Tim Barklow (SLAC)

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Mark Thomson's analysis of $\sigma(ZH)$ with $Z \rightarrow q\bar{q}$ uses two measurements to obtain the cross section:

$$\sigma(ZH) = \sigma(ZH) \cdot BR(visible) + \sigma(ZH) \cdot BR(invisible)$$

$\sigma(ZH) \cdot BR(visible)$

$\sigma(ZH) \cdot BR(invisible)$

Final(?) Results



Process	σ/fb	$\epsilon_{\text{pre sel}}$	$\epsilon_{\mathcal{L} > 0.70}$	$N_{\mathcal{L} > 0.70}$
q \bar{q}	25180	0.5 %	<0.1 %	6211
q \bar{q} lv	5914	6.4 %	0.1 %	3895
q \bar{q} q \bar{q}	5847	4.2 %	0.4 %	10818
q \bar{q} ll	1704	1.2 %	0.1 %	1218
q \bar{q} v \bar{v}	325	0.6 %	<0.1 %	35
Hv $_e\bar{v}_e$		- %		
<hr/>				
HZ	93.4	44.0 %	20.3 %	9493
<hr/>				
H \rightarrow invis.		0.6 %	<0.1 %	-
H \rightarrow q \bar{q} /gg		43.5 %	20.6 %	6211
H \rightarrow WW*		44.7 %	19.5 %	2240
H \rightarrow ZZ*		40.0 %	18.1 %	254
H \rightarrow $\tau^+\tau^-$		47.6 %	21.4 %	738
H \rightarrow $\gamma\gamma$		42.8 %	22.1 %	32
H \rightarrow Z γ		41.8 %	17.6 %	17
H \rightarrow $\mu^+\mu^-$		39.5 %	20.6 %	3

- ★ For optimal cut
 - signal ~9.5k events
 - background ~ 19k events

15 % improvement
c.f. LCWS analysis

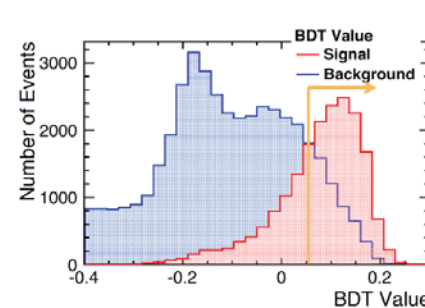
Efficiencies same
to ~10 % !!!

almost model
independent

BDT Selection



★ Preliminary results (7 variable BDT selection)



Signal		
Channel	Efficiency	
Z H \rightarrow qq invis.	20.7 %	
Backgrounds		
Channel	Efficiency	Events
qqlv	<0.1 %	900
qqll	<0.1 %	4
qqvv	1.5 %	2414

★ Assuming no invisible decays (1 sigma stat. error):

$$\Delta\sigma_{\text{invis}} = \pm 0.57 \%$$

(CLIC beam spectrum, 500 fb $^{-1}$ @ 350 GeV, no polarisation)

In order to use this cross section measurement in our Higgs analyses we have to quantify the penalty associated with the fact that $\sigma(ZH) \cdot BR(visible)$ is "almost model independent". By how much must we blow up $\Delta\sigma(ZH) \cdot BR(visible)$ to account for the fact that the efficiencies differ by 10% or more?



Model Independence



- ★ **Combining visible + invisible analysis: wanted M.I.**
 - **i.e. efficiency independent of Higgs decay mode**

Decay mode	$\epsilon_{\mathcal{L}>0.70}^{vis}$	$\epsilon_{BDT>0.08}^{invis}$	$\epsilon^{vis} + \epsilon^{invis}$
H → invis.	<0.1 %	20.7 %	20.7 %
H → q \bar{q} /gg	20.6 %	<0.1 %	20.6 %
H → WW*	19.5 %	<0.1 %	19.8 %
H → ZZ*	18.1 %	0.9 %	19.0 %
H → $\tau^+\tau^-$	21.4 %	0.1 %	21.5 %
H → $\gamma\gamma$	22.1 %	<0.1 %	22.1 %
H → Z γ	17.6 %	<0.1 %	17.1 %
H → $\mu^+\mu^-$	20.6 %	<0.1 %	20.6 %
<hr/>			
H → WW* → q \bar{q} q \bar{q}	19.3 %	<0.1 %	19.3 %
H → WW* → q \bar{q} lv	19.6 %	<0.1 %	19.6 %
H → WW* → q \bar{q} $\tau\nu$	19.9 %	<0.1 %	19.9 %
H → WW* → l ν lv	22.0 %	0.3 %	22.3 %
H → WW* → l ν $\tau\nu$	16.7 %	0.3 %	17.0 %
H → WW* → $\tau\nu$ $\tau\nu$	12.2 %	1.3 %	13.6 %

Very similar efficiencies

Look at wide range of WW topologies

We have used an approach where we use all of our $\sigma \cdot BR$ measurements for visible Higgs decays to obtain an estimate of the average signal efficiency for $\sigma(ZH) \cdot BR(visible)$. It is then straightforward to propagate the $\sigma \cdot BR$ errors to the error on $\sigma(ZH) \cdot BR(visible)$. This means that one must take into account the correlation between the $\sigma \cdot BR$ measurements and our $\sigma(ZH)$ measurement from hadronic Z decays when we fit for couplings and total width. **It also means that we must develop $\sigma \cdot BR$ analyses for all possible BSM Higgs decays**

Let

$$\Psi \equiv \sigma(ZH) \cdot BR(\text{visible})$$

Ω = Number of signal + background events in $\sigma(ZH) \cdot BR(\text{visible})$ analysis

B = Predicted number of background events in $\sigma(ZH) \cdot BR(\text{visible})$ analysis

Ξ = Average efficiency for signal events to pass $\sigma(ZH) \cdot BR(\text{visible})$ analysis

L = luminosity

$$\Psi = \frac{\Omega - B}{L \Xi} = \frac{1}{\Xi} \sum_i \psi_i \xi_i = \sum_i \psi_i \quad \text{where}$$

$$\psi_i = \sigma(ZH) \cdot BR_i$$

ξ_i = efficiency for events from Higgs decay i to pass $\sigma(ZH) \cdot BR(\text{visible})$ analysis

$$\Xi = \frac{\sum_i \psi_i \xi_i}{\sum_i \psi_i}$$

$$\psi_i = \frac{\omega_i - \beta_i}{L \eta_i}$$

ω_i = Number of signal + background events in $\sigma(ZH) \cdot BR_i$ analysis

β_i = Predicted number of background events in $\sigma(ZH) \cdot BR_i$ analysis

η_i = efficiency for Higgs decay i to pass $\sigma \cdot BR_i$ analysis

K_i = number of signal + background events common to had Z recoil
and $\sigma \cdot BR_i$ analyses

E = number of signal + background events unique to had Z recoil analysis

ε_i = number of signal + background events unique to $\sigma \cdot BR_i$ analysis

$$\Omega = E + \sum_i K_i \quad S \equiv \Omega - B \quad T \equiv \frac{\sqrt{S+B}}{S}$$

$$\omega_i = K_i + \varepsilon_i \quad s_i \equiv \omega_i - \beta_i \quad \tau_i \equiv \frac{\sqrt{s_i + \beta_i}}{s_i}$$

$$\lambda_i \equiv \frac{K_i}{\omega_i} \quad N \equiv L \sigma_{ZH} \quad r_i \equiv BR_i \quad \delta_i \equiv \xi_i - \Xi$$

$$\left(\frac{\Delta \Psi}{\Psi} \right)^2 = T^2 \left\{ 1 + \frac{N^2}{\Omega} \sum_i r_i^2 \tau_i^2 \left[\delta_i^2 - 2 \lambda_i \eta_i \delta_i \right] \right\}$$

This is our result for the error on $\sigma(ZH) \cdot BR(\text{visible})$ given the approach outlined on page 22

But, if we are confident in our measurement of $\sigma(ZH) \cdot BR(H \rightarrow BSM)$, why don't we simply calculate $\sigma(ZH)$ using $\sum_i \sigma(ZH) \cdot BR_i$?

In fact this is what Michael Peskin does in his fits when he uses the constraint $\sum_i BR_i = 1$. If this constraint implicitly includes an hadronic recoil ZH cross section measurement, then the importance of the leptonic recoil ZH cross section measurement should diminish when this constraint is imposed. This indeed is the case, as demonstrated in the following slides where Higgs couplings are calculated for different values of a $\Delta\sigma_{ZH}$ scale factor.

Energy Diff Lum Run Time* Int Lumi
 250 GeV 0.75 cm⁻²s⁻¹ 2.4 yr 280 fb¹

*No installation or ramp up; simply assume 50% eff. or 1.58×10^7 s per year.

$\Delta\sigma_{ZH}$ scale	1	1	2	2	4	4	8	8
$\sum_i BR_i = 1 ?$	no	yes	no	yes	no	yes	no	yes
ZZ	1.2%	0.74%	2.4%	0.87%	4.9%	0.91%	9.8%	0.93%
W^+W^-	4.5%	4.4%	5.0%	4.4%	6.5%	4.4%	10.8%	4.4%
$b\bar{b}$	4.9%	4.4%	5.4%	4.4%	6.8%	4.4%	10.9%	4.4%
$c\bar{c}$	6.3%	5.9%	6.6%	5.9%	7.9%	6.0%	11.6%	6.0%
gg	5.8%	5.5%	6.2%	5.5%	7.5%	5.5%	11.3%	5.5%
$\tau^+\tau^-$	5.3%	4.8%	5.7%	4.9%	7.1%	4.9%	11.1%	4.9%
$\gamma\gamma$	16.7%	16.6%	16.9%	16.6%	17.4%	16.6%	19.4%	16.6%
Γ_T	10.8%	8.3%	13.7%	8.4%	21.8%	8.4%	40.3%	8.4%

Energy	Diff Lum	Run Time*	Int Lumi
250 GeV	0.75 cm ⁻² s ⁻¹	16.9 yr	2000 fb ¹
350 GeV	0.75 cm ⁻² s ⁻¹	1.3 yr	200 fb ¹
500 GeV	0.75 cm ⁻² s ⁻¹	10.6 yr	3000 fb ¹

*No installation or ramp up; simply assume 50% eff. or 1.58×10^7 s per year.

$\Delta\sigma_{ZH}$ scale	1	1	2	2	4	4	8	8
$\sum_i BR_i = 1 ?$	no	yes	no	yes	no	yes	no	yes
ZZ	0.41%	0.21%	0.81%	0.23%	1.6%	0.24%	3.3%	0.24%
W^+W^-	0.46%	0.20%	0.85%	0.21%	1.7%	0.21%	3.3%	0.21%
$b\bar{b}$	0.63%	0.39%	0.95%	0.39%	1.7%	0.39%	3.3%	0.39%
$c\bar{c}$	1.1%	0.98%	1.3%	0.98%	1.9%	0.98%	3.4%	0.98%
gg	0.89%	0.77%	1.1%	0.78%	1.8%	0.78%	3.4%	0.78%
$\tau^+\tau^-$	0.88%	0.73%	1.1%	0.73%	1.8%	0.73%	3.4%	0.73%
$\gamma\gamma$	3.0%	2.9%	3.1%	2.9%	3.4%	2.9%	4.4%	2.9%
Γ_τ	2.0%	0.79%	3.5%	0.79%	6.6%	0.79%	13%	0.79%

It is clear from these results that $\sigma(ZH)$ is being calculated implicitly using $\sigma(ZH) = \sum_i \sigma(ZH) \cdot BR_i$ whenever the constraint $\sum_i BR_i = 1$ is imposed.

It appears that we have no need for a separate direct hadronic ZH recoil cross section measurement once we are confident that we have $\sigma(ZH) \cdot BR(H \rightarrow \text{visible BSM})$ under control. In fact, it appears that we don't even need the classic leptonic ZH recoil cross section measurement in this case!

Caveats :

These results assume that the true $BR(H \rightarrow \text{BSM})=0$ and that $BR(H \rightarrow \text{visible BSM}) < 0.9\%$ at 95% CL can be achieved with $\sqrt{s}=250 \text{ GeV} \ \& \ 250 \text{ fb}^{-1}$ or $\sqrt{s}=350 \text{ GeV} \ \& \ 500 \text{ fb}^{-1}$. (That is, it has been assumed that the same precision can be achieved for invisible decays and visible BSM decays.) This has yet to be demonstrated.

Also, we have to check how these conclusions are altered if the true $BR(H \rightarrow \text{BSM})=1\%$, or 10%.

215 page "Exotic Decays of the 125 GeV Higgs Boson" arXiv:1312.4992 : Is this a starting point for a complete $\sigma \bullet \text{BR}(H \rightarrow \text{BSM})$ analysis?

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Questions

- What is the required precision on $\sigma \cdot BR(H \rightarrow BSM)$ if we want to impose the constraint $\sum_i BR_i = 1$?
- What is the required precision on $\sigma \cdot BR(H \rightarrow BSM)$ if we want to use the direct hadronic recoil measurement of $\sigma(ZH)$?
- Do the 18 different $\sigma \cdot BR(H \rightarrow BSM_i)$ searches outlined in arXiv:1312.4992 cover all possible BSM decays? If not, what else is needed? And once we have answered that question, how do we prove that everything has indeed been covered?

Backup Slides

Let

$$\Psi \equiv \sigma(ZH) \cdot BR(\text{visible})$$

Ω = Number of signal + background events in $\sigma(ZH) \cdot BR(\text{visible})$ analysis

B = Predicted number of background events in $\sigma(ZH) \cdot BR(\text{visible})$ analysis

Ξ = Average efficiency for signal events to pass $\sigma(ZH) \cdot BR(\text{visible})$ analysis

L = luminosity

$$\Psi = \frac{\Omega - B}{L \Xi} = \frac{1}{\Xi} \sum_i \psi_i \xi_i = \sum_i \psi_i \quad \text{where}$$

$$\psi_i = \sigma(ZH) \cdot BR_i$$

ξ_i = efficiency for events from Higgs decay i to pass $\sigma(ZH) \cdot BR(\text{visible})$ analysis

$$\Xi = \frac{\sum_i \psi_i \xi_i}{\sum_i \psi_i}$$

$$\psi_i = \frac{\omega_i - \beta_i}{L\eta_i}$$

ω_i = Number of signal + background events in $\sigma(ZH)\cdot BR_i$ analysis

β_i = Predicted number of background events in $\sigma(ZH)\cdot BR_i$ analysis

η_i = efficiency for Higgs decay i to pass $\sigma\cdot BR_i$ analysis

K_i = number of signal + background events common to had Z recoil
and $\sigma\cdot BR_i$ analyses

E = number of signal + background events unique to had Z recoil analysis

ε_i = number of signal + background events events unique to $\sigma\cdot BR_i$ analysis

$$\Omega = E + \sum_i K_i \quad S \equiv \Omega - B \quad T \equiv \frac{\sqrt{S+B}}{S}$$

$$\omega_i = K_i + \varepsilon_i \quad s_i \equiv \omega_i - \beta_i \quad \tau_i \equiv \frac{\sqrt{s_i + \beta_i}}{s_i}$$

$$\lambda_i \equiv \frac{K_i}{\omega_i} \quad N \equiv L\sigma_{ZH} \quad r_i \equiv BR_i \quad \delta_i \equiv \xi_i - \Xi$$

$$(\Delta\Psi)^2 = \left(\frac{\partial\Psi}{\partial\Omega}\right)^2 V_{\Omega\Omega} + \left(\frac{\partial\Psi}{\partial\Xi}\right)^2 V_{\Xi\Xi} + 2\frac{\partial\Psi}{\partial\Omega}\frac{\partial\Psi}{\partial\Xi} V_{\Omega\Xi}$$

$$\frac{\partial\Psi}{\partial\Omega} = \frac{1}{L\Xi} = \frac{\Psi}{\Omega} \left(1 - \frac{B}{\Omega}\right)^{-1} \quad \frac{\partial\Psi}{\partial\Xi} = -\frac{\Omega - B}{L\Xi^2} = -\frac{\Psi}{\Xi}$$

$$V_{\Omega\Omega} = E + \sum_i K_i = \Omega$$

$$V_{\Xi\Xi} = \frac{1}{L^2\Psi^2} \sum_i \frac{(\xi_i - \Xi)^2}{(\eta_i)^2} (\varepsilon_i + K_i)$$

$$V_{\Omega\Xi} = \frac{1}{L\Psi} \sum_i \frac{\xi_i - \Xi}{\eta_i} K_i$$

$$\begin{aligned}
\left(\frac{\Delta\Psi}{\Psi}\right)^2 &= \frac{1}{\Omega^2}\left(1-\frac{B}{\Omega}\right)^{-2} V_{\Omega\Omega} + \frac{1}{\Xi^2} V_{\Xi\Xi} - \frac{2}{\Omega\Xi}\left(1-\frac{B}{\Omega}\right)^{-1} V_{\Omega\Xi} \\
&= \frac{1}{\Omega}\left(1-\frac{B}{\Omega}\right)^{-2} + \frac{1}{L^2\Xi^2\Psi^2} \sum_i \frac{(\xi_i - \Xi)^2}{(\eta_i)^2} (\varepsilon_i + K_i) - \frac{2}{L\Omega\Xi\Psi}\left(1-\frac{B}{\Omega}\right)^{-1} \sum_i \frac{\xi_i - \Xi}{\eta_i} K_i \\
&= \frac{1}{\Omega}\left(1-\frac{B}{\Omega}\right)^{-2} + \frac{1}{L^2\Xi^2\Psi^2} \sum_i \frac{(\xi_i - \Xi)^2}{(\eta_i)^2} (L\eta_i\psi_i + \beta_i) - \frac{2}{L\Omega\Xi\Psi}\left(1-\frac{B}{\Omega}\right)^{-1} \sum_i \frac{\xi_i - \Xi}{\eta_i} \lambda_i (L\eta_i\psi_i + \beta_i) \\
&= \frac{1}{\Omega}\left(1-\frac{B}{\Omega}\right)^{-2} \left[1 + \frac{L}{\Omega} \sum_i \frac{(\xi_i - \Xi)^2}{\eta_i} \psi_i \left(1 + \frac{\beta_i}{s_i}\right) - \frac{2L}{\Omega} \sum_i (\xi_i - \Xi) \psi_i \lambda_i \left(1 + \frac{\beta_i}{s_i}\right) \right] \\
&= \frac{S+B}{S^2} \left\{ 1 + \frac{L}{\Omega} \sum_i (\xi_i - \Xi) \psi_i \left(\frac{s_i + \beta_i}{s_i^2}\right) [(\xi_i - \Xi)L\psi_i - 2\lambda_i s_i] \right\} \\
&= T^2 \left\{ 1 + \frac{N^2}{\Omega} \sum_i r_i^2 \tau_i^2 [\delta_i^2 - 2\lambda_i \eta_i \delta_i] \right\}
\end{aligned}$$

What if we don't do a hadronic Z recoil measurement and instead only use $\sigma(ZH) \cdot BR_i$ to calculate $\sigma(ZH) \cdot BR(\text{visible}) = \sum_i \sigma(ZH) \cdot BR_i$?

$$\Psi' = \sum_i \psi_i \quad \psi_i = \frac{\omega_i - \beta_i}{L \xi_i}$$

$$(\Delta\Psi')^2 = \sum_i \left(\frac{\partial\Psi'}{\partial\omega_i} \right)^2 \omega_i, \quad \frac{\partial\Psi'}{\partial\omega_i} = \frac{1}{L\eta'_i}$$

$$(\Delta\Psi')^2 = \frac{1}{L^2} \sum_i = \frac{1}{L^2} \sum_i \frac{s_i + \beta_i}{\xi_i^2}$$

$$\begin{aligned} \left(\frac{\Delta\Psi'}{\Psi'} \right)^2 &= \left(\sum_i \frac{\omega_i - \beta_i}{L \xi_i} \right)^{-2} \frac{1}{L^2} \sum_i \frac{s_i + \beta_i}{\xi_i^2} \\ &= \frac{S+B}{S^2} \frac{L}{\Omega} \Xi^2 \sum_i \frac{\psi_i}{\xi_i} \left(1 + \frac{\beta_i}{s_i} \right) \end{aligned}$$

Compare this now with our formula for $\left(\frac{\Delta\Psi}{\Psi} \right)^2$ for $\lambda_i = 1$:

$$\begin{aligned} \left(\frac{\Delta\Psi}{\Psi} \right)^2 &= \frac{S+B}{S^2} \left\{ 1 + \frac{1}{\Omega} \sum_i \omega_i \left[\left(1 - \frac{\Xi}{\xi_i} \right)^2 - 2 \left(1 - \frac{\Xi}{\xi_i} \right) \right] \right\} \\ &= \frac{S+B}{S^2} \left\{ 1 + \frac{1}{\Omega} \sum_i \omega_i \left[1 - \frac{2\Xi}{\xi_i} + \left(\frac{\Xi}{\xi_i} \right)^2 - 2 + 2 \frac{\Xi}{\xi_i} \right] \right\} = \left(\frac{\Delta\Psi'}{\Psi'} \right)^2 \end{aligned}$$

